CONVERGENCE OF FORMAL SERIES OF JACOBI FORMS

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Abstract. We give an overview of the proof for general paramodular level that formal series of scalar Jacobi forms with an involution condition converge.

1. Introduction.

This talk discusses the results in [3], which is joint work with Hiroki Aoki and Tomoyoshi Ibukiyama. I would like to thank the American Institute of Mathematics for its support. Our time together at AIM SQuaREs was critical to the success of this project.

This work proves that formal series of Jacobi forms can be forced to converge by imposing symmetry conditions. Such a convergent series then defines a paramodular form and the Fourier-Jacobi expansion of this paramodular form will be the originally given formal series. Our result is the best possible in the sense that the imposed symmetries are induced by a single involution. Other important approaches to the convergence of formal series of Jacobi forms [7, 19, 8, 6, 16, 10] impose symmetries induced by infinitely many elements of the symplectic group.

The main result is a regularity theorem: less information is required to produce a certain kind of paramodular form than is included in their definition. In this sense, the main result is a generalization of the Koecher principle. The main result also proves the theoretical soundness of the method of Jacobi restriction: a computational method to rigorously compute spaces of paramodular forms [14, 5, 18, 17]. Our main result is not effective and only promises that the method of Jacobi restriction will eventually work.

That symmetries can force a formal series to converge is curious and these proceedings present a special opportunity to highlight the concepts and powerful theorems that are used in the proof. One ingredient

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is the new bound on the vanishing order of Jacobi forms that is deduced from a theorem of Aoki [2]. This bound enables us to prove that the dimension of spaces of formal series of Jacobi forms of weight k satisfying an involution condition grows as $O(k^3)$. This implies that the graded ring of formal series with a positive involution sign is algebraic over the corresponding subring of paramodular forms. A second ingredient, equally important, is the proof that formal series of Jacobi forms do converge on a collection of holomorphic curves dense in the Siegel upper half space when we impose $\Gamma^0(N)$-symmetries. Inductive bounds on the growth of the Jacobi coefficients proved too difficult, and this dense set of curves, resulting in specializations to holomorphic functions of one variable, in conjunction with $\Gamma^0(N)$-symmetries, may be thought of as a technical alternative to growth bounds. Once we know that a formal series converges on a dense subset of the Siegel upper half space, our attention becomes focused on showing that the partial sums of the formal series are locally bounded. A powerful theorem from several complex variables asserts that a sequence of holomorphic functions that is locally bounded, and convergent on a dense subset of a domain, in fact converges on the entire domain. As above, the involution condition alone implies that a formal series $\mathfrak{f}$ satisfies a polynomial relation with coefficients that are Fourier-Jacobi expansions of honest paramodular forms. If we adjust the formal series to make the polynomial relation monic, then a monic polynomial relation and holomorphic specializations on a dense set of curves is sufficient to demonstrate that the partial sums of the formal series are locally bounded.

We can make another use of the polynomial relation. We can use it to prove that any formal series $\mathfrak{f}$ satisfying it must have *some* $\Gamma^0(N)$-symmetries. More precisely, $\mathfrak{f}$ will satisfy symmetries for a subgroup $\Gamma \subseteq \Gamma^0(N)$ of finite index. The group Γ is a Fuchsian group of the first kind and its limit set $\Lambda(\Gamma)$ accordingly satisfies an ergodic property. This ergodic property suffices to show that formal series $\mathfrak{f}$ with Γ-symmetries have their own holomorphic specializations on a dense set of curves. Accordingly, we do not need to take $\Gamma^0(N)$-symmetries as a hypothesis on formal series with an involution at all, but can recover all needed deductions from just the single involution symmetry.

In order to obtain a monic polynomial relation, we need to adjust our given formal series $\mathfrak{f}$ to another formal series $\mathfrak{g} = \mathrm{FJ}(f_0)\mathfrak{f}$, where $\mathrm{FJ}(f_0)$ is the Fourier-Jacobi expansion of an honest paramodular form f_0. It is the formal series $\mathfrak{g}$ that is proven to be locally bounded and convergent and so $\mathfrak{g} = \mathrm{FJ}(G)$ for some paramodular form G. Thus, the final step in our argument is to show that the equation $\mathrm{FJ}(G) = \mathrm{FJ}(f_0)\mathfrak{f}$

implies that G/f_0 is holomorphic. This question, postponed until the end, may have worried the reader from the beginning. If a meromorphic paramodular form, G/f_0, can have a Fourier-Jacobi expansion, $\mathrm{FJ}(G/f_0) = \mathrm{FJ}(G)/\mathrm{FJ}(f_0) = \mathfrak{f}$, of the type of formal series we are considering, then our main result is false. Consider an irreducible component V of the divisor of f_0. Using the Weierstrauss preparation theorem, our dense set of holomorphic curves where $\mathfrak{f}$ converges, and the flexibility of the paramodular group, we find one holomorphic curve that meets the component V transversally at a regular point where G has no other vanishing than V. This is enough to show that the order of vanishing of G on V is at least the order of f_0, and that G/f_0 is holomorphic.

In summary, the main highlights of the proof involve new bounds on the vanishing order of Jacobi forms, a dense set of curves where a formal series with involution converges to a holomorphic function, local boundedness and convergence theorems from several complex variables, ergodic results for the limit set of Fuchsian groups, and the compatibility of the dense set of curves with the Weierstrauss preparation theorem.

2. Notation.

- σ', the transpose of the matrix σ.
- $t[\sigma] = \sigma' t \sigma$ for an appropriate matrix t.
- $\mathcal{H}_n$, the Siegel upper half space of degree n.
- $J_{k,m}$, the vector space of Jacobi forms of weight k and index m.
- $J_{k,m}(\nu) = \{\phi \in J_{k,m} : \mathrm{ord}\phi \geq \nu\}$.
- $K(N)$, the paramodular group of level N.
- μ_N, the paramodular Fricke involution.
- $K(N)^+ = \langle K(N), \mu_N \rangle$.
- $M_k\left(K(N)\right)^\epsilon$, the vector space of weight k paramodular forms with Fricke sign ϵ.
- $S_k\left(K(N)\right)^\epsilon$, the subspace of $M_k\left(K(N)\right)^\epsilon$ consisting of cusp forms.
- $\bar{\mathcal{X}}(N) = \{\text{semidefinite } \left(\begin{smallmatrix} n & r/2 \\ r/2 & Nm \end{smallmatrix}\right) : n, r, m \in \mathbb{Z}\}$.

3. Main Result

A paramodular form is a Siegel modular form of degree two for the group

$$K(N)=\mathrm{Sp}(4,\mathbb{Q})\cap\begin{pmatrix}\mathbb{Z}&N\mathbb{Z}&\mathbb{Z}&\mathbb{Z}\\ \mathbb{Z}&\mathbb{Z}&\mathbb{Z}&\frac{1}{N}\mathbb{Z}\\ \mathbb{Z}&N\mathbb{Z}&\mathbb{Z}&\mathbb{Z}\\ N\mathbb{Z}&N\mathbb{Z}&N\mathbb{Z}&\mathbb{Z}\end{pmatrix}=\mathrm{Stab}_{\mathrm{Sp}(4,\mathbb{Q})}\begin{pmatrix}\mathbb{Z}\\ \mathbb{Z}\\ \mathbb{Z}\\ N\mathbb{Z}\end{pmatrix}.$$

Paramodular forms $f\in M_k\left(K(N)\right)$ have Fourier expansions in three variables but perhaps more pertinent is the Fourier-Jacobi expansion of a paramodular form, which, using the components $\Omega=\left(\begin{smallmatrix}\tau&z\\ z&\omega\end{smallmatrix}\right)\in\mathcal{H}_2$, recollects the Fourier series in powers of $e(\omega)=e^{2\pi i\omega}$,

$$f\left(\begin{smallmatrix}\tau&z\\ z&\omega\end{smallmatrix}\right)=\sum_{m=0}^{\infty}\phi_m(\tau,z)\,e\left(Nm\omega\right).$$

Each coefficient is a Jacobi form, $\phi_m\in J_{k,Nm}$, and has its own Fourier expansion $\phi_m(\tau,z)=\sum_{n\ge0,r\in\mathbb{Z}}c(n,r;\phi_m)e(n\tau+rz)$. The Fourier-Jacobi expansion thus defines a map to formal series of Jacobi forms $\mathrm{FJ}:M_k\left(K(N)\right)\to\mathbb{M}(k,N)=\prod_{m=0}^{\infty}J_{k,Nm}$. This map is not surjective because the codomain is infinite dimensional. One source of consistency conditions among the Fourier-Jacobi coefficients (ϕ_m) arises from a normalizing involution of $K(N)$, the paramodular Fricke involution $\mu_N=\begin{pmatrix}{}^tF_N^{-1}&0\\ 0&F_N\end{pmatrix}$, where $F_N=\frac{1}{\sqrt{N}}\left(\begin{smallmatrix}0&1\\ -N&0\end{smallmatrix}\right)$ is the Fricke involution on $\Gamma_0(N)$. This involution splits $M_k\left(K(N)\right)$ into plus and minus forms, $M_k\left(K(N)\right)=M_k\left(K(N)\right)^{+}\oplus M_k\left(K(N)\right)^{-}$. The block diagonal form of μ_N gives a simple action on the Fourier series and consequently gives the following *involution condition* on the Fourier-Jacobi coefficients of any $f\in M_k\left(K(N)\right)^{\epsilon}$, $\epsilon\in\{\pm1\}$:

(1) For all semidefinite $\begin{pmatrix}n&r/2\\ r/2&Nm\end{pmatrix}$ with $n,r,m\in\mathbb{Z}$,

$$c(n,r;\phi_m)=(-1)^k\epsilon\,c(m,r;\phi_n).$$

Let $\mathbb{M}(k,N,\epsilon)$ be the subspace of formal series of Jacobi forms satisfying the involution condition:

$$\mathbb{M}(k,N,\epsilon)=\{\mathfrak{f}\in\mathbb{M}(k,N):\mathfrak{f}=\sum_{m=0}^{\infty}\phi_m\xi^{Nm}\text{ satisfies condition (1)}\}.$$

The Fourier-Jacobi expansion gives $\mathrm{FJ}:M_k\left(K(N)\right)^{\epsilon}\to\mathbb{M}(k,N,\epsilon)$. It seems a bit greedy to hope that we have hereby found all the consistency conditions among the Fourier coefficients of the Jacobi forms ϕ_m but we prove it here.

Theorem 3.1 (Main Theorem). *Let $N \in \mathbb{N}$, $k \in \mathbb{N}_0$, and $\epsilon \in \{\pm 1\}$. The map* $\mathrm{FJ} : M_k(K(N))^{\epsilon} \to \mathbb{M}(k, N, \epsilon)$ *is an isomorphism.*

Our main result has applications to computation. Jacobi restriction has provided rigorous optimal upper bounds for $\dim S_k(K(N))^{\epsilon}$, even though no one could guarantee in advance that the method would work. The following corollary proves that any schema for computing spaces of paramodular forms that spans spaces of Jacobi forms and imposes the involution condition is in principle sound.

Corollary 3.2. *For $d \in \mathbb{N}$, define the complex vector space $\mathbb{M}(k, N, \epsilon)[d]$ as $\{(\phi_m) \in \prod_{m=0}^{d} J_{k,Nm} : (\phi_m)$ satisfies (1) for all $n, m \le d\}$. The sequence $\dim_{\mathbb{C}} \mathbb{M}(k, N, \epsilon)[d]$ is monotonically decreasing for $d \ge \frac{1}{6}Nk$, and we have $\lim_{d \to +\infty} \dim_{\mathbb{C}} \mathbb{M}(k, N, \epsilon)[d] = \dim_{\mathbb{C}} M_k(K(N))^{\epsilon}$.*

In particular, we have $\dim_{\mathbb{C}} M_k(K(N))^{\epsilon} \le \dim_{\mathbb{C}} \mathbb{M}(k, N, \epsilon)[d]$ for $d > \frac{1}{6}Nk$, and we have equality for sufficiently large d.

4. Bounds on the vanishing order of Jacobi forms.

It is an issue even to prove that the space of formal series $\mathbb{M}(k, N, \epsilon)$ is finite dimensional. Consider a partial sum $\sum_{m=0}^{\ell-1} \phi_m \xi^{Nm}$ of a putative formal series in $\mathbb{M}(k, N, \epsilon)$ and ask how many choices there are for $\phi_\ell \in J_{k,N\ell}$. By the involution condition 1, ϕ_ℓ is determined to order $q^{\ell-1}$ by $\phi_0, \phi_1, \ldots, \phi_{\ell-1}$, because, for $n < \ell$,

$$c(n, r; \phi_\ell) = (-1)^k \epsilon\, c(\ell, r; \phi_n). \tag{2}$$

If such a solution to equation 2, $\phi_\ell \in J_{k,N\ell}$, does exist, we obtain all such solutions by adding elements of $J_{k,N\ell}(\ell)$. Thus, proving the limit

$$\lim_{\nu \to +\infty} \dim J_{k,N\nu}(\nu) = 0 \tag{3}$$

would prove the finite dimensionality of $\mathbb{M}(k, N, \epsilon)$. The standard estimate [9, 12] for the vanishing order of Jacobi forms: for $\phi \in J_{k,m}$, we have $\mathrm{ord}_q \phi \le (k + 2m)/12$, is woefully inadequate to prove 3. It gives that for $\phi \in J_{k,N\nu}(\nu)$, we have $\nu \le (k + 2N\nu)/12$, or $(12 - 2N)\nu \le k$, a useless estimate for $N \ge 6$. However, it does imply the finite dimensionality of $\mathbb{M}(k, N, \epsilon)$ for $N \le 5$. Beginning with this, in 2000, Aoki proved [1] the main result here for $N = 1$. He discovered the following remarkable argument. We have

$$\dim \mathbb{M}(k, 1, +) \le \sum_{m=0}^{\mathrm{floor}(k/10)} \dim J_{k,m}(m)$$

since $J_{k,m}(m) = \{0\}$ unless $m \le k/10$. Then, using estimates from [9], he calculated that, giving only the case of even weight k here,

$$\sum_{m=0}^{\text{floor}(k/10)} \dim J_{k,m}(m) = \dim M_k \left(K(1)\right)^+ .$$

This immediately gives the generating series of Siegel modular forms of level one in degree two, a result first proven at length by Igusa [15].

In the meantime, the involution condition of equation 2 was being exploited for computational purposes for general N. The method of Jacobi restriction recursively solves for $\phi_\ell \in J_{k,N\ell}$ one Fourier-Jacobi coefficient at a time. The spaces of Jacobi forms were spanned by using the method of theta blocks [13]. In practice we find that $J_{k,N\nu}(\nu) = \{0\}$ for $\nu \ge \nu_0$, for relatively small ν_0; moreover, the dimensions of the solution spaces of finite formal series satisfying equation 2 experimentally stabilize pretty quickly to $\dim M_k \left(K(N)\right)^\epsilon$. In 2013, the article [14] showed that Aoki's method also works for $N \in \{2, 3, 4\}$ but not for $N = 5$. Due to these considerations, it was explicitly asked in [14] whether or not the main result, Theorem 3.1 here, holds for general levels N.

In order to go further, one needs a superior bound on the vanishing order of Jacobi forms, one that performs better for fixed weight k and large indices m. Aoki [2] gave such an improved bound in 2022. Here I just give the half that applies to even weights k.

Definition 4.1. *For $j \in \mathbb{N}$ we define $\psi_j : \mathbb{N} \to \mathbb{Q}^+$ by*

$$\psi_j(u) = \begin{cases} 1 & \text{if } u = 1, \\ \prod_{p|u:\ p \text{ prime}} \left(1 - \frac{1}{p^j}\right) & \text{if } u \ge 2. \end{cases}$$

Definition 4.2. *We define $\psi : \mathbb{N} \setminus \{1\} \to \mathbb{Q}^+$ by*

$$\psi(t) = \begin{cases} t^2\psi_2(t) = 3 & \text{if } t = 2, \\ \frac{1}{2}t^2\psi_2(t) = \frac{1}{2}t^2\prod_{p|t:\ p \text{ prime}} \left(1 - \frac{1}{p^2}\right) & \text{if } t \ge 3. \end{cases}$$

Theorem 4.3 ([2])**.** *Let $k \in 2\mathbb{N}, m \in \mathbb{N}$. Let $\phi \in J_{k,m}$ have $\operatorname{ord}(\phi) = \mu$. We have*

$$\min\left(m, m - 6\mu + \frac{k}{2}\right) \ge \sum_{t:(\spadesuit)} \psi(t),$$

where t runs over all natural numbers satisfying the condition

$$(\spadesuit) \quad t \ne 1 \text{ and } \sum_{c=0}^{t-1} \psi_1\left(\gcd(t,c)\right) \max\left(\mu - \frac{mc(t-c)}{t^2}, 0\right) > \frac{kt}{12}\psi_2(t).$$

As a corollary of this theorem, Aoki proved that for $\phi \in J_{k,m}(\nu)$, we have $\nu \leq (k+1)(1+\sqrt{2m+1})/6$. This bound, for fixed k, grows as $O(\sqrt{m})$ instead of $O(m)$. Applying this result to $J_{k,N\nu}(\nu)$ we calculate that

$$J_{k,N\nu}(\nu) = \{0\} \text{ for } \nu > \frac{N}{18}(k+1)^2 + \frac{1}{3}(k+1). \tag{4}$$

This bound proves equation 3 and the finite dimensionality of $\mathbb{M}(k, N, \epsilon)$.

The first new result of [3] is the following. I include the proof as it illustrates how to apply Aoki's intricate Theorem 4.3.

Proposition 4.4. *Let $k, N, \nu \in \mathbb{N}$. If $\nu > \frac{1}{6}Nk$ then $J_{k,N\nu}(\nu) = \{0\}$.*

Lemma 4.5. *Let $t, \mu, m \in \mathbb{N}$. If $\mu < m$, we have*

$$\sum_{c=0}^{t-1} \max\left(\mu - \frac{mc(t-c)}{t^2}, 0\right) > \frac{\mu^2 t}{2m}.$$

Proof. The case of odd k follows from that of even k. If $\phi \in J_{k,N\nu}(\nu)$ then $\phi^2 \in J_{2k,N(2\nu)}(2\nu)$. Since $\nu > \frac{1}{6}Nk$ we have $2\nu > \frac{1}{6}N(2k)$ and so $\phi^2 = 0$ assuming the result for even weights; hence $\phi = 0$.

Suppose that there is a nontrivial $\phi \in J_{k,N\nu}(\nu)$ with vanishing order $\mu = \text{ord}\phi \geq \nu > \frac{1}{6}Nk$ and k even. We will obtain a contradiction to Theorem 4.3. For $m = N\nu$, we will contradict

$$\min\left(m, m - 6\mu + \frac{k}{2}\right) \geq \sum_{t:(\spadesuit)} \psi(t).$$

Since ψ has a positive minimum, it suffices to show that the set of positive integers satisfying ($\spadesuit$) is infinite.

We show that all sufficiently large primes p satisfy

$$\sum_{c=0}^{p-1} \psi_1\left(\gcd(p,c)\right) \max\left(\mu - \frac{mc(p-c)}{p^2}, 0\right) > \frac{kp}{12}\psi_2(p). \tag{$\spadesuit$}$$

We know that $a \mid t$ implies $\psi_1(a) \geq \psi_1(t)$. Since $\gcd(p,c) \mid p$, we have $\psi_1(\gcd(p,c)) \geq \psi_1(p)$. Therefore

$$\begin{aligned}&\sum_{c=0}^{p-1} \psi_1\left(\gcd(p,c)\right) \max\left(\mu - \frac{mc(p-c)}{p^2}, 0\right)\\ \geq &\psi_1(p) \sum_{c=0}^{p-1} \max\left(\mu - \frac{mc(p-c)}{p^2}, 0\right).\end{aligned}$$

Lemma 4.5 says that for $t, m, \mu \in \mathbb{N}$ with $\mu < m$ we have

$$\sum_{c=0}^{t-1} \max\left(\mu - \frac{mc(t-c)}{t^2}, 0\right) > \frac{\mu^2 t}{2m}.$$

We use the linear bound $\operatorname{ord}\phi \leq \frac{k+2m}{12}$ of [12], Proposition 3.2, to check the hypothesis $\mu < m$:

$$\mu = \operatorname{ord}\phi \leq \frac{k+2m}{12} < \frac{\frac{6\nu}{N} + 2m}{12} = \frac{\frac{6}{N^2} + 2}{12} m < m.$$

Thus by Lemma 4.5 we have

$$\sum_{c=0}^{p-1} \psi_1\left(\gcd(p,c)\right) \max\left(\mu - \frac{mc(p-c)}{p^2}, 0\right)$$
$$> \psi_1(p)\frac{\mu^2 p}{2m} \geq \psi_1(p)\frac{\nu^2 p}{2m} = \psi_1(p)\frac{\nu p}{2N}.$$

Thus a sufficient condition for a prime p to satisfy (♠) is

$$\psi_1(p)\frac{\nu p}{2N} > \frac{kp}{12}\psi_2(p).$$

This simplifies to $\nu > \frac{Nk}{6}\left(1 + \frac{1}{p}\right)$, or $\nu - \frac{1}{6}Nk > \frac{Nk}{6}\frac{1}{p}$, which is true for all sufficiently large p because $\nu - \frac{1}{6}Nk$ is positive. □

The bound of Proposition 4.4 is strictly better than equation 4. In particular, the bound is directly proportional to k, which gives

$$\dim \mathbb{M}(k, N, \epsilon) \leq \sum_{\nu=0}^{\operatorname{floor}(Nk/6)} \dim J_{k,N\nu}(\nu) \sim \sum_{\nu=0}^{\operatorname{floor}(Nk/6)} kN\nu \in O\left(N^3 k^3\right).$$

The growth rate of $\dim M_k\left(K(N)\right)^\epsilon$ is also $O(k^3)$, which shows that the graded ring $\oplus_k \mathbb{M}(k, N, +)$ is algebraic over the graded subring $\oplus_k \mathrm{FJ}\left(M_k(K(N)^+)\right)$. Accordingly, every formal series $\mathfrak{f} \in \mathbb{M}(k, N, +)$ satisfies some polynomial relation in the ring of formal series

$$\mathrm{FJ}(f_0)\mathfrak{f}^d + \mathrm{FJ}(f_1)\mathfrak{f}^{d-1} + \cdots + \mathrm{FJ}(f_j)\mathfrak{f}^{d-j} + \cdots + \mathrm{FJ}(f_d) = 0,$$

for some $d \in \mathbb{N}$, $k_0 \in \mathbb{N}_0$, and some $f_j \in M_{k_0+jk}(K(N)^+)$ with f_0 not identically zero. Here it is helpful to break our troubles into two pieces by introducing an associated formal series $\mathfrak{g} = \mathrm{FJ}(f_0)\mathfrak{f}$ that satisfies a monic polynomial relation

$$\mathfrak{g}^d + \mathrm{FJ}(g_1)\mathfrak{g}^{d-1} + \cdots + \mathrm{FJ}(g_j)\mathfrak{g}^{d-j} + \cdots + \mathrm{FJ}(g_d) = 0$$

with $g_j = f_0^{j-1} f_j \in M_{(k+k_0)j}\left(K(N)^+\right)$ for $j = 1, \ldots, d$. This maneuvering postpones the issue of whether or not $\oplus_k \mathrm{FJ}\left(M_k(K(N)^+)\right)$ is integrally closed in $\oplus_k \mathbb{M}(k, N, +)$.

As a remark, an alternate approach can be adopted at this point. From the monic polynomial relation and the theory of Puiseux series it follows that the formal series $\mathfrak{g}$ converges in some neighborhood of infinity in $\mathcal{H}_2$. One may then proceed by analytic continuation, compare [4] by Aoki and Saito.

5. Additional Symmetries.

When Jacobi restriction is run to classify initial Fourier-Jacobi expansions of elements in $S_k(K(N))^{\epsilon}$, we typically include all known symmetries, not just the involution condition. In particular, we have $\Gamma^0(N)$-symmetries. Let $\Gamma \subseteq \Gamma^0(N)$ be a subgroup.

$$\mathbb{M}(k, N; \Gamma) = \{\mathfrak{f} = \sum_{m=0}^{\infty} \phi_m \xi^{Nm} \in \mathbb{M}(k, N) : \mathfrak{f} \text{ satisfies equation (5)}\},$$

$$\forall \sigma \in \Gamma, \forall t_1 = \begin{pmatrix} n_1 & r_1/2 \\ r_1/2 & Nm_1 \end{pmatrix}, t_2 = \begin{pmatrix} n_2 & r_2/2 \\ r_2/2 & Nm_2 \end{pmatrix} \in \bar{\mathcal{X}}(N), \tag{5}$$
$$\text{if } t_1[\sigma] = t_2 \text{ then } c(n_2, r_2; \phi_{m_2}) = c(n_1, r_1; \phi_{m_1}).$$

The $\Gamma^0(N)$-symmetries play an important role in theoretical considerations as well. As a consequence of the polynomial relation, we will show that each formal series $\mathfrak{f} \in \mathbb{M}(k, N, \epsilon)$ automatically satisfies some additional $\Gamma^0(N)$-symmetries.

Each formal series $\mathfrak{f}$ has an associated formal Fourier series,

$$\mathrm{AFS}(\mathfrak{f}) = \sum_{t=\begin{pmatrix} n & r/2 \\ r/2 & Nm \end{pmatrix} \in \bar{\mathcal{X}}(N)} c(n, r; \phi_m) q^t,$$

and the collection of formal Fourier series forms its own kind of integral domain. We convert the polynomial relation into one in this ring of formal Fourier series.

$$\mathrm{FS}(f_0)(\mathrm{AFS}(\mathfrak{f}))^d + \cdots + \mathrm{FS}(f_j)(\mathrm{AFS}(\mathfrak{f}))^{d-j} + \cdots + \mathrm{FS}(f_d) = 0. \tag{6}$$

The group $\Gamma^0(N)$ acts on formal Fourier series in the following way. For $\sigma \in \Gamma^0(N)$ and $\psi = \sum_{t \in \bar{\mathcal{X}}(N)} a(t; \psi) q^t$, set $\mathrm{j}(\sigma)\psi = \sum_{t \in \bar{\mathcal{X}}(N)} a(t[\sigma]; \psi) q^t$. In this way, $\Gamma^0(N)$ acts as a group of automorphisms on the integral domain of formal Fourier series. The coefficients of the polynomial relation 6 are invariant under this action, which therefore permutes the roots of the polynomial relation. The induced homomorphism from $\Gamma^0(N)$ to the finite symmetric group S_d has a kernel Γ of finite index in $\Gamma^0(N)$ that fixes the roots, including $\mathrm{AFS}(\mathfrak{f})$. Hence we deduce that $\mathfrak{f}$ has additional symmetries under a subgroup Γ of finite index in $\Gamma^0(N)$.

6. Locally bounded.

Both the involution condition 1 and the $\Gamma^0(N)$-symmetries 5 are "fair" hypotheses to place upon a formal series of Jacobi forms because they give linear relations that a computer can check. This is by way of contrast to a convergence hypothesis, which the computer cannot directly check. What theoretical purpose do the $\Gamma^0(N)$-symmetries serve for us?

It is difficult to bound the Jacobi coefficients ϕ_m of a formal series with involution in terms of a finite number of them. However, if we first restrict these Jacobi forms to elliptic modular forms, by the usual method of specializing (τ, z) to a torsion point $(\tau, x\tau + y)$ for $x, y \in \mathbb{Q}$, then the $\Gamma^0(N)$-symmetries allow us to bound all these elliptic forms in terms of a finite number of them. This is the theoretical use for us of the $\Gamma^0(N)$-symmetries. By technical steps we reduce to the case where $\mathfrak{f} = \sum_{m=1}^{\infty} \phi_m e(Nm\omega)$ a a cusp form, meaning $\phi_m \in J_{k,Nm}^{\text{cusp}}$ for each $m \in \mathbb{N}$. We write $\mathfrak{f} \in \mathbb{S}(k, N; \Gamma)$ to mean that $\mathfrak{f}$ is a cusp form and that $\mathfrak{f}$ satisfies the symmetries 5.

Definition 6.1. *Let $\Gamma \subseteq \mathrm{SL}(2, \mathbb{Z})$ be a subgroup of finite index. Let $U = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{GL}(2, \mathbb{Z})$. Define*

$$\mathcal{W}_0(\Gamma) = \{(x, y) \in \mathbb{Q}^2 : x \in (U\Gamma U)\text{-}orbit(\infty) \text{ and } y \,\mathrm{Denom}(x) \in \mathbb{Z}\},$$
$$\mathcal{W}_1(\Gamma) = \{(\tau, z) \in \mathcal{H}_1 \times \mathbb{C} : \exists (x, y) \in \mathcal{W}_0(\Gamma) : z = x\tau + y\},$$
$$\mathcal{W}(\Gamma) = \{\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) \in \mathcal{H}_2 : (\tau, z) \in \mathcal{W}_1(\Gamma)\}.$$

Corollary 6.2. *Let $\Gamma \subseteq \Gamma^0(N)$ be a subgroup of finite index. Let $\mathfrak{f} = \sum_{m=1}^{\infty} \phi_m \xi^{Nm} \in \mathbb{S}(k, N; \Gamma)$. The formal series $\mathfrak{f}$ converges absolutely on $\mathcal{W}(\Gamma)$. For $(\tau_1, z_1) \in \mathcal{W}_1(\Gamma)$ set $\eta_1 = (\mathrm{Im}(z_1))^2/\mathrm{Im}(\tau_1)$. The function $H(\tau_1, z_1, \mathfrak{f}) : N_\infty(\eta_1) \to \mathbb{C}$ defined by*

$$H(\tau_1, z_1, \mathfrak{f})(\omega) = \sum_{m=1}^{\infty} \phi_m(\tau_1, z_1) e\left(Nm\omega\right)$$

is holomorphic on $N_\infty(\eta_1) = \{\omega \in \mathcal{H}_1 : \mathrm{Im}(\omega) > \eta_1\}$.

We see from Corollary 6.2 that the formal series $\mathfrak{f}$ converges when restricted to the curve $\Omega_1(\omega) = \left(\begin{smallmatrix} \tau_1 & z_1 \\ z_1 & \omega \end{smallmatrix}\right)$, and there specializes to a holomorphic function $H(\tau_1, z_1, \mathfrak{f})$ of ω. What makes these specializations really useful, however, is that the $(\tau_1, z_1) \in \mathcal{W}_1(\Gamma)$ form a dense set in $\mathcal{H}_1 \times \mathbb{C}$ and the curves form a dense set in $\mathcal{H}_2$. This point relies on the theory of Fuchsian groups. As a subgroup of finite index, Γ is a Fuchsian group of the first kind with a limit set $\Lambda(\Gamma)$. The limits set is the set of points on the Riemann sphere where Γ does not act properly discontinuously.

Definition 6.3. *Let* $\Gamma \subseteq \mathrm{SL}(2,\mathbb{R})$. *The* limit set $\Lambda(\Gamma)$ *of* Γ *is the set of* $z \in \mathbb{P}^1(\mathbb{C})$ *such that there exists a* $w \in \mathbb{P}^1(\mathbb{C})$ *and a sequence of distinct* $\gamma_n \in \Gamma$ *with* $\lim_{n\to+\infty} \gamma_n\langle w\rangle = z$.

We use the following ergodic property of the limit set.

Lemma 6.4. *Let* $\Gamma \subseteq \mathrm{SL}(2,\mathbb{Z})$ *be a subgroup of finite index. The* Γ*-orbit of* ∞ *is dense in* $\Lambda(\Gamma) = \mathbb{P}^1(\mathbb{R})$.

It is this ergodic property that allows us to prove that $\mathfrak{f}$ converges to a holomorphic function on a dense set of curves.

Lemma 6.5. *Let* $\Gamma \subseteq \mathrm{SL}(2,\mathbb{Z})$ *be a subgroup of finite index. The sets* $\mathcal{W}_0(\Gamma)$, $\mathcal{W}_1(\Gamma)$, *and* $\mathcal{W}(\Gamma)$ *are dense in* $\mathbb{R}^2$, $\mathcal{H}_1 \times \mathbb{C}$, *and* $\mathcal{H}_2$, *respectively.*

The fact that $\mathfrak{f} \in \mathbb{S}(k, N; \Gamma)$ converges on a dense subset of $\mathcal{H}_2$ raises the possibility of using powerful convergence theorems from the theory of several complex variables. The following is Exercise 4a of section 4 in Chapter 1 of [11] and a proof can be found in [3].

Theorem 6.6. *Let* $U \subseteq \mathbb{C}^d$ *be open. Let* $\{f_j\}$ *be a locally bounded sequence of holomorphic functions on* U *that converges on a dense subset of* U. *Then the sequence* $\{f_j\}$ *converges on* U *and uniformly on compact subsets of* U.

This theorem shifts our attention to proving that the sequence of partial sums $\sum_{m=1}^{M} \phi_m(\tau, z)e(Nm\omega)$ is locally bounded on $\mathcal{H}_2$. These partial sums are continuous and so it suffices to demonstrate their local boundedness on a dense subset. We can achieve this, due to the following elementary bound, when the formal series $\mathfrak{g} = \mathrm{FJ}(f_0)\mathfrak{f}$ satisfies a monic polynomial so that the specializations $H(\tau_1, z_1, \mathfrak{g})$ also satisfy a monic polynomial.

Lemma 6.7. *Let* $d \in \mathbb{N}$ *and* $z \in \mathbb{C}$ *be given. Let a monic polynomial* $P \in \mathbb{C}[X]$ *of degree* d *be given by* $P(X) = X^d + \sum_{j=1}^{d} a_j X^{d-j}$ *for* $a_1, \ldots, a_d \in \mathbb{C}$. *If* $P(z) = 0$ *then* $|z| \le 1 + \sum_{j=1}^{d} |a_j|$.

The specializations $H(\tau_1, z_1, \mathfrak{g})$ are roots of a monic polynomial with holomorphic coefficients and so are locally bounded. The partial sums $\sum_{m=1}^{M} \phi_m(\tau_1, z_1)e(Nm\omega)$ of $H(\tau_1, z_1, \mathfrak{g})(\omega)$ are locally bounded because the function $H(\tau_1, z_1, \mathfrak{g})$ is holomorphic. Therefore the general partial sums $\sum_{m=1}^{M} \phi_m(\tau, z)e(Nm\omega)$ are locally bounded because these partial sums are continuous and locally bounded on a dense subset.

Thus we can show that the adjusted formal series $\mathfrak{g} = \mathrm{FJ}(f_0)\mathfrak{f}$ is convergent on $\mathcal{H}_2$ and $\mathfrak{g} = \mathrm{FJ}(G)$ for some paramodular form G. Finally, we must address a problem that we have postponed; namely, to prove that the holomorphy of G/f_0 follows from $\mathrm{FJ}(G) = \mathrm{FJ}(f_0)\mathfrak{f}$.

7. Divisor Theory.

Finally, we confront an obstacle that has always been a worry, that a properly meromorphic paramodular form might have a Fourier-Jacobi expansion that lies in $\mathbb{M}(k, N, \epsilon)$,

$$\mathrm{FJ}(G/f_0) := \mathrm{FJ}(G)/\mathrm{FJ}(f_0) = \mathfrak{f} \in \mathbb{M}(k, N, \epsilon).$$

We reduce to the case $\mathfrak{f} \in \mathbb{S}(k, N, \epsilon)$ by lemmas here suppressed.

We want to deduce that G/f_0 is holomorphic from the assumptions $\mathrm{FJ}(G) = \mathrm{FJ}(f_0)\mathfrak{f}$ and $\mathfrak{f} \in \mathbb{S}(k, N, \epsilon)$. It is the Weierstrauss preparation theorem and the particular form of our dense collection of holomorphic curves, $\Omega_1(\omega) = \left(\begin{smallmatrix} \tau_1 & x\tau_1+y \\ x\tau_1+y & \omega \end{smallmatrix}\right)$ for $(x, y) \in \mathcal{W}_0(\Gamma)$, on which $\mathfrak{f}$ specializes to a holomorphic function, that will allow us to prove that G/f_0 is holomorphic. We proceed by showing that, for any irreducible component V of the divisor of f_0, we have $\mathrm{ord}_V(G) \geq \mathrm{ord}_V(f_0)$. After a series of normalizations, using involutions of $K(N)^+$, we find a point $p = \left(\begin{smallmatrix} \tau_1 & z_1 \\ z_1 & \omega_1 \end{smallmatrix}\right) \in V$ and a local equation $\varpi_p = 0$ of V at p such that $G_p \in \varpi_p^{\mathrm{ord}_V(G)}\mathcal{O}_p^*$ and $(f_0)_p \in \varpi_p^{\mathrm{ord}_V(f_0)}\mathcal{O}_p^*$ in the local ring $\mathcal{O}_p$ of germs of holomorphic functions at p. By the Weierstrauss preparation theorem we may also choose p so that $p \in V \cap \mathcal{W}(\Gamma)$, i.e., one of our holomorphic curves passes through p. Moreover, V is the only irreducible component of G passing through p, ϖ is regular in ω at p, and so the specialization $\varpi\left(\Omega_1(\omega)\right)$ is not identically zero as a function of ω. By specializing the equation $\mathrm{FJ}(G) = \mathrm{FJ}(f_0)\mathfrak{f}$ we obtain $G\left(\Omega_1(\omega)\right) = H(\tau_1, z_1, \mathfrak{g})(\omega) f_0\left(\Omega_1(\omega)\right)$. Since V is the only component of the divisor of G passing through p, in the local ring $\mathcal{O}_p$ we have

$$(\mathrm{unit})\ \varpi_p\left(\Omega_1(\omega)\right)^{\mathrm{ord}_V(G)} \in H(\tau_1, z_1, \mathfrak{g})_p(\omega)\, \varpi_p\left(\Omega_1(\omega)\right)^{\mathrm{ord}_V(f_0)}\ \mathcal{O}_p.$$

This implies that $\mathrm{ord}_V(G) \geq \mathrm{ord}_V(f_0)$ and hence that G/f_0 is holomorphic. The Fourier-Jacobi expansion of the paramodular form G/f_0 is our original formal series $\mathfrak{f}$, and $\mathfrak{f}$ converges on $\mathcal{H}_2$.

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