ON CRITICAL VALUES OF THE TENSOR PRODUCT L-FUNCTIONS FOR $\mathrm{GSp}_4 \times \mathrm{GSp}_4$

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1. Main result

Let Σ and Π be cuspidal automorphic representations of $\mathrm{GSp}_4(\mathbb{A}_{\mathbb{Q}})$. We denote by

$$L(s, \Sigma \times \Pi) = \prod_v L(s, \Sigma_v \times \Pi_v), \quad \mathrm{Re}(s) \gg 0$$

the degree 16 tensor product L-function associated to Σ and Π. It admits meromorphic continuation to the whole complex plane and satisfies a functional equation relating $L(s, \Sigma \times \Pi)$ to $L(1-s, \Sigma^\vee \times \Pi^\vee)$. Assume Σ_∞ and Π_∞ are holomorphic discrete series representations of weights $(\underline{\kappa}; \mathsf{u})$ and $(\underline{\kappa}'; \mathsf{u}')$ respectively in $\mathbb{Z}^2 \times \mathbb{Z}$ with $\underline{\kappa} = (\kappa_1, \kappa_2)$, $\underline{\kappa}' = (\kappa_1', \kappa_2')$ such that

$$\kappa_1 \geqslant \kappa_2 \geqslant 3, \quad \kappa_1' \geqslant \kappa_2' \geqslant 3, \quad \kappa_1 + \kappa_2 \equiv \mathsf{u} \pmod 2, \quad \kappa_1' + \kappa_2' \equiv \mathsf{u}' \pmod 2.$$

In this case, for each $\sigma \in \mathrm{Aut}(\mathbb{C})$, the irreducible admissible representations

$${}^\sigma\Sigma := \Sigma_\infty \otimes {}^\sigma\Sigma_f, \quad {}^\sigma\Pi := \Pi_\infty \otimes {}^\sigma\Pi_f$$

are cuspidal automorphic. Let Σ_+ and Π_+ be the space of cusp forms in Σ and Π whose archimedean component are highest weight vectors in the minimal $\mathrm{U}(2)$-type of Σ_∞ and Π_∞ respectively. For $\sigma \in \mathrm{Aut}(\mathbb{C})$, we then have σ-linear $\mathrm{GSp}_4(\mathbb{A}_f)$-equivariant isomorphisms

$$\Sigma_+ \longrightarrow {}^\sigma\Sigma_+, \quad \varphi \longmapsto {}^\sigma\varphi$$
$$\Pi_+ \longrightarrow {}^\sigma\Pi_+, \quad \varphi \longmapsto {}^\sigma\varphi$$

defined via conjugation of σ on the Fourier coefficients of cusp forms along the Siegel parabolic subgroup of GSp_4. A critical point for $L(s, \Sigma \times \Pi)$ is an integer m such that the local factors $L(s, \Sigma_\infty \times \Pi_\infty)$ and $L(1-s, \Sigma_\infty^\vee \times \Pi_\infty^\vee)$ are holomorphic at $s = m$. Note that there exist critical points if and only if

$$\{\kappa_1 + \kappa_2 - 2,\ \kappa_1 - \kappa_2 + 2\} \cap \{\kappa_1' + \kappa_2' - 2,\ \kappa_1' - \kappa_2' + 2\} = \varnothing.$$

In particular, we must have $\kappa_1 - \kappa_2 \neq \kappa_1' - \kappa_2'$. Without loss of generality, we may assume $\kappa_1 - \kappa_2 > \kappa_1' - \kappa_2'$. We then have three critical types according to the relative position of $\kappa_1 + \kappa_2 - 2$, $\kappa_1 - \kappa_2 + 2$, $\kappa_1' + \kappa_2' - 2$, $\kappa_1' - \kappa_2' + 2$:

$$\begin{cases} \kappa_1' - \kappa_2' + 2 < \kappa_1 - \kappa_2 + 2 < \kappa_1' + \kappa_2' - 2 < \kappa_1 + \kappa_2 - 2 & \text{Case 1,} \\ \kappa_1' - \kappa_2' + 2 < \kappa_1 - \kappa_2 + 2 < \kappa_1 + \kappa_2 - 2 < \kappa_1' + \kappa_2' - 2 & \text{Case 2,} \\ \kappa_1' - \kappa_2' + 2 < \kappa_1' + \kappa_2' - 2 < \kappa_1 - \kappa_2 + 2 < \kappa_1 + \kappa_2 - 2 & \text{Case 3.} \end{cases}$$

Based on

- Deligne's conjecture [Del79] on critical values of motivic L-functions, in the special case for the tensor product of two motives of rank 4;
- the computation of Yoshida [Yos01, § 3 and § 4] on the fundamental periods of a motive of rank 4 associated to a Siegel cusp eigenform of degree two;

we expect that the algebraicity of critical values for $L(s, \Sigma \times \Pi)$ should be expressed in terms of a monomial (an automorphic analogue of Deligne's periods) in

$$\begin{aligned}
&2\pi\sqrt{-1}, \quad G(\omega_\Sigma), \quad G(\omega_\Pi) \quad \text{(Gauss sums)} \\
&(\sqrt{-1})^{\mathsf{u}} \cdot \langle f_\Sigma, f_\Sigma \rangle, \quad (\sqrt{-1})^{\mathsf{u}'} \cdot \langle f_\Pi, f_\Pi \rangle \quad \text{(Petersson norms)} \\
&(\sqrt{-1})^{\mathsf{u}} \cdot L^{(\infty)}(1, \Sigma, \mathrm{Ad}) \cdot \langle f_\Sigma, f_\Sigma \rangle^{-1} \quad \text{(adjoint } L\text{-value / Petersson norm).}
\end{aligned}$$

More precisely, we propose the following conjecture.

Conjecture. *Fix non-zero $f_\Sigma \in \Sigma_+$ and $f_\Pi \in \Pi_+$. For a critical point $m \in \mathbb{Z}$, we have*

$$\sigma\left(\frac{L^{(\infty)}(m, \Sigma \times \Pi)}{(2\pi\sqrt{-1})^{8m} \cdot q(\Sigma \times \Pi)}\right) = \frac{L^{(\infty)}(m, {}^\sigma\Sigma \times {}^\sigma\Pi)}{(2\pi\sqrt{-1})^{8m} \cdot q({}^\sigma\Sigma \times {}^\sigma\Pi)}, \qquad \sigma \in \mathrm{Aut}(\mathbb{C}).$$

Here

$$\begin{aligned}
&q(\Sigma \times \Pi) \\
&= (2\pi\sqrt{-1})^{4\mathsf{u}+4\mathsf{u}'} \cdot G(\omega_\Sigma)^4 G(\omega_\Pi)^4 \\
&\times \begin{cases}
(2\pi\sqrt{-1})^{\kappa_1'+\kappa_2'-6} \cdot (\sqrt{-1})^{\mathsf{u}'} \cdot L^{(\infty)}(1, \Sigma, \mathrm{Ad}) \cdot \langle f_\Pi, f_\Pi \rangle & \textit{Case 1,} \\
(2\pi\sqrt{-1})^{-\kappa_1-\kappa_2+2\kappa_1'+2\kappa_2'-6} \cdot (\sqrt{-1})^{\mathsf{u}} \cdot L^{(\infty)}(1, \Sigma, \mathrm{Ad}) \cdot \langle f_\Sigma, f_\Sigma \rangle^{-1} \cdot \langle f_\Pi, f_\Pi \rangle^2 & \textit{Case 2,} \\
(2\pi\sqrt{-1})^{-2\kappa_1-2\kappa_2-12} \cdot L^{(\infty)}(1, \Sigma, \mathrm{Ad})^2 \cdot \langle f_\Sigma, f_\Sigma \rangle^{-2} & \textit{Case 3.}
\end{cases}
\end{aligned}$$

In particular, if f_Σ and f_Π are rational over some number field E, then

$$\frac{L^{(\infty)}(m, \Sigma \times \Pi)}{(2\pi\sqrt{-1})^{8m} \cdot q(\Sigma \times \Pi)} \in E.$$

Our main result is the following theorem.

Theorem. *The conjecture holds under the following regularity conditions:*

(1) $\kappa_1 - \kappa_2 > \kappa_1' - \kappa_2' \geqslant 1$.

(2) $\min\left\{|\kappa_1 + \varepsilon(\kappa_2 - 2) - \kappa_1' - \varepsilon'(\kappa_2' - 2)|\,,\, 2\kappa_2 - 4,\, 2\kappa_2' - 4 \;\middle|\; \varepsilon, \varepsilon' \in \{\pm 1\}\right\}$

$$\geqslant \begin{cases} 4 & \textit{if } \mathsf{u} \textit{ and } \mathsf{u}' \textit{ are odd,} \\ 5 & \textit{if } \mathsf{u} + \mathsf{u}' \textit{ is odd,} \\ 6 & \textit{if } \mathsf{u} \textit{ and } \mathsf{u}' \textit{ are even.} \end{cases}$$

(3) *In <u>Case 1</u>,* $\kappa_2 \geqslant 7$, $\kappa_2' \geqslant 7$, *and* $\min\{\kappa_1 + \kappa_2 - \kappa_1' - \kappa_2',\, \kappa_1 - \kappa_2 - \kappa_1' + \kappa_2'\} \geqslant 5$.

(4) *In <u>Case 2</u>,* $\kappa_2 \geqslant 7$, $\kappa_2' \geqslant 7$, *and* $\kappa_1' + \kappa_2' - \kappa_1 - \kappa_2 \geqslant 5$.

(5) *In <u>Case 3</u>,* $\kappa_2 \geqslant 7$.

2. Sketch of proof

Let Ψ and Φ be the functorial lifts of Σ and Π to $\mathrm{GL}_4(\mathbb{A}_\mathbb{Q})$ with respect to the standard representation $\mathrm{GSp}_4(\mathbb{C}) \to \mathrm{GL}_4(\mathbb{C})$. In particular,

$$L(s, \Sigma \times \Pi) = L(s, \Psi \times \Phi),$$

where the right-hand side is the Rankin–Selberg L-function for $\mathrm{GL}_4 \times \mathrm{GL}_4$. Since $\kappa_1 > \kappa_2 \geqslant 4$ and $\kappa_1' > \kappa_2' \geqslant 4$, Σ and Π are non-CAP (cf. [ACI+23, Proposition A.3-(2)]). This implies that Ψ and Φ are regular algebraic in the sense of Clozel [Clo90] with essentially tempered archimedean components. Consider two auxiliary isobaric automorphic representations of $\mathrm{GL}_4(\mathbb{A}_\mathbb{Q})$

$$\Psi' := \Pi_1 \boxplus \Pi_2, \quad \Phi' := \Pi_3 \boxplus \Pi_4,$$

where $\Pi_1, \Pi_2, \Pi_3, \Pi_4$ are cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. Assume further that $\Psi_\infty \simeq \Psi'_\infty$ and $\Phi_\infty \simeq \Phi'_\infty$. By the cross-ratio formula [Che23, Theorem 4.3], under the regularity conditions (1) and (2), for any critical point $m \in \mathbb{Z}$ for $L(s, \Sigma \times \Pi)$ we have

$$\frac{L(m, \Psi \times \Phi) \cdot L(m, \Psi' \times \Phi')}{L(m, \Psi \times \Phi') \cdot L(m, \Psi' \times \Phi)} \sim 1,$$

provided the denominator is non-zero. The formula can be rewrite as

$$L(m, \Sigma \times \Pi) \sim \frac{L(m, \Sigma \times \Pi_3) \cdot L(m, \Sigma \times \Pi_4) \cdot L(m, \Pi \times \Pi_1) \cdot L(m, \Pi \times \Pi_2)}{\prod_{1 \leqslant i \leqslant 2,\, 3 \leqslant j \leqslant 4} L(m, \Pi_i \times \Pi_j)}. \tag{2.1}$$

For the denominator of the right-hand side of (2.1), we have the algebraicity result of Shimura [Shi76] for $\mathrm{GL}_2 \times \mathrm{GL}_2$. Therefore, we are reduced to the study of algebraicity of critical values for $\mathrm{GSp}_4 \times \mathrm{GL}_2$.

Let Σ^{gen} and Π^{gen} be the globally generic cuspidal automorphic representations which are nearly equivalent to Σ and Π respectively. By comparing the rational structures via the global Whittaker models and the interior coherent cohomology of the Siegel modular threefold, we can define the de Rham–Whittaker periods

$$p_1(\Sigma^{\mathrm{gen}}),\ p_2(\Sigma^{\mathrm{gen}}),\ p_1(\Pi^{\mathrm{gen}}),\ p_2(\Pi^{\mathrm{gen}})$$

associated to Σ^{gen} and Π^{gen}. Here the subscripts refer to the cohomological degree. Based on the integral representation for $\mathrm{GSp}_4 \times \mathrm{GL}_2$ suggested by Novodvorsky in [Nov79, § 3] (see also [PSS84, Theorem 1.1]) and its cohomological interpretation, we prove that the algebraicity of the numerator of the right-hand side of (2.1) can be expressed in terms of the de Rham–Whittaker periods. Note that in the unbalanced case for $\mathrm{GSp}_4 \times \mathrm{GL}_2$, the algebraicity follows from our result [Che23, Theorem 5.7] applied to $\mathrm{GL}_4 \times \mathrm{GL}_2$. Finally, by

- comparing with the algebraicity results of Morimoto [Mor14], [Mor18] for $\mathrm{GSp}_4 \times \mathrm{GL}_2$;
- the algebraicity of the Petersson norms of cusp forms in Σ^{gen} and Π^{gen} in terms of their adjoint L-values proved in [CI23] and [Che22];

we obtain period relations between the de Rham–Whittaker periods and $\langle f_\Sigma, f_\Sigma \rangle$, $\langle f_\Pi, f_\Pi \rangle$, $L(1, \Sigma, \mathrm{Ad})$. The regularity conditions (3)-(5) are imposed so that various assumptions in the above mentioned algebraicity results have non-empty intersection.

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