CONCRETE CONSTRUCTIONS OF AUTOMORPHIC REPRESENTATIONS AND CENTRAL VALUES OF L-FUNCTIONS

BIN XU

Abstract. This expository article is based on the talk delivered at the RIMS conference "Arithmetic Aspects of Automorphic Forms and Automorphic Representations". We begin by reviewing certain relations among global packets, automorphic periods, and automorphic L-functions. Then, in the last part, we introduce the automorphic descent method, which provides a concrete construction of automorphic representations of classical groups. This method offers a new approach to show the existence of different quadratic twists of cuspidal automorphic representations of $\mathrm{PGL}_2(\mathbb{A})$ with non-zero central L-values.

1. Background

The L-function is one of the most important objectives in number theory, and its special value gives fruitful arithmetic information. For example, for the Riemann ζ-function, the first example of L-functions, one has

$$\zeta(2n) = (-1)^{n+1}\frac{(2\pi)^{2n}B_{2n}}{2(2n)!}, \quad (n \in \mathbb{Z}_{\geq 0}) \tag{1.1}$$

where B_m is the m-th Bernoulli number. In particular, one has $\zeta(2) = \pi^2/6$, and $\zeta(4) = \pi^4/90$. Moreover, let $p \equiv 7 \bmod 8$ be a prime number, and let $\chi_p(\cdot) = \left(\frac{\cdot}{p}\right)$ be the Legendre symbol modulo p. Then one has the formula for the special value of the Dirichlet L-function $L(s, \chi_p)$ at $s = 0$:

$$L(0, \chi_p) = \sum_{0<a<p/2} \chi_p(a) = h_{-p}, \tag{1.2}$$

where h_{-p} is the class number of $\mathbb{Q}(\sqrt{-p})$.

Now we consider the L-functions of modular forms. Denote by $S_k(N)$ the space of cusp modular forms of weight $k \in 2\mathbb{N}$ and level $\Gamma_0(N)$ (with trivial character). For a newform $f(z) \in S_k(N)$, one defines its (complete) L-function by

$$L(s, f) = (2\pi)^{-s}\Gamma(s)\sum_{n=1}^{\infty}\frac{a_n}{n^s}, \quad (s \in \mathbb{C})$$

where a_n's are the Fourier coefficients of f in its q-expansion. The above L-function has a functional equation with respect to

$$s \mapsto k - s,$$

hence its central value is at $s = k/2$. A basic question is the following:

Question 1.1. *For a new form $f \in S_k(N)$, are there Dirichlet characters η such that*

$$L\left(\frac{k}{2}, f \otimes \eta\right) \neq 0?$$

Here $L(s, f\otimes\eta)$ *is the twist of* $L(s,f)$ *by* η*, which is defined by*

$$L\left(s, f\otimes\eta\right) = (2\pi)^{-s}\Gamma(s)\sum_{n=1}^{\infty}\frac{\eta(n)a_n}{n^s}.$$

Such results are useful in many arithmetic applications. For example, let $f\in S_2(32)$ be the modular form associated with the elliptic curve

$$E: y^2 = x^3 - x$$

defined over $\mathbb{Q}$. Then $n\in\mathbb{Z}_{>0}$ is not a *congruent number* (see [8]) if

$$L\left(1, f\otimes\eta_n\right)\neq 0,$$

here η_n is the quadratic Dirichlet character associated with the quadratic extension $\mathbb{Q}(\sqrt{n})/\mathbb{Q}$ via class field theory.

Under suitable normalization, a newform $f\in S_k(N)$ above corresponds to an irreducible cuspidal automorphic representations τ of $\mathrm{PGL}_2(\mathbb{A}_{\mathbb{Q}})$, such that

$$L\left(s-\frac{k-1}{2},\tau\right) = L(s,f).$$

It follows that Question 1.1 can be considered in the context of automorphic representation theory.

From now on, let F be a number field and let $\mathbb{A}$ be its adele ring. One may consider the following more general question:

Question 1.2. *For an irreducible cuspidal automorphic representation* τ *of* $\mathrm{PGL}_2(\mathbb{A})$*, are there automorphic (Hecke) characters* $\eta: F^\times\backslash\mathbb{A}^\times\longrightarrow\mathbb{C}^\times$ *such that*

$$L\left(\frac{1}{2},\tau\otimes\eta\right)\neq 0?$$

The above questions are first studied by Shimura (see [45]), followed by Goldfeld-Hoffstein-Patterson ([25]), Waldspurger (see [49, 50]), Jacquet (see [28]), Rohrlich (see [43, 44]), Bump-Friedberg-Hoffstein (see [3, 4]) and Friedberg-Hoffstein (see [9]).

In this article, we will focus on the case where the twist η is *quadratic*. In this case, we have the functional equation

$$L(s,\tau\otimes\eta) = \varepsilon(s,\tau\otimes\eta)L(1-s,\tau\otimes\eta),$$

here $\varepsilon(s,\tau\otimes\eta)$ is the global epsilon factor. Moreover, for $L(s,\tau\otimes\eta)\neq 0$ at the central point $s=1/2$, one requires that

$$\varepsilon\left(\frac{1}{2},\tau\otimes\eta\right) = 1.$$

The following theorem, due to Waldspurger and Friedberg-Hoffstein, gives an answer to Question 1.2 for quadratic twists:

Theorem 1.3 (Waldspurger [50], Friedberg-Hoffstein [9])**.** *Let* τ *be an irreducible cuspidal automorphic representation of* $\mathrm{PGL}_2(\mathbb{A})$*. Suppose that there exists a quadratic character* η_0 *such that* $\varepsilon(1/2,\tau\otimes\eta_0)=1$*, then there exist (infinitely many) quadratic automorphic characters* η *such that*

$$L\left(\frac{1}{2},\tau\otimes\eta\right)\neq 0.$$

In fact, the result of Friedberg-Hoffstein ([9]) is even stronger. They consider all cuspidal representations of $\mathrm{GL}_2(\mathbb{A})$, and can control the ramification of η. The proof of the above theorem is based on both very explicit study of representations of $\mathrm{GL}_2(\mathbb{A})$, and also precise analytic techniques. For this reason, there are still very few results for higher rank groups. If not requiring the twist to be quadratic, Luo proves a result for $\mathrm{GL}_3(\mathbb{A}_\mathbb{Q})$ (see [36]), and recently, Radziwiłł-Yang prove a result for $\mathrm{GL}_4(\mathbb{A}_\mathbb{Q})$ (see [42]).

On the other hand, it is known that Question 1.2, which requires η to be quadratic, is closed related to the following two topics in the study of automorphic representation theory:

(1) The Shimura-Waldspurger correspondence and the global packets for metaplectic groups.
(2) The global Gan-Gross-Prasad conjecture.

Then it is natural to expect that the information of central L-values can be derived if one knows enough of the above two aspects. In the following parts of this article, we will recall these two topics, and show how to derive the non-vanishing of central L-values (up to quadratic twists) by a new approach based on concrete construction of automorphic representations.

2. The Shimura-Waldspurger correspondence

In early 1970s, Shimura established a lifting (see [46]) from Hecke eigenforms of weight $(k+1/2)$ and level $\Gamma_0(4)$, to Hecke eigenforms of weight $2k$ and level $\mathrm{SL}_2(\mathbb{Z})$. This pioneered the study of half integral weight modular forms.

Later, Niwa (see [38]) and Shintani (see [47]) explicitly constructed the Shimura lifting and its inverse by using the so-called theta series lifting. The theory of theta series lifting was then largely developed by Howe in the late 1970s (see [26]). He introduces the notion of *reductive dual pairs*, so that the lifting gives a correspondence between representations of the members in a reductive dual pair.

In 1980s, Waldspurger studied the Shimura correspondence systematically in the framework of automorphic representation theory (see [48, 50]). Two very nice references summarizing Waldspurger's work are [39] and [10]. Waldspurger's work is based on a thorough study of the theta correspondence with respect to the dual pair $(\widetilde{\mathrm{SL}}_2(\mathbb{A}), \mathrm{SO}_3(\mathbb{A}) \simeq \mathrm{PGL}_2(\mathbb{A}))$, here $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ is the metaplectic double cover of $\mathrm{SL}_2(\mathbb{A})$:

$$1 \longrightarrow \{\pm 1\} \longrightarrow \widetilde{\mathrm{SL}}_2(\mathbb{A}) \longrightarrow \mathrm{SL}_2(\mathbb{A}) \longrightarrow 1. \tag{2.1}$$

In particular, using theta correspondence between the above groups, one can describe the cuspidal spectrum of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ in terms of that of $\mathrm{PGL}_2(\mathbb{A})$. More precisely, for an irreducible cuspidal automorphic representation τ of $\mathrm{PGL}_2(\mathbb{A})$, one has the notion of the theta lift $\Theta_\psi(\tau)$ of τ to $\widetilde{\mathrm{SL}}_2(\mathbb{A})$, which depends on a non-trivial additive character $\psi : F\backslash\mathbb{A} \longrightarrow \mathbb{C}^\times$. In particular, a key fact is the following:

Theorem 2.1 (Waldspurger). *For an irreducible cuspidal automorphic representation τ of $\mathrm{PGL}_2(\mathbb{A})$,*

$$L\left(\frac{1}{2}, \tau\right) \neq 0$$

if and only if $\Theta_\psi(\tau) \neq 0$ for any non-trivial additive character $\psi : F\backslash\mathbb{A} \longrightarrow \mathbb{C}^\times$. Moreover, $\Theta_\psi(\tau)$ is a cuspidal representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$.

Based on the above result, Waldspurger shows that irreducible genuine cuspidal representations of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ can be parametrized by irreducible cuspidal representations τ of $\mathrm{PGL}_2(\mathbb{A})$

such that

$$L\left(\frac{1}{2}, \tau \otimes \eta_\alpha\right) \neq 0$$

for some square class $[\alpha] \in F^\times/(F^\times)^2$, here η_α denotes the quadratic automorphic character associated with the quadratic extension $F(\sqrt{\alpha})/F$ via class field theory. Denote by

$$\Pi_\psi^\tau\left(\widetilde{\mathrm{SL}}_2(\mathbb{A})\right)$$

the set of (equivalence classes of) irreducible cuspidal representations of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ parametrized by τ, this parametrization also depends on ψ. We call it the *global packet for* $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ *with respect to the parameter* (τ, ψ) for simplicity. More precisely, one has:

- Elements in $\Pi_\psi^\tau(\widetilde{\mathrm{SL}}_2(\mathbb{A}))$ come from theta lifts $\Theta_{\psi^\alpha}(\tau \otimes \eta_\alpha)$ with $L(1/2, \tau \otimes \eta_\alpha) \neq 0$. Here $\psi^\alpha(\cdot) = \psi(\alpha \cdot)$.
- One can also define the L-function $L_\psi(s, \widetilde{\sigma})$ for $\widetilde{\sigma} \in \Pi_\psi^\tau(\widetilde{\mathrm{SL}}_2(\mathbb{A}))$, which depends on ψ as well (see [39]). In fact, one has
$$L_\psi(s, \widetilde{\sigma}) = L(s, \tau),$$
and
$$L_{\psi^\beta}(s, \widetilde{\sigma}) = L(s, \tau \otimes \eta_\beta)$$
for any $\beta \in F^\times$. Here we remark that $L_\psi(s, \widetilde{\sigma})$ can be defined using the doubling method (see [13]), and this definition is compatible with the above properties (see [10, Theorem 7.8]).

Now we have seen that every element $\widetilde{\sigma} \in \Pi_\psi^\tau(\widetilde{\mathrm{SL}}_2(\mathbb{A}))$ corresponds to a non-zero L-value $L(1/2, \tau \otimes \eta_\alpha)$ for some $[\alpha] \in F^\times/(F^\times)^2$. It is natural to ask the following question:

Question 2.2. *Let* τ *be an irreducible cuspidal representation of* $\mathrm{PGL}_2(\mathbb{A})$ *such that the corresponding global packet* $\Pi_\psi^\tau(\widetilde{\mathrm{SL}}_2(\mathbb{A}))$ *is non-empty. Can one tell that*

$$L\left(\frac{1}{2}, \tau \otimes \eta\right) \neq 0$$

for some (may be different) quadratic character η*?*

3. The Gan-Gross-Prasad conjecture

The Gan-Gross-Prasad (GGP) conjecture (see [11]) is another important topic in the study of automorphic representations, which are closely related to the central value of L-functions as well. The GGP conjecture concerns the following relation:

central values of (Rankin-Selberg) L-functions	$\leftrightsquigarrow$	certain period integrals of automorphic forms

Such relationship also appears in Dirichlet class number formula for imaginary quadratic fields we have mentioned in (1.2). The first example of GGP conjecture is the so-called *Waldspurger formula* (see [49]), which relates the central values of quadratic base change L-functions for GL_2 to toric periods.

We introduce GGP in the case of $\mathrm{SL}_2(\mathbb{A}) \times \widetilde{\mathrm{SL}}_2(\mathbb{A})$ as an example, which will also be used in the final part of this article. This is a basic example for the so-called *symplectic-metaplectic* case. One may refer to [11], [51] or [35] for more detailed information.

Fix a non-trivial additive character $\psi : F\backslash\mathbb{A} \longrightarrow \mathbb{C}^\times$ as before. Let π be an irreducible cuspidal representation of $\mathrm{SL}_2(\mathbb{A})$, and $\widetilde{\sigma}$ be an irreducible genuine cuspidal representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$. For $\phi \in \pi$, $\widetilde{\phi} \in \widetilde{\sigma}$ and $\beta \in F^\times$, one defines the *Fourier-Jacobi period*

$$\widetilde{\mathcal{P}}^{\mathrm{FJ}}_{\psi,\beta,\varphi}(\phi,\widetilde{\phi}) = \int_{\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A})} \phi(g)\widetilde{\phi}(\widetilde{g})\theta^{\varphi}_{\psi^{-\beta}}(\widetilde{g})\,\mathrm{d}g, \tag{3.1}$$

where

$$\theta^{\varphi}_{\psi^\alpha}(\mathrm{h}\cdot\widetilde{g}) = \sum_{\xi\in F}\omega_{\psi^\alpha}(\mathrm{h}\cdot\widetilde{g})\,\varphi(\xi)$$

is the theta series on $\mathcal{H}_3(\mathbb{A}) \rtimes \widetilde{\mathrm{SL}}_2(\mathbb{A})$ associated the Weil representation ω_{ψ^α} with respect to ψ^α. Here $\varphi \in \mathcal{S}(\mathbb{A})$, the space of Schwartz functions on $\mathbb{A}$, $\widetilde{g}$ is the inverse image of g in $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ via (2.1) , and $\mathcal{H}_3$ is the dimension 3 Heisenberg group over F. The convergence of (3.1) is guaranteed by the cuspidality.

On the other hand, one has the Rankin-Selberg L-function

$$L_\psi(s,\pi\times\widetilde{\sigma}),$$

which again depends on ψ (see [11]). Note that if $\widetilde{\sigma}$ is ψ-generic, one also has an integral presentation of $L_\psi(s,\pi\times\widetilde{\sigma})$ (see [17]). We remark that irreducible cuspidal representations of $\mathrm{SL}_2(\mathbb{A})$ can be parametrized by automorphic representations τ' of $\mathrm{GL}_3(\mathbb{A})$ of orthogonal type, i.e., $L(s,\tau',\mathrm{Sym}^2)$, the symmetric square L-function of τ', has a pole at $s = 1$ (see [1, 11]). We denote such a packet by

$$\Pi^{\tau'}(\mathrm{SL}_2(\mathbb{A})).$$

Then, for any $\pi \in \Pi^{\tau'}(\mathrm{SL}_2(\mathbb{A}))$ and $\widetilde{\sigma} \in \Pi^{\tau}_{\psi}(\widetilde{\mathrm{SL}}_2(\mathbb{A}))$, one has (see [11, 22])

$$L_\psi(s,\pi\times\widetilde{\sigma}) = L(s,\tau'\times\tau).$$

The global Gan-Gross-Prasad conjecture is known in the case (the rank one symplectic-metaplectic case) we are discussing. We state it in a simplified form in the following theorem:

Theorem 3.1 (global GGP for $\mathrm{SL}_2\times\widetilde{\mathrm{SL}}_2$). *Let τ be an irreducible cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$, and τ' be an irreducible generic automorphic representation of $\mathrm{GL}_3(\mathbb{A})$. Then*

$$L\left(\frac{1}{2},\tau'\times\tau\right) \neq 0$$

if and only if there exist $\pi \in \Pi^{\tau'}(\mathrm{SL}_2(\mathbb{A}))$ and $\widetilde{\sigma} \in \Pi^{\tau}_{\psi^\beta}(\widetilde{\mathrm{SL}}_2(\mathbb{A}))$ such that the Fourier-Jacobi period

$$\widetilde{\mathcal{P}}^{\mathrm{FJ}}_{\psi,\beta,\varphi}(\phi,\widetilde{\phi})$$

is not identically zero on $\pi\times\widetilde{\sigma}$ for some choice of data. Moreover, the distinguished pair $(\pi,\widetilde{\sigma})$ is unique, and we call it the GGP pair associated with the parameters $\left(\tau',(\tau,\psi^\beta)\right)$.

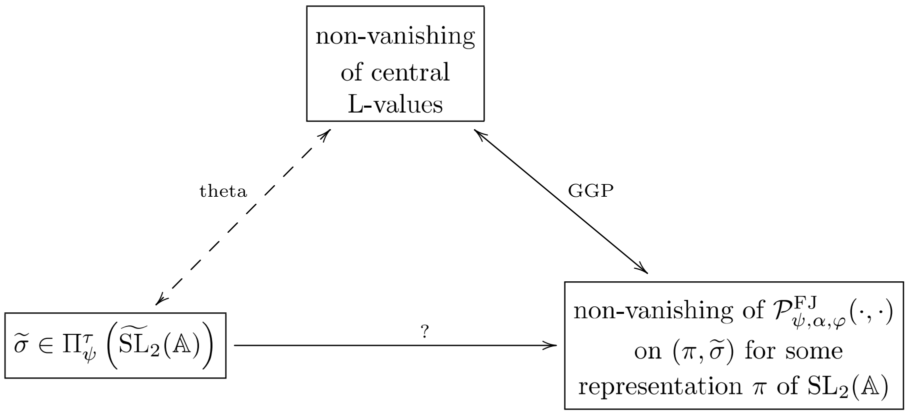
It is shown in a work of Xue (see [51]) that the above theorem can be deduced from the Waldspurger formula (see [49]) using theta correspondence. We remark that the above theorem is also included in a work of Qiu (see [40]).

Remark 3.2. *We give some remarks on the GGP conjecture.*

(1) *In general, one can also consider global GGP conjecture for $\mathrm{Sp}_{2n}\times\widetilde{\mathrm{Sp}}_{2r}$, which can be formulated similarly (see [11]). It is still open at present.*

(2) *There is also a refined version of GGP conjecture, due to Ichino-Ikeda (see [27]), which gives a formula involving periods and central L-values. For the symplectic-metaplectic case, the refined GGP is formulated by Xue (see [51]).*
(3) *The local counterpart of the above conjecture is called the local GGP conjecture. For the symplectic-metaplectic non-Archimedean case, it is proved by Atobe (see [2]). The uniqueness part in Theorem 3.1 requires a local uniqueness result for* $\mathrm{SL}_2 \times \widetilde{\mathrm{SL}}_2$ *in Archimedean case, and a proof can be found in [35, §6].*

In summary, at this stage, we have the following relations:



It follows that, if we can realize the horizontal arrow, then we can obtain the information for the non-vanishing of central L-values from the global packet $\Pi_\psi^\tau(\widetilde{\mathrm{SL}}_2(\mathbb{A}))$. This leads to the following constructive task, which can be viewed as a *reciprocal branching problem* for automorphic representations (see [29]):

Question 3.3. *For a given* $\widetilde{\sigma} \in \Pi_\psi^\tau(\widetilde{\mathrm{SL}}_2(\mathbb{A}))$*, can one construct a cuspidal automorphic representation* π *of* $\mathrm{SL}_2(\mathbb{A})$*, whose global parameter can be determined, such that* $(\pi, \widetilde{\sigma})$ *has non-zero Fourier-Jacobi period?*

As can be seen in the works [29] and [35], the above question can be solved using the *automorphic descent* method, which we will describe in the next section.

4. Automorphic descent method and applications

4.1. **A brief introduction to automorphic descent method.** The automorphic descent method, developed by Ginzburg, Rallis and Soudry in their series of papers [20, 19, 18, 21, 22], gives rise to an inverse map of the functorial lift (see [5, 6, 7]) of automorphic representations of classical groups. More precisely, starting from irreducible generic automorphic representations of general linear groups with certain (symmetric) properties, this method constructs (generic) cuspidal automorphic representations of quasi-split classical groups, by taking various Fourier coefficients on certain residual representations obtained from Siegel Eisenstein series. A complete and detailed reference of this theory is [23].

In the works [30, 33, 32, 29, 34], and also in some early considerations in [16, 14, 15], a twisted version of automorphic descent is developed. A complete framework of this is given in the work of Jiang-Zhang (see [34]). In this case, not just starting from an irreducible generic isobaric sum automorphic representation τ of a general linear group, an irreducible cuspidal automorphic representation σ of a classical group is also involved in the initial data, and the descent is constructed by taking Fourier coefficients on certain residual representations obtained from Eisenstein series supported on maximal parabolic subgroups of non-Siegel type. The point is, guided by the endoscopic classification theory (see [1]), the twisted automorphic descent provides a systematic way to concretely construct more members (e.g. the non-generic ones) in the global Arthur packets of classical groups parametrized by generic global parameters. Moreover, the twisted automorphic descent method is capable of constructing representations of pure inner forms of classical groups, hence has potential to recover the global Vogan packets. From this point of view, based on more understandings of these global packets, more applications are expected under this framework.

We briefly introduce the strategy of the (twisted) descent method at first. One may refer to [29, 35, 33] for more details. Let $\mathbf{G}$ be a classical group, and take a representation τ' of GL_N which parametrizes representations of $\mathbf{G}$. We also take a representation σ of a classical group $\mathbf{H}_0$ which is relevant to $\mathbf{G}$ and of smaller size, satisfying certain compatibility properties with τ'. The representation σ is called the *twisting data* for the construction. We also call (τ', σ) the *initial data* of the construction.

The descent construction can be outlined as follows:

(4.1) $\boxed{(\tau', \sigma)}$ $\rightsquigarrow$ [a "source representation" $\Pi_{\tau'\otimes\sigma}$ of a larger group] $\rightsquigarrow$ [a tower of representations $\{\pi_\ell\}_\ell$ of certain groups $\{G_\ell\}_\ell$] $\rightsquigarrow$ [the first occurrence π_{ℓ^*}]

Here G_ℓ's are of the same type as $\mathbf{G}$. It is expected that the *first occurrence* π_{ℓ^*} is cuspidal, and is parametrized by τ' or "a variance" of τ'. Moreover, (π_{ℓ^*}, σ) has a non-zero GGP period.

4.2. **An automorphic descent construction for symplectic groups.** Now we consider an example, which is studied in [35]. We also fix a non-trivial additive character $\psi : F\backslash\mathbb{A} \longrightarrow \mathbb{C}^\times$. Let n be a positive integer, and

$$\tau' = \tau_1 \boxplus \tau_2 \boxplus \cdots \boxplus \tau_t \tag{4.2}$$

be an irreducible generic isobaric sum automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$. Here, τ_i's are distinct irreducible unitary cuspidal automorphic representations of $\mathrm{GL}_{n_i}(\mathbb{A})$ such that $\sum_{i=1}^t n_i = 2n$. We assume that each τ_i is of orthogonal type, i.e. the L-function $L(s, \tau_i, \mathrm{Sym}^2)$ has a pole at $s = 1$. On the other hand, we take an irreducible cuspidal automorphic representation $\widetilde{\sigma}$ of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ such that

$$L_\psi\left(\frac{1}{2}, \tau' \times \widetilde{\sigma}\right) \neq 0.$$

The representation $\widetilde{\sigma}$ serves as the twisting data in this case. We remark that the condition for $\widetilde{\sigma}$ comes from the corresponding Rankin-Selberg construction (see [33, 32]) and the global GGP conjecture.

To be more precise, we introduce some more notation. Let V be a symplectic space of dimension $(4n + 2)$ defined over F, with symplectic form $\langle\ ,\ \rangle$. Then V has a polarization

$$V = V^+ \oplus V^-,$$

where V^+ is a maximal totally isotropic subspace of V with dimension $(2n+1)$. Fix a maximal flag

$$\mathcal{F} : \quad 0 \subset V_1^+ \subset V_2^+ \subset \cdots \subset V_{2n+1}^+ = V^+$$

in V^+, and choose a basis $\{e_1, \ldots, e_{2n+1}\}$ of V^+ over F such that

$$V_i^+ = \mathrm{Span}\{e_1, \ldots, e_i\}$$

for $1 \le i \le 2n+1$. Let $\{e_{-1}, \ldots, e_{-(2n+1)}\}$ be a basis for V^-, which is dual to $\{e_1, \ldots, e_{2n+1}\}$, i.e., $\langle e_i, e_{-j}\rangle = \delta_{i,j}$ for $1 \le i, j \le 2n+1$. Let $\mathrm{Sp}_{4n+2} = \mathrm{Sp}(V)$ be the symplectic group over F associated with V. Note that with the above choice of basis, the form of V is associated with the skew-symmetric matrix $J_{4n+2} = \begin{pmatrix} & w_{2n+1} \\ -w_{2n+1} & \end{pmatrix}$, where w_i is an $(i \times i)$-matrix with 1's on the anti-diagonal and zero's elsewhere.

We fix a Borel subgroup $\mathbf{B}_0 = \mathbf{T}_0 \cdot \mathbf{U}_0$ of $\mathrm{Sp}(V)$ consisting of upper-triangular matrices, and call a parabolic subgroup standard if it contains $\mathbf{B}_0$. We denote by P the standard parabolic subgroup of $\mathrm{Sp}(V)$ which stabilizes the partial flag $0 \subset V_{2n}^+ \subset V^+$. Then P has a Levi decomposition $P = M \cdot U$, with Levi subgroup $M \simeq \mathrm{GL}_{2n} \times \mathrm{SL}_2$.

For $1 \le \ell \le 2n$, let P_ℓ be the standard parabolic subgroup of $\mathrm{Sp}(V)$ which stabilizes the partial flag

$$\mathcal{F}_\ell: \quad 0 \subset V_1^+ \subset V_2^+ \subset \cdots \subset V_\ell^+.$$

It has a Levi decomposition $P_\ell = M_\ell \cdot N_\ell$ with Levi subgroup $M_\ell \simeq \mathrm{GL}_1^\ell \times \mathrm{Sp}(V^{(\ell)})$. Here $V^{(\ell)}$ is the subspace of V which sits into the decomposition

$$V = V_\ell^+ \oplus V^{(\ell)} \oplus V_\ell^-,$$

where $V_\ell^- = \mathrm{Span}\{e_{-\ell}, \ldots, e_{-1}\}$. The unipotent subgroup N_ℓ consists of elements of the form

$$u = u_\ell(z, x, y) = \begin{pmatrix} z & z \cdot x & y \\ & I_{4n-2\ell+2} & x' \\ & & z^* \end{pmatrix},$$

where $z \in Z_\ell$, $x \in \mathrm{Mat}_{\ell \times (4n-2\ell+2)}$, $x' = J_{4n-2\ell+2}{}^t x w_\ell$, and $y \in \mathrm{Sym}_{\ell \times \ell}$. Here Z_ℓ is the maximal upper-triangular unipotent subgroup of GL_ℓ, and $\mathrm{Sym}_{\ell\times\ell}$ is the set of $(\ell \times \ell)$-symmetric matrices. Define a homomorphism $\chi_\ell : N_\ell \longrightarrow \mathbb{G}_a$ by

$$\chi_\ell(u) = \sum_{i=2}^{\ell+1} \langle u \cdot e_i, e_{-(i-1)}\rangle. \tag{4.3}$$

Here we view e_i's as column vectors, and $u \cdot e_i$ is the multiplication of matrices. Note that we also allow $\ell = 0$, in which case χ_0 is trivial.

Let $\widetilde{\mathrm{Sp}}_{4n+2} := \widetilde{\mathrm{Sp}}(V)$ be the metaplectic double cover of $\mathrm{Sp}(V)$. The same as above, we denote by $\widetilde{P}$ the parabolic subgroup of $\widetilde{\mathrm{Sp}}(V)$ which is the inverse image of $P \subset \widetilde{\mathrm{Sp}}(V)$. Note that we have the Levi decomposition $\widetilde{P} = \widetilde{M} \cdot U$, where $\widetilde{M} \simeq \mathrm{GL}_{2n} \times \widetilde{\mathrm{Sp}}(V^{(2n)})$ (see [12, §2.3]).

We also need some notation on Weil representations and theta series. For a symplectic space $(W, \langle\ ,\ \rangle)$ of dimension $2r$ over F with polarization $W = W^+ \oplus W^-$, we realize the Heisenberg group $\mathcal{H}_W$ (or $\mathcal{H}_{2r+1}$ if we just need to emphasize the dimension) corresponding to $(W, 2\langle\ ,\ \rangle)$ as $\mathcal{H}_W = W \oplus F$. For any $\alpha \in F^\times$, we denote by $\omega_{\psi^\alpha}^{(r)}$ the (global) Weil representation of $\mathcal{H}_W(\mathbb{A}) \rtimes \widetilde{\mathrm{Sp}}(W)(\mathbb{A})$ (see, for example, [23, §1.2]) on the Schwartz space $\mathcal{S}(W^+(\mathbb{A})) \simeq \mathcal{S}(\mathbb{A}^r)$ with respect to ψ^α (the Schrödinger model). We define the corresponding theta series by

$$\theta_{\psi^\alpha}^{\varphi}(\mathrm{h} \cdot \widetilde{h}) = \sum_{\xi \in W^+(F)} \omega_{\psi^\alpha}^{(r)}\left(\mathrm{h} \cdot \widetilde{h}\right) \varphi(\xi),$$

here $\mathrm{h} \in \mathcal{H}_W(\mathbb{A})$ and $\widetilde{h} \in \widetilde{\mathrm{Sp}}(W)(\mathbb{A})$.

As indicated in the above diagram (4.1), we have the following data and steps:

The initial data. In this case, our initial data is $(\tau', \widetilde{\sigma})$, where τ' is the same as in (4.2), and $\widetilde{\sigma}$ is an irreducible cuspidal automorphic representation of $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ such that

$$L_\psi\left(\frac{1}{2}, \tau' \times \widetilde{\sigma}\right) \neq 0.$$

The source representation. To obtain the source representation for our construction, we consider the Eisenstein series on $\widetilde{\mathrm{Sp}}_{4n+2}(\mathbb{A})$ supported on $(\widetilde{P}, \tau' \otimes \widetilde{\sigma})$, here $\widetilde{P}$ is defined in the previous page. More precisely, let $\mathcal{A}(\widetilde{M}(F)U(\mathbb{A})\backslash\widetilde{\mathrm{Sp}}_{4n+2}(\mathbb{A}))$ be the space of automorphic forms on $\widetilde{M}(F)U(\mathbb{A})\backslash\widetilde{\mathrm{Sp}}_{4n+2}(\mathbb{A})$, and let $\mathcal{A}(\widetilde{M}(F)U(\mathbb{A})\backslash\widetilde{\mathrm{Sp}}_{4n+2}(\mathbb{A}))_{\mu_\psi(\tau'\otimes\widetilde{\sigma})}$ be its $\mu_\psi(\tau'\otimes\widetilde{\sigma})$-isotypic subspace, here $\mu_\psi = \prod_v \mu_{\psi_v}$ is a (global) genuine character, whose local components can be defined by the Rao cocycle (see [41]). One may refer to [23, §2.2] or [35, §3.1] for a more precise definition.

For $s \in \mathbb{C}$ and an automorphic function

$$\Phi_{\tau'\otimes\widetilde{\sigma}} \in \mathcal{A}\left(\widetilde{M}(F)U(\mathbb{A})\backslash\widetilde{\mathrm{Sp}}_{4n+2}(\mathbb{A})\right)_{\mu_\psi(\tau'\otimes\widetilde{\sigma})},$$

one can define $\lambda_s\Phi_{\tau'\otimes\widetilde{\sigma}} := (\lambda_s \circ m_P)\Phi_{\tau'\otimes\widetilde{\sigma}}$ following [37, §II.1], where $\lambda_s \in X_M^{\mathrm{Sp}_{4n+2}} \simeq \mathbb{C}$ (see [37, §I.1] for the definition of $X_M^{\mathrm{Sp}_{4n+2}}$ and the map m_P). Then the corresponding Eisenstein series on $\widetilde{\mathrm{Sp}}_{4n+2}(\mathbb{A})$ is defined to be

$$\widetilde{E}(s, \Phi_{\tau'\otimes\widetilde{\sigma}}, \widetilde{h}) = \sum_{\gamma\in P(F)\backslash\mathrm{Sp}_{4n+2}(F)} \lambda_s\Phi_{\tau'\otimes\widetilde{\sigma}}(\gamma\widetilde{h}), \tag{4.4}$$

which converges absolutely for $\mathrm{Re}(s) \gg 0$, and has a meromorphic continuation to the whole complex plane ([37, §IV]).

With the above choice of data, $\widetilde{E}(s, \Phi_{\tau'\otimes\widetilde{\sigma}}, \cdot)$ has a simple pole at $s = 1/2$ (see [31, 24]). Denote by

$$\widetilde{\mathcal{E}}_{\tau'\otimes\widetilde{\sigma}} = \left\langle \mathrm{Res}_{s=\frac{1}{2}}\widetilde{E}(s, \Phi_{\tau'\otimes\widetilde{\sigma}}, \cdot)\right\rangle,$$

the representation of $\widetilde{\mathrm{Sp}}_{4n+2}(\mathbb{A})$ generated by the residues at $s = 1/2$. It is square integrable by the L^2-criterion in [37], and provides the *source representation* for our construction.

The descent tower. The descent construction is obtained by taking certain *Fourier coefficients* of automorphic forms $f \in \widetilde{\mathcal{E}}_{\tau'\otimes\widetilde{\sigma}}$.

For $1 \leq \ell \leq 2n$, define a character

$$\psi_\ell : N_\ell(F)\backslash N_\ell(\mathbb{A}) \longrightarrow \mathbb{C}^\times$$

by $\psi_\ell = \psi \circ \chi_\ell$ (see (4.3)). In matrix form, we have

$$\psi_\ell(u_\ell(z, x, y)) = \psi_{Z_\ell}(z)\psi(x_{\ell,1}). \tag{4.5}$$

If $\ell = 0$, then ψ_0 is trivial. We extend ψ_ℓ trivially to $N_{\ell+1}(\mathbb{A})$. Note that $N_\ell\backslash N_{\ell+1}$ is isomorphic to the Heisenberg group $\mathcal{H}_{V^{(\ell+1)}}$. We will fix such an isomorphism

$$j_\ell : N_\ell\backslash N_{\ell+1} \simeq \mathcal{H}_{V^{(\ell+1)}}. \tag{4.6}$$

Let $\widetilde{\Pi}$ be an irreducible genuine automorphic representation of $\widetilde{\mathrm{Sp}}_{4n+2}(\mathbb{A})$. For an automorphic form $f \in \widetilde{\Pi}$ and $\beta \in F^\times$, one defines its *β-Fourier-Jacobi coefficient of depth ℓ* to be (see also [23, §3.2])

$$\mathcal{FJ}^{\varphi}_{\psi_\ell,\beta}(f)(h) = \int_{N_{\ell+1}(F)\backslash N_{\ell+1}(\mathbb{A})} f(u\widetilde{h})\psi_\ell^{-1}(u)\theta^{\varphi}_{\psi^{-\beta}}(j_\ell(u)\cdot\widetilde{h})\,\mathrm{d}u, \tag{4.7}$$

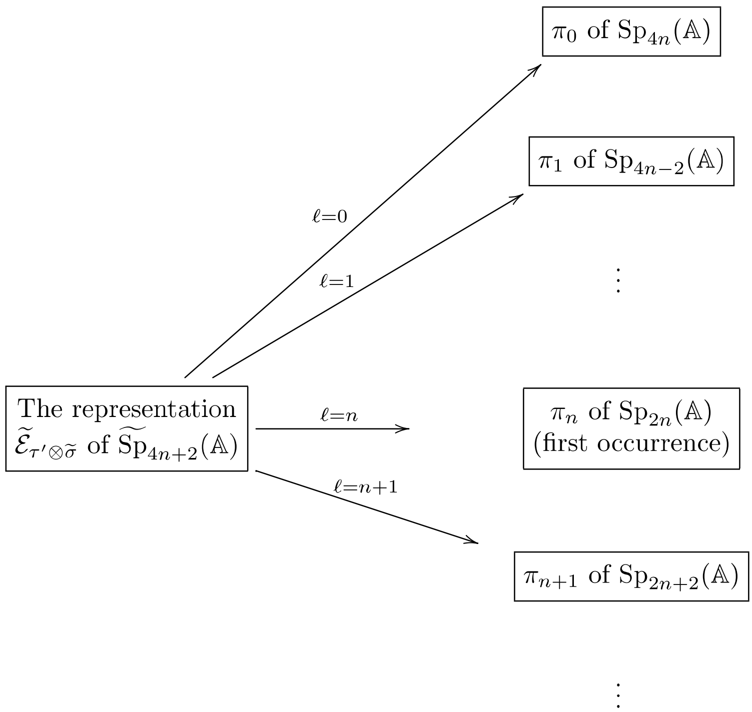


FIGURE 1. The descent tower

where $\varphi \in \mathcal{S}(V^{(\ell+1),+}) \simeq \mathcal{S}(\mathbb{A}^{2n-\ell})$, $\omega_{\psi^{-\beta}}^{(2n-\ell)}$ is the global Weil representation of $\mathcal{H}_{V^{(\ell+1)}}(\mathbb{A}) \rtimes \widetilde{\mathrm{Sp}}(V^{(\ell+1)})(\mathbb{A})$, $\theta_{\psi^{-\beta}}^{\varphi}$ is the corresponding theta series, and $\widetilde{h}$ is a projection (the same as in (2.1)) pre-image of h in $\widetilde{\mathrm{Sp}}(V^{(\ell+1)})(\mathbb{A})$. It is easy to see that $\mathcal{FJ}_{\psi_\ell,\beta}^{\varphi}(f)(h)$ is an automorphic function on $\mathrm{Sp}(V^{(\ell+1)})(\mathbb{A}) = \widetilde{\mathrm{Sp}}_{4n-2\ell}(\mathbb{A})$.

Remark 4.1. *One can further define the corresponding Fourier-Jacobi period $\mathcal{P}_{\psi,\beta,\varphi}^{\mathrm{FJ}}(\cdot,\cdot)$, which is an integration of the above Fourier-Jacobi coefficient against an automorphic form on a symplectic group. The depth of the coefficients involved depends on the co-rank of the groups. On the other hand, one can also define the Fourier-Jacobi coefficient of an automorphic form on a symplectic group, which turns out to be an automorphic form on metaplectic groups (see [23, §3.2]), and also the corresponding Fourier-Jacobi periods $\widetilde{\mathcal{P}}_{\psi,\beta,\varphi}^{\mathrm{FJ}}(\cdot,\cdot)$ as in (3.1) (see, for example, [35, §2.3]).*

As explained in the diagram (4.1), we will take Fourier-Jacobi coefficients of the source representation $\widetilde{\mathcal{E}}_{\tau'\otimes\widetilde{\sigma}}$ in the automorphic descent construction. Define

$$\pi_{\ell,\beta} = \widetilde{\mathrm{Sp}}_{4n-2\ell}(\mathbb{A}) - \mathrm{Span}\left\{ \mathcal{FJ}_{\psi_\ell,\beta}^{\varphi}(f)\Big|_{\widetilde{\mathrm{Sp}}_{4n-2\ell}(\mathbb{A})} \;\Big|\; f \in \widetilde{\mathcal{E}}_{\tau'\otimes\widetilde{\sigma}},\ \varphi \in \mathcal{S}(\mathbb{A}^{2n-\ell}) \right\}. \tag{4.8}$$

Leaving the depth ℓ to vary, we obtain a tower of automorphic modules of $\widetilde{\mathrm{Sp}}_{4n-2\ell}(\mathbb{A})$, and this is our *automorphic descent tower*. The descent tower can be expressed by Figure 1.

The basic information for the descent tower is given in the following proposition (see [35]):

Proposition 4.2. *For the descent tower $\{\pi_{\ell,\beta}\}_\ell$, we have:*

(1) (*Tower property, non-vanishing and cuspidality*) *The representation $\pi_{\ell,\beta}$ in the descent tower vanishes identically for any $n+1 \le \ell \le 2n$. Moreover, there exists $\beta \in F^\times$ such that $\pi_\beta = \pi_{n,\beta}$ is a non-zero cuspidal representation of $\mathrm{Sp}_{2n}(\mathbb{A})$.*

(2) (*Relation to GGP periods*) *The representation π_β is irreducible if we presume the corresponding local GGP in Archimedean case, and the Fourier-Jacobi period $\widetilde{\mathcal{P}}^{\mathrm{FJ}}_{\psi,\beta,\varphi}(\cdot,\cdot)$ is not identically zero on $\pi_\beta \times \widetilde{\sigma}$ for some choice of data.*

(3) (*Global parameter*) *Suppose moreover that the central character $\omega_{\tau'}$ of τ' is trivial, then the representation π_β is parametrized by $(\tau' \otimes \eta_\beta) \boxplus \mathbf{1}$.*

By the above proposition, we call the *first occurrence* π_β the *the Fourier-Jacobi type automorphic descent of τ', twisted by $\widetilde{\sigma}$*. We note here that the data $\beta \in F^\times$ in the first occurrence (Part (1) of Proposition 4.2) will be crucial in exploring the non-vanishing of quadratic twists of L-values later.

As a corollary, we have a GGP type result in sympletic-metaplectic case:

Theorem 4.3 (Theorem 6.2 of [35])**.** *Let τ' be the same as above with $\omega_{\tau'} = \mathbf{1}$, τ be an irreducible cuspidal representation of $\mathrm{PGL}_2(\mathbb{A})$ such that $L(1/2, \tau' \times \tau) \neq 0$, and $\widetilde{\sigma} \in \Pi^\tau_\psi(\widetilde{\mathrm{SL}}_2(\mathbb{A}))$. Then there exist $[\beta] \in F^\times/(F^\times)^2$ and $\pi \in \Pi^{(\tau'\otimes\eta_\beta)\boxplus\mathbf{1}}(\mathrm{Sp}_{2n}(\mathbb{A}))$ such that the Fourier-Jacobi period*

$$\widetilde{\mathcal{P}}^{\mathrm{FJ}}_{\psi,\beta,\varphi}(\phi, \widetilde{\phi})$$

is not identically zero on $\pi \times \widetilde{\sigma}$ for some choice of data.

4.3. **Applications to non-vanishing of central values.** Now we explain how to decode the non-vanishing of quadratic twists of central values from elements $\widetilde{\sigma} \in \Pi^\tau_\psi(\widetilde{\mathrm{SL}}_2(\mathbb{A}))$. At first, we recall that cuspidal representations τ of $\mathrm{PGL}_2(\mathbb{A})$ which make $\widetilde{\sigma} \in \Pi^\tau_\psi(\widetilde{\mathrm{SL}}_2(\mathbb{A})) \neq \emptyset$ must satisfy the condition that (see [39, 50]) there exists a quadratic character η_0 such that

$$\varepsilon\left(\frac{1}{2}, \tau \otimes \eta_0\right) = 1.$$

Part I: existence of a quadratic twist with non-vanishing central L-value. We use automorphic descent construction to show that there exists a quadratic character $\eta : F^\times\backslash\mathbb{A}^\times \longrightarrow \mathbb{C}^\times$ such that $L(1/2, \tau \otimes \eta) \neq 0$. There are two steps.

Step 1: Starting point.

Take a character χ of an anisotropic $\mathrm{SO}_2^\delta(\mathbb{A})$ (here $\delta \in F^\times/F^\times)^2$), such that

$$L\left(\frac{1}{2}, \tau \times \chi\right) \neq 0.$$

This can be done by considering the spectral decomposition of $\tau|_{\mathrm{SO}_2^\delta(\mathbb{A})}$, and then using the Waldspurger formula (see [49]). Then, one can lifts χ to an irreducible cuspidal representation τ_δ of $\mathrm{GL}_2(\mathbb{A})$ of orthogonal type (see [7]), with (non-trivial) central character η_δ, such that

$$L\left(\frac{1}{2}, \tau \times \tau_\delta\right) \neq 0.$$

Finally, we obtain the initial data $(\tau_\delta, \widetilde{\sigma})$ in the end of this step.

Step 2: Automorphic descent.

From the initial data $(\tau_\delta, \widetilde{\sigma})$, one can construct a non-zero irreducible cuspidal representation π'_α of $\mathrm{SL}_2(\mathbb{A})$, such that (see [35, §3–§5]):

- It is parametrized by $\tau_\gamma \boxplus \eta_\gamma$ for some generic automorphic representation τ_γ of $\mathrm{GL}_2(\mathbb{A})$ and quadratic character η_γ. Note that $\omega_{\tau_\gamma} = \eta_\gamma$ with $\gamma \in F^\times/(F^\times)^2$.
- The corresponding Fourier-Jacobi period $\widetilde{\mathcal{P}}^{\mathrm{FJ}}_{\psi,\alpha,\varphi}(\cdot,\cdot)$ is not identically zero on $\pi'_\alpha \times \widetilde{\sigma}$.

Note that we have the relation

$$\Pi^{\tau}_{\psi}\left(\widetilde{\mathrm{SL}}_2(\mathbb{A})\right) = \Pi^{\tau\otimes\eta_\alpha}_{\psi^\alpha}\left(\widetilde{\mathrm{SL}}_2(\mathbb{A})\right),$$

then by Theorem 3.1 (global GGP), $\mathcal{P}^{\mathrm{FJ}}_{\psi,\alpha,\varphi}(\cdot,\cdot)$ is not identically zero on $\pi'_\alpha \times \widetilde{\sigma}$ implies that $(\pi_\alpha, \widetilde{\sigma})$ is the unique GGP pair associated with the parameters

$$(\tau_\gamma \boxplus \eta_\gamma, (\tau \otimes \eta_\alpha, \psi^\alpha)),$$

and also

$$L\left(\frac{1}{2}, (\tau_\gamma \boxplus \eta_\gamma) \times (\tau \otimes \eta_\alpha)\right) \neq 0.$$

In particular, one has

$$L\left(\frac{1}{2}, \tau \otimes \eta_\alpha\eta_\gamma\right) \neq 0.$$

Then we have shown that there exists a quadratic twist (possibly trivial) of τ, with non-zero central L-value.

Part II: existence of different quadratic twists with non-vanishing central L-values. In fact, by the Shimura-Waldspurger correspondence, we expect that different elements $\widetilde{\sigma} \in \Pi^\tau_\psi(\widetilde{\mathrm{SL}}_2(\mathbb{A}))$ give different quadratic twists of τ with non-zero central L-values. This can also be verified by applying the automorphic descent method. The same as above, there are also two steps.

Step 1: Starting point.

Based on the above, one can take a *generic* automorphic representation τ_1 of $\mathrm{GL}_2(\mathbb{A})$ of orth type with trivial central character, such that

$$L\left(\frac{1}{2}, \tau_1 \times \tau\right) \neq 0.$$

In fact, we can view $\eta_0 = \eta_\alpha\eta_\gamma$ as a representation of F-split $\mathrm{SO}_2(\mathbb{A})$, and lift η_0 to an irreducible generic automorphic representation τ_1 of $\mathrm{GL}_2(\mathbb{A})$ (see [5, 7]), which is of orthogonal type and has trivial central character. Moreover, the condition $L(1/2, \tau \otimes \eta_0) \neq 0$ implies that $L(1/2, \tau_1 \times \tau) \neq 0$.

Then we obtain another initial data $(\tau_1, \widetilde{\sigma})$.

Step 2: Automorphic descent.

From the above initial data $(\tau_1, \widetilde{\sigma})$, one can construct a (non-zero) irreducible csupidal representation π_β of $\mathrm{SL}_2(\mathbb{A})$ such that (see Proposition 4.2):

- The representation π_β is parametrized by $(\tau_1 \otimes \eta_\beta) \boxplus \mathbf{1}$.

- The Fourier-Jacobi period $\widetilde{\mathcal{P}}^{\mathrm{FJ}}_{\psi,\beta,\varphi}(\cdot,\cdot)$ is non-zero on $\pi_\beta\times\widetilde{\sigma}$. Then $(\pi_\beta,\widetilde{\sigma})$ is the unique GGP pair (see Theorem 3.1) associated with the pair of parameters $((\tau_1\otimes\eta_\beta)\boxplus\mathbf{1},(\tau\otimes\eta_\beta,\psi^\beta))$.

As before, $\widetilde{\mathcal{P}}^{\mathrm{FJ}}_{\psi,\beta,\varphi}(\cdot,\cdot)$ is not identically zero on $\pi_\beta\times\widetilde{\sigma}$ implies

$$L\left(\frac{1}{2},[(\tau_1\otimes\eta_\beta)\boxplus\mathbf{1}]\times(\tau\otimes\eta_\beta)\right)\neq 0,$$

and hence

$$L\left(\frac{1}{2},\tau\otimes\eta_\beta\right)\neq 0.$$

Take another element $\widetilde{\sigma}'\in\Pi_\psi^\tau(\widetilde{\mathrm{SL}}_2(\mathbb{A}))$ $(\widetilde{\sigma}'\not\cong\widetilde{\sigma})$. From the initial data $(\tau_1,\widetilde{\sigma}')$, the same as above, one can construct representation π_ξ $(\xi\in F^\times)$ of $\mathrm{SL}_2(\mathbb{A})$ such that $(\pi_\xi,\widetilde{\sigma}')$ is the GGP pair associated with the following pair of parameters:

$$\left((\tau_1\otimes\eta_\xi)\boxplus\mathbf{1},(\tau\otimes\eta_\xi,\psi^\xi)\right),$$

which gives

$$L\left(\frac{1}{2},\tau\otimes\eta_\xi\right)\neq 0.$$

By the uniqueness of the GGP pair, one must have $[\xi]\neq[\beta]\in F^\times/(F^\times)^2$, otherwise one obtains $\widetilde{\sigma}\simeq\widetilde{\sigma}'$, which is a contradiction. Then, there are at least

$$\sharp\Pi_\psi^\tau\left(\widetilde{\mathrm{SL}}_2(\mathbb{A})\right)$$

different quadratic twists with non-vanishing central L-values.

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School of Mathematics, Sichuan University, No. 29 Wangjiang Road, Chengdu 610064, P. R. China

E-mail address: binxu@scu.edu.cn