On mod p^m singular vector-valued modular forms

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Abstract

We generalize the notion of mod p^m singular Siegel modular forms of p-rank r to the vector-valued case and we show that also in this case a congruence mod $(p-1)p^{m-1}$ between the scalar weight and the p-rank must hold. In some sense our proof is even simpler than the one given in [3].

1 Introduction

Integrality properties of Fourier coefficients and of Hecke eigenvalues and congruences among them are important topics in the arithmetic theory of modular forms. The recent impressive works of [1] show that one should consider such questions also in the context of vector-valued Siegel modular forms.

We study the following question:

Can it happen that a degree n Siegel modular form with integral Fourier coefficients has all its rank n Fourier coefficients divisible by p if there is at least a rank n-1 Fourier coefficient coprime to p ?

In [3] we studied this problem for the case of scalar-valued automorphy factors. We showed that this is only possible if n and the weight k are tied together by a congruence mod $p-1$. Our aim is a generalization of this statement for the vector-valued case. We adopt some techniques from the theory of vector-valued singular modular forms over $\mathbb{C}$ as created by Freitag [7] and from our work on (scalar-valued) mod p^m singular modular forms [4]. Here is our main result (notations will be explained below).

Theorem: *Let ρ be an irreducible ℓ-dimensional polynomial representation of $GL(n,\mathbb{C})$, of scalar weight k and realized integrally on $\mathbb{C}^\ell$ and p an odd prime. Let F be an integral modular form of automorphy factor ρ for $\Gamma^n = Sp(n,\mathbb{Z})$. If F is mod p^m singular of rank $r < n$ then we have*

$$2k - r \equiv 0 \bmod (p-1)p^{m-1}.$$

For simplicity, we only consider level one forms here. However, in the same way as in [3] we may extend our results to cases of congruence subgroups with quadratic nebentypus characters mod p, possibly half-integral weights and also to dyadic cases and to the more delicate case of mod $\mathfrak{p}^m$ singular forms where $\mathfrak{p}$ is a prime ideal over p in some number field, see also [5].

2 Preliminaries

2.1 Siegel modular forms

Let n be a positive integer, $\mathbb{H}_n$ the Siegel upper half space of degree n, Γ^n the Siegel modular group of degree n and $\Gamma_0^n(N)$ the subgroup defined by the congruence "$C \equiv 0_n \bmod N$".
Let $\rho : GL(n,\mathbb{C}) \longrightarrow GL(V)$ be an irreducible ℓ-dimensional polynomial representation, we will always use a version in coordinates, i.e., $V = \mathbb{C}^\ell$. We define the slash-operator for V-valued functions on $\mathbb{H}_n$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R})$ by

$$F \mid_\rho M = \rho(CZ+D)^{-1} F(M < Z >).$$

For a representation ρ as above, we define the space of modular forms of automorphy factor ρ for $\Gamma = \Gamma^n$ as

$$M_\rho^n(\Gamma) := \{f : \mathbb{H}_n \longrightarrow V \mid \forall \gamma \in \Gamma : f \mid_\rho g = f,\ f \text{ holomorphic}\}$$

(and the standard additional condition for $n = 1$).
For general properties of Siegel modular forms we refer to the books [7, 11].
We use an "integral version" of ρ, i.e., we assume that - after choosing appropriate coordinates in the representation space V - we have

$$\rho(GL(n,\mathbb{Z})) \subset GL(\ell,\mathbb{Z}).$$

In particular, the polynomial functions defining ρ have coefficients in $\mathbb{Q}$. Such a choice is always possible, see [9].
Any modular form $F \in M^n_\rho(\Gamma)$ has a Fourier expansion

$$F(Z) = \sum_{T\in\Lambda^n} a_F(T) e^{2\pi i tr(TZ)}$$

where

$$\Lambda^n = \{T = (t_{ij}) \in Sym_n(\mathbb{Q}) \mid t_{ii}, 2t_{ij} \in \mathbb{Z}, \quad T \,\text{positive-semidefinit}\}.$$

Furthermore, for $0 \leq r \leq n$ we denote by Λ^n_r the subset of all T with $rank(T) = r$.
Later on we also need a Fourier-Jacobi-expansion of F; for that purpose we we decompose $Z \in \mathbb{H}_n$ into block matrices $Z = \begin{pmatrix} z_1 & z_2 \\ z_2^t & z_4 \end{pmatrix}$ with $z_4 \in \mathbb{H}_r$ and we get a partial Fourier series

$$F(Z) = \sum_{T\in\Lambda^r} \Phi_T(z_1, z_2) e^{2\pi i tr(Tz_4)}.$$

Under the integrality assumption from above for ρ we can talk about integral Siegel modular forms by requesting all Fourier coefficients to be in $\mathbb{Z}^\ell$.
We also mention that $M^n_\rho = M^n_\rho(\mathbb{Z}) \otimes \mathbb{C}$ always holds (see [2] for a simple proof in the level one case), but such a property will not be needed in our work.

We call such a modular form $F \in M^n_\rho(\mathbb{Z})$ mod p^m singular of rank r if all Fourier coefficients $a_F(T)$ with $rank(T) > r$ are in $p^m \cdot \mathbb{Z}^\ell$ and there exists some Fourier coefficient of rank r which is not congruent zero mod p.
By general representation theory there is a weight

$$\mathbf{k} = (k_1 \geq k_2 \geq \cdots \geq k_n)$$

associated to ρ ("highest weight").
A mod p^m singular modular form of rank r is in particular a noncusp form, it remains nonzero after $(n-r)$-fold application of Siegel's Φ-operator and it remains mod p^m singular of rank r (and degree $r+1$) after applying Φ^{n-r-1}.
A theorem of Weissauer [12] tells us that

$$k_n = \cdots = k_{n-r+1}$$

must hold. We call this $k := k_n$ the (scalar) weight of ρ and we write $\rho = \rho_0 \otimes \det^k$. Note that ρ_0 is still polynomial.
Conclusion: *From the facts mentioned above, we see that it suffices to prove the theorem for the case* $r = n - 1$.

3 On some partial Fourier series of F

We study some subseries of the Fourier expansion of a modular form, defined by the rank of T and by divisibility properties of the entries of T (respectively). This section is independent of congruence properties.

3.1 The subseries defined by fixing the rank of T

We want to avoid the use of the theta decomposition of the Fourier-Jacobi coefficients of F, which was used in [3] as a main tool.
Instead we start, for an arbitrary modular form $F \in M^n_\rho(\Gamma)$ with a description of the "rank r"-part of the Fourier expansion, following the procedure in [4]:
We express the partial Fourier series

$$F^* := \sum_{T\in\Lambda_n, rank(T)=r} a_F(T) e^{2\pi i tr(TZ)}$$

by

$$F^0 := \sum_{T\in\Lambda_r^r} a_F(\begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}) e^{2\pi i tr(Tz_4)},$$

using the relation $a_F(UTU^t) = \rho(U)a_F(T)$, which holds for all T and all $U \in GL(n, \mathbb{Z})$.
We obtain, using the submatrices z_1, z_2, z_4 introduced above,

$$F^* = \sum_U \sum_{T\in\Lambda_r^r} \rho(U) a_F(\begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}) e^{2\pi i tr(U\cdot \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix} U^t\cdot Z)}.$$

The expression in the exponential is then equal to

$$tr(u_2 T u_2^t z_1 + 2 u_2 T u_4 z_2) + tr(u_4 T u_4^t z_4).$$

The summation over U goes over $GL(n,\mathbb{Z})/GL(n,r,\mathbb{Z})$ with

$$GL(n,r,\mathbb{Z}) = \{U = \begin{pmatrix} u_1 & 0 \\ u_3 & u_4 \end{pmatrix} \mid u_1 \in GL(n-r,\mathbb{Z}), u_3 \in \mathbb{Z}^{r,n-r}, u_4 \in Aut(T)\}.$$

3.2 Imposing congruence conditions on the upper right block submatrix of T

In the context of the doubling method we already used a kind of "partial twists" of modular forms to modify the primitive nebentypus character of the modular form in question, see [6]. In loc.cit., remark 2.2. we mentioned that there is also a version for trivial character. We will use it in the following context:

Observation: *Let F be a modular form of degree n (possibly vector-valued) with Fourier expansion*

$$F(Z) = \sum_S a(S) e^{2\pi i tr(SZ)}.$$

Then for any $t \geq 0$ the subseries $\tilde{F}(Z)$, where S runs only over those matrices, which are of the form

$$S = \begin{pmatrix} S_1 & S_2 \\ S_2^t & S_4 \end{pmatrix} \quad \textit{with} \quad S_1 \in \Lambda_{n-r}, S_4 \in \Lambda_r \quad \textit{and} \quad S_2 \equiv 0 \bmod p^t$$

is still modular, in particular for

$$\Gamma_0^{n-r}(p^{2t}) \times \Gamma_0^r(p^{2t}) \subset \Gamma_0^n(p^{2t})$$

Here we identify the product $Sp(n-r) \times Sp(r)$ with a subgroup of $Sp(n)$ via the natural embedding defined by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \times \begin{pmatrix} a & b \\ c & d \end{pmatrix} \hookrightarrow \begin{pmatrix} A & 0 & B & 0 \\ 0 & a & 0 & b \\ C & 0 & D & 0 \\ 0 & c & 0 & d \end{pmatrix}.$$

4 Proof of theorem

We now apply the consideration from above to a modular form $F \in M^n_\rho(\Gamma)_{\mathbb{Z}}$, which is mod p^m singular of rank $r = n-1$. We extract for this F the Fourier-Jacobi coefficient mod p^m for $T \in \Lambda^r_r$ with $a(\begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix})$ not congruent 0 mod p and $\det(T)$ minimal with this property. Then the summation over U just becomes summation over u_2.
For such T we get

$$\phi_T(z_1, z_2) \equiv \sum_{u_2} \rho(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}) \cdot a(\begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}) e^{2\pi i tr(u_2 T u_2^t z_1 + 2u_2 T z_2)} \bmod p^m. \tag{1}$$

Everything would be much easier if we could restrict the summation to those u_2 for which $\rho(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix})$ would just be congruent mod p^m to the identity. To achieve this, we use the observation from the previous section to switch from F to $\tilde{F}$ with sufficiently large t. This is still singular mod p^m of rank r. The condition "$S_2 \equiv 0 \bmod p^t$" from the observation now becomes "$T \cdot u_2 \equiv 0 \bmod p^t$ ", which can be rephrased as

$$T \cdot u_2 \in T \cdot \mathbb{Z}^r \cap p^t \cdot \mathbb{Z}^r$$

i.e.

$$u_2 \in \mathbb{Z}^r \cap p^t \cdot T^{-1} \cdot \mathbb{Z}^r = R \cdot \mathbb{Z}^r \tag{2}$$

for a suitable integral $r \times r$ matrix R of maximal rank. Then for sufficiently large t (depending on T) we have $u_2 \equiv 0 \bmod p^m$ for such u_2.
Now we look at the Fourier-Jacobi coefficient $\tilde{\Phi}$ mod p^m for $\tilde{F}$. The expression in (1) for $\tilde{\phi_T}(z_1, 0)$ then runs only over u_2 with $u_2 \equiv 0 \bmod p^t$; on such u_2 the value of ρ is just the identity mod p^m.
Then we obtain

$$\tilde{\Phi}_T(z_1, 0) \equiv \sum_{u_2 \in R \cdot \mathbb{Z}^r} a(\begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}) e^{2\pi i tr(u_2 T u_2^t z_1)} \bmod p^m.$$

We may rewrite this, using (2) and putting $\mathcal{R} := R^t T S$

$$\tilde{\Phi}_T(z_1, 0) \equiv \theta^{(n-r)}_{\mathcal{R}}(z_1) \cdot a(\begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}) \bmod p^m \tag{3}$$

with the theta series

$$\theta_{\mathcal{R}}^{(n-r)}(z_1) := \sum_{X \in \mathbb{Z}^{(r,n-r)}} e^{2\pi i tr(X^t \mathcal{R} X \cdot z_1)}.$$

The change from T to $\mathcal{R} := S^t \cdot T \cdot S$ comes from the condition (2).
To understand $F_{|z_2=0}$ and $\Phi_T(z_1, 0)$ as modular forms, we decompose the representation space $V = \mathbb{C}^\ell$ for ρ_0 as a $GL(1) \times GL(n-1)$-module. The branching laws (e.g., Goodman-Wallach [8, Theorem 8.1.2]) give

$$V = \oplus V_i \otimes W_i \tag{4}$$

where V_j is one dimensional with $x \in GL(1, \mathbb{C})$ acting on V_i by multiplication by x^i. We do not need the explicit description of the $GL(n-1, \mathbb{C})$- module W_i here. We also emphasize that the decomposition (4) will be needed just over $\mathbb{C}$.
Using the elementary divisor theorem, we may choose a basis

$$\mathfrak{a}_1, \ldots, \mathfrak{a}_\ell$$

and natural numbers $\alpha_1 \mid \cdots \mid \alpha_m$ such that $\mathfrak{a}_1, \ldots, \mathfrak{a}_\ell$ is a $\mathbb{Z}$ basis of $V(\mathbb{Z}) = \mathbb{Z}^\ell$ and $\alpha_1 \cdot \mathfrak{a}_1, \ldots, \alpha_r \cdot \mathfrak{a}_m$ is a $\mathbb{Z}$-basis of $(V_0 \otimes W_0) \cap V(\mathbb{Z})$. We do not claim that m is the dimension of the $\mathbb{C}$-vector space $V_0 \otimes W_0$, but it is important to remark that $a(\begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}) \neq 0$ implies that $m \geq 1$.

How to interpret the congruence (3) ??

We observe that $\tilde{\Phi}_T(z_1, 0)$ is a V -valued function, more precisely

$$\tilde{\Phi}_T(z_1, 0) = \sum_i g_i(z_1) \cdot \mathfrak{a}_i$$

where the g_i are modular forms of weight k for $i \leq r$ and of higher weight otherwise (all of degree one).
On the other hand, $a(\begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix})$ is $(V_0 \otimes W_0) \cap V(\mathbb{Z})$-valued and therefore

$$\theta_{\mathcal{R}}(z_1) \cdot a(\begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}) = \theta_{\mathcal{R}}(z_1) \sum_{j=1}^{r} \beta_j \cdot \alpha_j \mathfrak{a}_j$$

with integer coefficients β_j and at least one $\beta_{j_0} \cdot \alpha_{j_0}$ not congruent zero mod p.
The congruence (3) then implies that for such j_0
the congruence

$$g_{j_0} \equiv \beta_{j_0} \cdot \alpha_{j_0} \cdot \theta_{\mathcal{R}} \bmod p$$

holds.
This is a congruence mod p^m between a degree one modular form $g := g_{j_0}$ of level $\Gamma_1(N) \cap \Gamma_0(p^{2t})$ with possible quadratic nebentypus mod p and a theta series $\theta_{\mathcal{R}}$ of weight $\frac{r}{2}$. To avoid the case of half-integral weights, we consider the congruence between g^2 and the theta series for $\mathcal{R} \perp \mathcal{R}$. This theta series of weight r then may have nebentypus $\left(\frac{(-1)^r \det(R)^2}{\cdot}\right)$ which is a nontrivial character at most modulo 4. After possibly applying level change mod p^m to g^2 and the θ-series (their weights would not change mod $(p-1)p^{m-1}$, we arrive at a congruence between two modular forms of level coprime to p. We may now argue as in [10, Corollary 4.4.2], see also [3, section 5]. We obtain the desired congruence

$$2k \equiv r \bmod (p-1)p^{m-1}.$$

Final remarks:

- Irreducibility of ρ is not really needed; indeed the proof from above works (for $n = r + 1$) in the case of arbitrary ρ, provided that all the irreducible components are of the same scalar weight k
- There seems to be a possibility of generalizing our structure theorem on mod p^m singular forms [4] to the vector-valued case. For that purpose however we cannot just switch to "$T \cdot u_2 \equiv 0 \bmod p^t$" as above; we would have to work directly with series of type (1).

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