TOWARDS THE p-ADIC DERIVED HECKE ALGEBRA FOR WEIGHT ONE FORMS

ROBIN ZHANG

ABSTRACT. This note outlines an approach to defining p-adic Shimura classes and p-adic derived Hecke operators on the completed cohomology of modular curves, with more details to follow in a forthcoming paper by the author. After reviewing the modulo-p constructions of Harris and Venkatesh, we formulate a conjecture relating the action of p-adic derived Hecke operators on cusp forms of weight 1 and level $\Gamma_1(N)$ to the p-adic logarithm of the Stark unit for the corresponding adjoint Deligne–Serre representation. This new p-adic conjecture can be viewed as complementary to the Harris–Venkatesh conjecture.

CONTENTS

1. INTRODUCTION

Let $f = \sum_n a_n q^n$ be a newform of weight 1 and level $\Gamma_1(N)$, and denote its field of coefficients by $\mathbb{Q}(f) \subset \mathbb{C}$ with ring of integers $\mathbb{Z}[f]$. There is a finite Galois extension $E/\mathbb{Q}$ and an associated 3-dimensional complex representation $\mathrm{Ad}(\rho_f)$ of $\mathrm{Gal}(E/\mathbb{Q})$ obtained by taking the adjoint action of the Deligne–Serre representation ρ_f of f on trace-free 2×2 complex matrices. The trace-free adjoint representation has an associated dual space of units $\mathcal{U}(\mathrm{Ad}(\rho_f)) := \mathrm{Hom}_{\mathrm{Gal}(E/\mathbb{Q})}(\mathrm{Ad}(\rho_f), \mathcal{O}_E^\times \otimes \mathbb{Z}[f])$ with a $(\mathbb{Z}/p\mathbb{Z})^\times$-regulator map

$$\mathrm{Reg}_{(\mathbb{Z}/p\mathbb{Z})^\times} : \mathcal{U}\big(\mathrm{Ad}(\rho_f)\big) \longrightarrow (\mathbb{Z}/p\mathbb{Z})^\times \otimes \mathbb{Z}\left[f, \frac{1}{6N}\right]$$

for each prime $p \nmid 6N$. Let Cusp denote the cuspidal divisor on the modular curve $X_1(N)$. Using constructions (especially the Shimura covering/subgroup/class) with origins in the works of Shimura [Shi63, Shi67], Mazur [Maz77], and Merel [Mer96] among others, the study of derived Hecke operators on the coherent cohomology of modular curves

$$T_{(\mathbb{Z}/p\mathbb{Z})^\times,N} : H^0\big(X_1(N), \omega(-\mathrm{Cusp})\big)\otimes(\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow H^1\big(X_1(N), \omega(-\mathrm{Cusp})\big)\otimes(\mathbb{Z}/p\mathbb{Z})^\times$$

was initiated by Harris and Venkatesh [HV19], who posed a conjecture relating the action of $T_{(\mathbb{Z}/p\mathbb{Z})^\times,N}$ to the action of a group of units. For the newform $f^* = \sum_n \overline{a_n} q^n$ obtained from f by complex conjugation of Fourier coefficients, the Serre-duality pairing $\langle f^*, T_{(\mathbb{Z}/p\mathbb{Z})^\times,N}(f)\rangle_{\mathrm{SD}}$ (i.e. "pseudo-eigenvalue" of $T_{(\mathbb{Z}/p\mathbb{Z})^\times,N}$) was packaged as the Harris–Venkatesh "norm" $||f||^2_{(\mathbb{Z}/p\mathbb{Z})^\times} \in (\mathbb{Z}/p\mathbb{Z})^\times \otimes \mathbb{Z}[f]$ in [Zha23b], which drew an analogy with the Petersson norm $||f||^2_{\mathbb{R}} = \int_{X_0(N)} |f|^2 \frac{dx\,dy}{y} \in \mathbb{R}$ and the conjectures of Stark [Sta71, Sta75, Sta76, Sta80].

Conjecture 1 (The Harris–Venkatesh conjecture). Let f be a newform of weight 1 and level $\Gamma_1(N)$. There exists a $u \in \mathcal{U}(\mathrm{Ad}(\rho_f)) \otimes \mathbb{Q}$ and a prime p_0 such that for all primes $p \geq p_0$,

$$||f||^2_{(\mathbb{Z}/p\mathbb{Z})^\times} = \mathrm{Reg}_{(\mathbb{Z}/p\mathbb{Z})^\times}(u)$$

or equivalently,

$$\big\langle f^*, T_{(\mathbb{Z}/p\mathbb{Z})^\times,N}(f)\big\rangle_{\mathrm{SD}} = \mathrm{Reg}_{(\mathbb{Z}/p\mathbb{Z})^\times}(u).$$

The Harris–Venkatesh conjecture has been proved for imaginary dihedral forms in [DHRV22, Lec23, Zha23a], and for certain real dihedral forms in [DHRV22]. Numerical evidence for some exotic f has been given in [Mar21]. The Harris–Venkatesh conjecture has been refined in [Zha23b] to be compatible with the Stark conjecture (i.e. u can be taken to be the Stark unit u_{Stark}) and in [Zha23c] with explicit local data (i.e. the minimal p_0).

This note outlines key constructions and a conjecture from upcoming work of the author on p-adic derived Hecke operators. Section 2 briefly reviews the definition of the Stark unit group and the regulator maps defined on it. Section 3 reviews the modulo-p derived Hecke operator and the Harris–Venkatesh conjecture. Section 4 gives the construction of an element $\widehat{f^*}$ of the completed cohomology group

$$\widehat{H}^0\big(X_0(p^\infty), \omega(-\mathrm{Cusp})\big) = \varinjlim_n H^0\big(X_0(p^n), \omega(-\mathrm{Cusp})\big),$$

which gives rise to an element $f \cdot \widehat{f^*} \in \widehat{H}^0(X_0(p^\infty), \Omega)$ that will be paired with the p-adic Shimura class and serve as an analogue of the weight-2 cusp form $\mathrm{Tr}^{\Gamma_0(p)\cap\Gamma_1(N)}_{\Gamma_0(p)}(f(z)f^*(pz))$ in [DHRV22, Section 1.2]. Section 5 gives the construction of p-adic Shimura classes

$$\mathfrak{S}_{\mathbb{Z}_p^\times} \in \mathrm{Hom}_{\mathbb{Z}_p^\times}(\widehat{H}^0(X_0(p^\infty), \Omega), \mathbb{Z}_p^\times)$$
$$\mathfrak{S}_{\mathbb{Z}_p} \in \mathrm{Hom}_{\mathbb{Z}_p}(\widehat{H}^0(X_0(p^\infty), \Omega), \mathbb{Z}_p)$$

via projective limits of flat cohomology classes obtained from modular curve coverings

$$\pi_n : X_1(p^n) \longrightarrow X_0(p^n).$$

Since $\mathfrak{S}_{\mathbb{Z}_p^\times} = \mathfrak{S}_{(\mathbb{Z}/p\mathbb{Z})^\times} \times \mathfrak{S}_{\mathbb{Z}_p}$, we choose to focus on the $\mathbb{Z}_p$-component that is complementary to the Harris–Venkatesh setting. Serre duality is applied to $f \cdot \widehat{f}^*$ and the Shimura class $\mathfrak{S}_{\mathbb{Z}_p}$ to define the $\mathbb{Z}_p$ derived Hecke operator

$$T_{\mathbb{Z}_p,N} : H^0\big(X_1(N), \omega(-\mathrm{Cusp})\big)^{\mathrm{ord}} \longrightarrow H^1\big(X_1(N), \omega(-\mathrm{Cusp})\big)^{\mathrm{ord}},$$

assuming ordinariness when $p = 2$ or 3, i.e. $\mathrm{ord}_v(a_p) = 0$ for a good prime v over p; this gives a a p-adic "norm" $||f||^2_{\mathbb{Z}_p} \in \mathbb{Z}_p \otimes \mathcal{O}_v$ where $\mathcal{O}_v$ is the completion of $\mathbb{Z}[f]$ at a finite prime v over p. Inspired by the Harris–Venkatesh conjecture, we propose the following p-adic conjecture in terms of the p-adic regulator $\mathrm{Reg}_{\mathbb{Z}_p}$, which is the p-adic logarithm of a unit after evaluation at a distinguished element $w_v \in \mathrm{Ad}(\rho_f)$ (cf. the relationship between $\mathrm{Reg}_{\mathbb{R}}$ and the classical logarithm in [Tat84, Zha23b]).

Conjecture 2 (p-adic conjecture). Let f be a newform of weight 1 and level $\Gamma_1(N)$. There exists an element $u \in \mathcal{U}(\mathrm{Ad}(\rho_f)) \otimes \mathbb{Q}$ and a prime p_0 such that for all primes $p \geq p_0$,

$$||f||^2_{\mathbb{Z}_p} = \mathrm{Reg}_{\mathbb{Z}_p}(u),$$

or equivalently,

$$\big\langle f^*, T_{\mathbb{Z}_p,N}(f)\big\rangle = \mathrm{Reg}_{\mathbb{Z}_p}(u).$$

In the sense that the original conjecture of Harris–Venkatesh [HV19] is over $(\mathbb{Z}/p\mathbb{Z})^\times$ and that there are isomorphisms $\mathbb{Z}_p^\times \cong \mu_{p-1} \times (1+p\mathbb{Z}_p) \cong (\mathbb{Z}/p\mathbb{Z})^\times \times \mathbb{Z}_p$, Conjecture 2 is a complement to the Harris–Venkatesh conjecture.

Remark 3. There is a unique element $u_f \in \mathcal{U}(\mathrm{Ad}(\rho_f)) \otimes \mathbb{Q}$ associated to f by [Zha23b, Conjecture 1]. This u_f is a multiple of the Stark unit u_{Stark} from the Stark conjecture for $\mathrm{Ad}(\rho_f)$ by a Rankin–Selberg constant $c_{f,\mathrm{RS}}$ (cf. [Zha23b, Theorem 3], [Zha23c]). It would be nice to specify how the u in the p-adic conjecture 2 is related to u_f, as was done for the mod-p setting in [Zha23b]. The author would like to speculate that perhaps in this way, one could draw a precise local–global picture for Stark units via the derived Hecke algebra. Furthermore, the author hopes to relate the new p-adic conjecture to special values of p-adic L-functions; this would be an analogue to the Gross–Stark conjecture with units instead of p-units, perhaps answering the speculations of Rivero [Riv23, Sections 1.3 and 6].

Acknowledgements. The author is grateful to Kenichi Namikawa and Keiichi Gunji for the opportunity to present this work at the "Arithmetic aspects of automorphic forms and automorphic representations" conference at the Research Institute for Mathematical Sciences, Kyoto University (RIMS) and for the inclusion of this report in its proceedings. The author thanks Michael Harris, Loïc Merel, Gyujin Oh, and Akshay Venkatesh for helpful discussions on this topic.

This material is based on work supported by the National Science Foundation under Grant No. DMS-2303280.

2. The Stark unit group

The "Stark unit group" defined in this section follows the treatment and notation of [Zha23b], which itself is adapted from [Tat84, HV19, DHRV22]. The elements of this group appear in the Stark conjecture, Conjecture 1, and Conjecture 2.

For a modular form f of weight 1 and level $\Gamma_1(N)$, there is a finite Galois extension $E/\mathbb{Q}$ and an associated Artin representation $\rho_f : \mathrm{Gal}(E/\mathbb{Q}) \longrightarrow \mathrm{GL}(V)$ realized on a free module V of rank 2 over $\mathbb{Z}[f]$ by Deligne–Serre [DS74]. From ρ_f, one obtains a 3-dimensional complex representation $\mathrm{Ad}(\rho_f)$ from the action of $\mathrm{Gal}(E/\mathbb{Q})$ by conjugation through ρ_f on trace-free 2×2 complex matrices. For $\sigma \in \mathrm{Gal}(E/\mathbb{Q})$, let $x_\sigma := 2\rho_f(\sigma) - \mathrm{Tr}(\rho_f(\sigma)) \cdot \mathrm{Id}_{2\times 2}$ denote the trace-zero 2×2 matrix obtained from $\rho_f(\sigma)$.

The (trace-free) adjoint representation $\mathrm{Ad}(\rho_f)$ has an integral model given by $M := \mathbb{Z}[f] \cdot \{x_\sigma \mid \sigma \in \mathrm{Gal}(E/\mathbb{Q})\}$. There is a (dual) space of units

$$\mathcal{U}\big(\mathrm{Ad}(\rho_f)\big) := \mathrm{Hom}_{\mathrm{Gal}(E/\mathbb{Q})}\big(M, \mathcal{O}_E^\times \otimes \mathbb{Z}[f]\big),$$

which is called the "Stark unit group" in [HV19] because the Stark conjectures [Sta71, Sta75, Sta76, Sta80] predict the existence of a Stark unit $u_{\mathrm{Stark}} \in \mathcal{U}(\mathrm{Ad}(\rho_f)) \otimes \mathbb{Q}$ that is unique up to multiplication by roots of unity and whose Stark regulator $\mathrm{Reg}_{\mathbb{R}}(u_{\mathrm{Stark}})$ gives the leading term of the Artin L-function $L(\mathrm{Ad}(\rho_f), s)$ at $s = 0$. As detailed in [HV19, Lemma 2.1] and [Hor23, Corollary 2.6], the space $\mathcal{U}(\mathrm{Ad}(\rho_f))$ is a rank-1 $\mathbb{Z}[f, \frac{1}{6N}]$-module that does not depend on the choice of E.

For each place w of E, there is

- an embedding $\iota_w : E \hookrightarrow E_w$ where E_w is $\mathbb{R}$, $\mathbb{C}$, or a p-adic field;
- a Frobenius element Frob_w given by the p-power map or complex conjugation; and
- a distinguished element $x_{\mathrm{Frob}_w} \in M$ that is invariant under the action of the decomposition group $\langle \mathrm{Frob}_w \rangle \subset \mathrm{Gal}(E/\mathbb{Q})$.

The Stark regulator map $\mathrm{Reg}_{\mathbb{R}}$ can be defined by

(1) picking an archimedean place w of E,
(2) taking the composition of the evaluation-at-x_{Frob_w} map, ι_w, and the logarithm map:

$$\mathrm{Reg}_{\mathbb{R}} : \mathcal{U}\big(\mathrm{Ad}(\rho_f)\big) \xrightarrow[]{\substack{\text{evaluation} \\ \text{at } x_{\mathrm{Frob}_w}}} \big(\mathcal{O}_E^\times\big)^{\mathrm{Frob}_w} \otimes \mathbb{Z}[f] \xrightarrow{\log \circ \iota_w} \mathbb{R} \otimes \mathbb{Z}[f].$$

For each prime p, there is a $(\mathbb{Z}/p\mathbb{Z})^\times$ regulator map (called "reduction of a Stark unit" in [HV19, DHRV22]) defined by

(1) picking a non-archimedean place w of E above p,
(2) taking the composition of the evaluation-at-x_{Frob_w} map, ι_w, and reduction modulo the ideal of w:

$$\mathrm{Reg}_{(\mathbb{Z}/p\mathbb{Z})^\times} : \mathcal{U}\big(\mathrm{Ad}(\rho_f)\big) \xrightarrow[]{\substack{\text{evaluation} \\ \text{at } x_{\mathrm{Frob}_w}}} \big(\mathcal{O}_E^\times\big)^{\mathrm{Frob}_w} \otimes \mathbb{Z}[f] \xrightarrow{\iota_w} \mathcal{O}_{E_w}^\times \otimes \mathbb{Z}[f] \xrightarrow[]{\substack{\text{reduction} \\ \text{mod } w}} (\mathbb{Z}/p\mathbb{Z})^\times \otimes \mathbb{Z}[f].$$

Harris and Venkatesh [HV19, Section 2.8] observed that $\mathrm{Reg}_{(\mathbb{Z}/p\mathbb{Z})^\times}$ is independent of the choice of w over p: the vector $x_{\mathrm{Frob}_w} \in M$ is invariant under the decomposition group $\langle \mathrm{Frob}_w \rangle$ and the maps used in defining $\mathrm{Reg}_{(\mathbb{Z}/p\mathbb{Z})^\times}$ are $\mathrm{Gal}(E/\mathbb{Q})$-equivariant, so changing w corresponds to a Galois conjugation which leaves the regulator map invariant.

By omitting the "reduction modulo w" step in the construction of $\mathrm{Reg}_{\mathbb{Z}/p\mathbb{Z}^\times}$, we can define two p-adic regulators (with an additional application of the p-adic logarithm function): $\mathrm{Reg}_{\mathbb{Z}_p}$ and $\mathrm{Reg}_{\mathbb{Z}_p^\times}$.

3. The modulo-p Shimura class and derived Hecke operator

This section is a condensed overview of the remaining ingredients (Shimura class, norm, and derived Hecke operator) necessary to state Conjecture 1 following the presentation in [Zha23a, Zha23b].

3.1. The modulo-p Shimura class. Let p be a prime not dividing $6N$. The Shimura covering is a finite étale Galois covering of modular curves (more precisely, the maximal étale intermediate extension) with deck group $(\mathbb{Z}/p\mathbb{Z})^\times$:

$$\begin{array}{c} X_1(p) \\ \Big\downarrow{\scriptstyle (\mathbb{Z}/p\mathbb{Z})^\times} \\ X_0(p) \end{array}$$

which induces an element $\mathfrak{S}_{(\mathbb{Z}/p\mathbb{Z})^\times,\text{ét}} \in H^1_{\text{ét}}(X_0(p), (\mathbb{Z}/p\mathbb{Z})^\times)$. After taking

(a) the base change $X_0(p)_{\mathbb{Z}/(p-1)\mathbb{Z}} := X_0(p) \otimes \operatorname{Spec}(\mathbb{Z}/(p-1)\mathbb{Z})$,
(b) the pushforward of the étale sheaf $\mathbb{Z}/(p-1)\mathbb{Z} \longrightarrow \mathcal{O}_{X_0(p)_{\mathbb{Z}/(p-1)\mathbb{Z}}}$,
(c) the comparison of Zariski and étale cohomology for quasi-coherent sheaves, and
(d) the pairing given by Serre duality,

$$\begin{array}{c} \big(H^1(X_0(p)_{\mathbb{Z}/(p-1)\mathbb{Z}}, \mathcal{O}) \otimes (\mathbb{Z}/p\mathbb{Z})^\times\big) \otimes \big(H^0(X_0(p)_{\mathbb{Z}/(p-1)\mathbb{Z}}, \Omega^1) \otimes (\mathbb{Z}/p\mathbb{Z})^\times\big) \\ \Big\downarrow{\scriptstyle \langle\cdot,\cdot\rangle_{\mathrm{SD}}} \\ (\mathbb{Z}/p\mathbb{Z})^\times, \end{array}$$

we have a composition of the respective maps:

$$\begin{array}{ccc} \mathfrak{S}_{(\mathbb{Z}/p\mathbb{Z})^\times,\text{ét}} & \in & H^1_{\text{ét}}(X_0(p), (\mathbb{Z}/p\mathbb{Z})^\times) \\ \Big\downarrow & & \Big\downarrow{\scriptstyle (a)} \\ & & H^1_{\text{ét}}(X_0(p), \mathbb{Z}/(p-1)\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/p\mathbb{Z})^\times \\ & & \Big\downarrow{\scriptstyle (b)} \\ & & H^1_{\text{ét}}\big(X_0(p)_{\mathbb{Z}/(p-1)\mathbb{Z}}, \mathbb{G}_a\big) \otimes (\mathbb{Z}/p\mathbb{Z})^\times \\ & & \Big\downarrow{\scriptstyle (c)} \\ \mathfrak{S}_{(\mathbb{Z}/p\mathbb{Z})^\times} & \in & H^1_{\mathrm{Zar}}\big(X_0(p)_{\mathbb{Z}/(p-1)\mathbb{Z}}, \mathcal{O}\big) \otimes (\mathbb{Z}/p\mathbb{Z})^\times \\ \Big\downarrow & & \Big\downarrow{\scriptstyle (d)} \\ \big\langle \cdot, \mathfrak{S}_{(\mathbb{Z}/p\mathbb{Z})^\times} \big\rangle_{\mathrm{SD}} & \in & \mathrm{Hom}\big(H^0(X_0(p), \Omega^1), (\mathbb{Z}/p\mathbb{Z})^\times\big). \end{array}$$

Hence, the image of $\mathfrak{S}_{(\mathbb{Z}/p\mathbb{Z})^\times,\text{ét}}$ under the first four maps furnishes an element $\mathfrak{S}_{(\mathbb{Z}/p\mathbb{Z})^\times} \in H^1\big(X_0(p)_{\mathbb{Z}/(p-1)\mathbb{Z}}, \mathcal{O}\big) \otimes (\mathbb{Z}/p\mathbb{Z})^\times$. Since there is an isomorphism $S_2(\Gamma_0(p)) \cong H^0\big(X_0(p), \Omega^1\big)$, (see [Shi94, Corollary 2.17]), we can view $\mathfrak{S}_{(\mathbb{Z}/p\mathbb{Z})^\times}$ as acting on weight-2 cusp forms via $\langle \cdot, \mathfrak{S}_{(\mathbb{Z}/p\mathbb{Z})^\times}\rangle_{\text{SD}} \in \text{Hom}(S_2(\Gamma_0(p)), (\mathbb{Z}/p\mathbb{Z})^\times)$.

3.2. **The modulo-p norm.** For a cusp form $f = \sum_n a_n q^n$ of weight 1 and level $\Gamma_1(N)$ with coefficients in $\mathbb{Q}(f) \subset \mathbb{C}$, applying complex conjugation to the Fourier coefficients gives the dual cusp form $f^* = \sum_n \overline{a_n} q^n$ of weight 1 and level $\Gamma_1(N)$. Define $\Gamma_{0,1}(p, N) := \Gamma_0(p) \cap \Gamma_1(N)$ and denote its modular curve by $X_{0,1}(p, N) = X_{\Gamma_{0,1}(p,N)}$.

For a prime p not dividing $6N$, the product $f(z)f^*(pz)$ is a cusp form of weight 2 and level $\Gamma_0(p) \cap \Gamma_1(N)$ whose trace $\text{Tr}^{\Gamma_{0,1}(p,N)}_{\Gamma_0(p)}(f(z)f^*(pz))$ is a cusp form of weight 2, level $\Gamma_0(p)$, trivial nebentypus, and with coefficients in $\mathbb{Z}[f, \frac{1}{6N}]$. [Zha23b] defines a Harris–Venkatesh "norm"

$$||f||^2_{(\mathbb{Z}/p\mathbb{Z})^\times} := \mathfrak{S}_p\left(\text{Tr}^{Np}_p(f(z)f^*(pz))\right) \in (\mathbb{Z}/p\mathbb{Z})^\times \otimes \mathbb{Z}\left[f, \frac{1}{6N}\right].$$

This leads to the norm formulation of Conjecture 1.

3.3. **The modulo-p derived Hecke operator.** Harris and Venkatesh [HV19] define a derived Hecke operator $T_{(\mathbb{Z}/p\mathbb{Z})^\times,N}$ on the space of cusp forms of weight 1 and level N coprime to p by adding $\cup\mathfrak{S}_p$ to the definition of the usual Hecke operator via the pullback and pushforward of two projection maps $\pi, \pi' : X_{\Gamma_1(N)\cap\Gamma_0(p)} \longrightarrow X_{\Gamma_1(N)}$:

$$\begin{array}{ccc}
H^0\big(X_1(N)_{\mathbb{Z}/(p-1)\mathbb{Z}}, \omega(-\text{Cusp})\big) \otimes (\mathbb{Z}/p\mathbb{Z})^\times & \overset{T_{(\mathbb{Z}/p\mathbb{Z})^\times,N}}{\dashrightarrow} & H^1\big(X_1(N)_{\mathbb{Z}/(p-1)\mathbb{Z}}, \omega(-\text{Cusp})\big) \otimes (\mathbb{Z}/p\mathbb{Z})^\times \\
\big\downarrow \pi^* & & \big\uparrow \pi'_* \\
H^0\big(X_{0,1}(p,N)_{\mathbb{Z}/(p-1)\mathbb{Z}}, \omega(-\text{Cusp})\big) \otimes (\mathbb{Z}/p\mathbb{Z})^\times & \xrightarrow{\cup\mathfrak{S}_{(\mathbb{Z}/p\mathbb{Z})^\times}} & H^1\big(X_{0,1}(p,N)_{\mathbb{Z}/(p-1)\mathbb{Z}}, \omega(-\text{Cusp})\big) \otimes (\mathbb{Z}/p\mathbb{Z})^\times
\end{array}$$

The Harris–Venkatesh norm concretely encapsulates the action of the derived Hecke operator:

$$||f||^2_{(\mathbb{Z}/p\mathbb{Z})^\times} = \big\langle f^*, T_{(\mathbb{Z}/p\mathbb{Z})^\times,N}(f)\big\rangle_{\text{SD}}.$$

This leads to the derived Hecke operator formulation of Conjecture 1.

4. The p-adic system of weight one forms

In the modulo-p setting, we applied the Shimura class $\mathfrak{S}_{(\mathbb{Z}/p\mathbb{Z})^\times}$ to the weight-2 cusp form $f(z)f^*(pz)$. In this section, we construct the p-adic analogue by looking at spaces generated by $f^*(p^n z)$. Note that this is the only section of this note in which p is allowed to be 2 or 3.

4.1. **Non-vanishing and uniqueness.** Let $f = \sum_n a_n q^n$ be a newform for $\Gamma_1(N)$ of weight 1 with central character ω and dual form $f^* = \sum_n \overline{a_n} q^n$. Let p be a prime that does not divide N. For the field $\mathbb{Q}(f) = \mathbb{Q}(\chi_{\rho_f})$ of coefficients of f with ring of integers $\mathbb{Z}[f]$, let ν be a place of $\mathbb{Q}(f)$ over p and let $\mathcal{O}_\nu$ be the completion of $\mathbb{Z}[f]$ at ν. Consider the intersection $\Gamma_{0,1}(p^n, N) := \Gamma_0(p^n) \cap \Gamma_1(N)$ and its modular curve $X_{0,1}(p^n, N)$.

For each integer $n \geq 0$, define the modular form $f_n^*(z) := f^*(p^n z) = f^*(z)|_1 \begin{pmatrix} p^n & \\ & 1 \end{pmatrix}$. The subspace of cusp forms of weight 1 and level p^n generated by f^* over $\mathcal{O}_\nu$ is given by

$$V_{f^*,n} := \sum_{i=0}^{n} \mathcal{O}_\nu f_i^* \subset H^0(X_{0,1}(p^n, N)_{\mathcal{O}_\nu}, \omega(-\mathrm{Cusp})),$$

where Cusp is the cuspidal divisor. The spaces $\{V_{f^*,n}\}_{n\geq 0}$ form a projective system under trace maps; take the projective limit,

$$\widehat{V}_{f^*} := \varprojlim_n V_{f^*,n}.$$

The polynomial $X^2 - a_p X + \omega(p)p$ (which should not be mistaken for the Hecke polynomial for weight 1) has a unique root α in $\mathcal{O}_\nu^\times$ by Hensel's lemma; it appears in calculations of the trace map between levels of our projective system.

Theorem 4. *The space $\widehat{V}_{f^*}$ is non-zero only if ν is ordinary. In this case, it is one-dimensional and generated by an element $\widehat{f^*} \in \widehat{V}_{f^*}$ whose image in $V_{f^*,n}$ is given by*

$$\widehat{f^*}_n := \alpha^{1-n} f_n^* - \alpha^{-n}\omega(p) f_{n-1}^*.$$

Remark 5. In the proof of Theorem 4, we will only use the fact that f^* has eigenvalue a_p under T_p. Thus, Theorem 4 actually holds for any old form φ generated by f^* such that the level of φ is coprime to p. Also, the condition on ordinariness (i.e. $\mathrm{ord}_\nu(a_p) = 0$) is only relevant for $p = 2$ or 3 since a newform of weight 1 is ordinary at the other primes.

The key to proving Theorem 4 is the calculation of the trace map from level $n+1$ to level n.

Lemma 6. *Consider the trace map*

$$\mathrm{Tr}_{n+1} : V_{f^*,n+1} \longrightarrow V_{f^*,n}, \qquad n \geq 1.$$

(a) Then for all $i \leq n$,

$$\mathrm{Tr}_{n+1}(f_i^*) = p f_i^*;$$

(b) and

$$\mathrm{Tr}_{n+1}(f_{n+1}^*) = a_p f_n^* - \omega(p) f_{n-1}^*.$$

Proof. (a): This is due to the fact that $[\Gamma_{0,1}(p^n, N) : \Gamma_{0,1}(p^{n+1}, N)] = p$.

(b): Since $\Gamma_{0,1}(p^{n+1}, N)\backslash\Gamma_{0,1}(p^n, N)$ is represented by elements $\gamma_i := \begin{pmatrix} 1 & 0 \\ p^n i N & 1 \end{pmatrix}$ for $0 \leq i \leq p-1$,

$$\mathrm{Tr}_{n+1}(f_{n+1}^*) := \sum_{\gamma \in \Gamma_{0,1}(p^{n+1},N)\backslash\Gamma_{0,1}(p^n,N)} f_{n+1}^*|_1\gamma = \sum_i f^*|_1 \left(\begin{pmatrix} p^{n+1} & \\ & 1 \end{pmatrix} \gamma_i \right).$$

Notice that

$$\begin{pmatrix} p^{n+1} & \\ & 1 \end{pmatrix} \gamma_i = \begin{pmatrix} p^{n+1} & \\ p^n iN & 1 \end{pmatrix} = \begin{pmatrix} p & \\ iN & 1 \end{pmatrix} \begin{pmatrix} p^n & \\ & 1 \end{pmatrix},$$

and the Hecke operator T_p is presented by the left $\Gamma_1(N)$-cosets represented by $\begin{pmatrix} p & \\ iN & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & \\ & p \end{pmatrix}$. Then

$$\begin{aligned} \mathrm{Tr}_{n+1}(f^*_{n+1}) &= \sum_i f^*|_1 \left(\begin{pmatrix} p^{n+1} & \\ & 1 \end{pmatrix} \gamma_i \right) \\ &= \sum_i f^*|_1 \left(\begin{pmatrix} p & \\ iN & 1 \end{pmatrix} \begin{pmatrix} p^n & \\ & 1 \end{pmatrix} \right) \\ &= \left(T_p f^* - f^*|_1 \begin{pmatrix} 1 & \\ & p \end{pmatrix} \right) \Big|_1 \begin{pmatrix} p^n & \\ & 1 \end{pmatrix} \\ &= T_p f^*|_1 \begin{pmatrix} p^n & \\ & 1 \end{pmatrix} - f^*|_1 \begin{pmatrix} p^n & \\ & p \end{pmatrix} \\ &= a_p f^*_n - \omega(p) f^*_{n-1}. \end{aligned}$$

□

Proof of Theorem 4. For any $m \geq n$, let $\mathrm{Tr}_{m,n} : V_{f^*,m} \longrightarrow V_{f^*,n}$ denote the $(m-n)$-fold composition of trace maps

$$\mathrm{Tr}_{m,n} : \quad V_{f^*,m} \xrightarrow{\mathrm{Tr}_m} V_{f^*,m-1} \xrightarrow{\mathrm{Tr}_{m-1}} \dots \xrightarrow{\mathrm{Tr}_{n+1}} V_{f^*,n}.$$

Let $\widehat{V}_{f^*,n}$ denote the intersection of the images $\mathrm{Im}(\mathrm{Tr}_{m,n})$ over all $m \geq n$. Then the projective limit $\widehat{V}_{f^*} \neq 0$ if and only if $\widehat{V}_{f^*,n} \neq 0$ for some n.

Let π be a uniformizer of $\mathcal{O}_v$. Consider the case that $\mathrm{ord}_v(a_p) > 1$. Then by Lemma 6, $\mathrm{Im}(\mathrm{Tr}_{n+2,n}) \subset \pi V_{f^*,n}$ for all $n \geq 1$. This shows that $\mathrm{Im}(\mathrm{Tr}_{n+2k,n}) \subset \pi^k V_{f^*,n}$, so $\widehat{V}_{f^*,n} := \bigcap_{m \geq n} \mathrm{Im}(\mathrm{Tr}_{m,n}) = 0$ and $\widehat{V}_{f^*} = 0$ too.

Now consider the case $\mathrm{ord}_v(a_p) = 0$. Take the basis for $V_{f^*,n+1}$ given by,

$$\begin{aligned} (\widehat{f^*})_0 &:= f^*_0, \\ (\widehat{f^*})_i &:= \alpha^{1-i} f^*_i - \alpha^{-i}\omega(p) f^*_{i-1} \qquad \text{for } 0 < i \leq n+1. \end{aligned}$$

By Lemma 6, $\mathrm{Tr}_{n+1}(f^*_i) = p f^*_i$ for $i < n+1$ and $\mathrm{Tr}_{n+1}(f^*_{n+1}) = a_p f^*_n - \omega(p) f^*_{n-1}$, so

$$\begin{aligned} \mathrm{Tr}_{n+1}\left((\widehat{f^*})_{n+1}\right) &= \alpha^{1-n-1}\mathrm{Tr}_{n+1}(f^*_{n+1}) - \alpha^{-n-1}\omega(p)\mathrm{Tr}_{n+1}(f^*_n) \\ &= \alpha^{-n}\big(a_p {f^*}_n - \omega(p) f^*_{n-1}\big) - \alpha^{-n-1}\omega(p) p {f^*}_n \\ &= \alpha^{-n-1}(\alpha a_p - \omega(p)p) f^*_n - \alpha^{-n}\omega(p) {f^*}_{n-1} \\ &= \alpha^{-n-1}\alpha^2 f^*_n - \alpha^{-n}\omega(p) f^*_{n-1} \\ &= (\widehat{f^*})_n. \end{aligned}$$

Then the sequence $\{(\widehat{f^*})_n\}_{n\geq 0}$ defines an element $\widehat{f^*} \in \widehat{V}_{f^*}$ and each $\widehat{V}_{f^*,n}$ is equal to $\mathcal{O}_v \cdot (\widehat{f^*})_n$. □

4.2. **An extended remark on ordinary primes.** The ordinariness of a_p in Theorem 4 can be characterized as follows. Recall that f corresponds to a two-dimensional complex representation ρ_f by Deligne–Serre with $a_p = \mathrm{Tr}(\rho_f(\mathrm{Frob}_p))$. In fact, the p-th Fourier coefficient a_p of f is the sum of two roots of unity whose order divides $\#\mathrm{Im}(\rho_f)$:

$$a_p = \zeta_1 + \zeta_2 = \zeta_1 \cdot (1 - \xi),$$

where $\xi = -\frac{\zeta_2}{\zeta_1}$. It is known that nonzero a_p is not invertible for some p-adic norm on $\mathbb{Q}(f)$ if and only if the order of ξ is a power of p. This shows that p is ordinary if $a_p \neq 0$ and $p \nmid 2\#\mathrm{Im}(\rho_f)$.

When ρ_f has dihedral image, more can be said about when a_p is non-zero based on the splitting of p in K. In the dihedral case, $\rho_f \cong \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}}(\chi)$ for a finite character χ of $\mathrm{Gal}(K^{ab}/K)$ and a quadratic number field $K/\mathbb{Q}$. Here, $L(f,s) = L(\chi,s)$ and

$$f = \sum_{\mathfrak{a}\in S_\chi} \chi(\mathfrak{a}) q^{N(\mathfrak{a})},$$

where $\mathfrak{a}$ runs through the set S_χ of ideals of $\mathcal{O}_K$ coprime to the conductor of χ. Therefore,

$$a_n = \sum_{\mathfrak{a}:N(\mathfrak{a})=n} \chi(\mathfrak{a}),$$

so $a_p = 0$ when p is inert in K and $a_p = \chi(\mathfrak{p}) + \chi(\bar{\mathfrak{p}})$ when $p\mathcal{O}_K = \mathfrak{p} + \bar{\mathfrak{p}}$.

5. The p-adic Shimura class and derived Hecke operator

In this section, we introduce the p-adic Shimura classes which play a central role in studying the action of p-adic Hecke operators on modular forms of weight 1. We then define the p-adic norm and p-adic derived Hecke operators given a p-adic Shimura class. This completes the formulation of Conjecture 2. $\mathbb{Z}_p$ is used throughout this section for notational convenience, but one may also take $\mathbb{Z}_p \otimes \mathbb{Z}[f]$ (analogously to $(\mathbb{Z}/p\mathbb{Z})^\times \otimes \mathbb{Z}[f]$).

5.1. **The $\mathbb{Z}_p^\times$ Shimura class.** Let p be a prime. Consider the tower of coverings of modular curves for $n \in \mathbb{N}$,

$$\pi_n : X_1(p^n) \longrightarrow X_0(p^n).$$

These covers are not étale when $n \geq 2$ even after inverting $6N$, but are finite flat (cf. [Maz77, Proposition 11.6] for the $n = 1$ case) with structure group denoted G_n and define flat cohomology classes,

$$\mathfrak{S}_{(\mathbb{Z}/p^n\mathbb{Z})^\times, \mathrm{fl}} \in H^1_{\mathrm{fl}}(X_0(p^n), G_n).$$

We sketch a 1-cocycle construction of the Shimura class $\mathfrak{S}_{(\mathbb{Z}/p^n\mathbb{Z})^\times, \mathrm{fl}}$ if G_n behaves like $(\mathbb{Z}/p^n\mathbb{Z})^\times$ or some quotient $(\mathbb{Z}/p^n\mathbb{Z})^\times_a$ thereof. In the étale case, an unramified cover of connected $X = X_1(p)$ over $Y = X_0(p)$ with Galois group $(\mathbb{Z}/p\mathbb{Z})^\times$ gives a morphism from the pointed fundamental group $\pi_0(Y, y_0)$ to $(\mathbb{Z}/p\mathbb{Z})^\times$, which gives an element of $H^1_{\text{ét}}(Y, (\mathbb{Z}/p\mathbb{Z})^\times) \cong \mathrm{Hom}(\pi_1^{\text{ét}}(Y, y_0), (\mathbb{Z}/p\mathbb{Z})^\times)$ since the Galois group is abelian. Now for $X = X_1(p^n)$ over $Y = X_0(p^n)$ with structure group $(\mathbb{Z}/p^n\mathbb{Z})^\times$ and $n \geq 2$, the cover is not étale but is finite flat. The fiber product $X \times_Y X$ is a union of subvarieties isomorphic to X with a group action by some quotient of the original group $(\mathbb{Z}/p^n\mathbb{Z})^\times$. Fix one copy of X in $X \times_Y X$; the other copies of X come from actions of the quotient group, and $X \times_Y X$ is a group scheme over X. This gives a cochain from $X \times_Y X$ to the quotient group and an element of $H^1_{\text{fl}}(X_0(p^n), ((\mathbb{Z}/p^n\mathbb{Z})^\times))$. The compatibility of $\{\pi_n\}_{n\geq 0}$ gives an element $\mathfrak{S}_{\mathbb{Z}_p^\times, \text{fl}}$ in the (partially) complete cohomology

$$\widehat{H}^1_{\text{fl}}\big(X_0(p^\infty), \mathbb{Z}_p^\times\big) := \varinjlim_n \varprojlim_m H^1_{\text{fl}}\big(X_0(p^n), \big((\mathbb{Z}/p^m\mathbb{Z})^\times\big)_a\big).$$

One can then view the Shimura class as an element of Zariski cohomology, similar to what was done for the $n = 1$ étale case of Harris–Venkatesh [HV19, Section 3.1].

5.2. **The $\mathbb{Z}_p$ Shimura class.** We can obtain a $\mathbb{Z}_p$ Shimura class from a $\mathbb{Z}_p^\times$ Shimura class. Consider the natural decomposition

$$\Delta \times \mathbb{Z}_p \xrightarrow{\sim} \mathbb{Z}_p^\times,$$

where Δ is the torsion subgroup of $\mathbb{Z}_p^\times$. In particular, Δ is isomorphic to $\mu_{p-1} \cong (\mathbb{Z}/p\mathbb{Z})^\times$ when p is odd, and to μ_2 when $p = 2$. In the other component, the map $\mathbb{Z}_p \longrightarrow \mathbb{Z}_p^\times$ is given by $x \mapsto (1+p)^x$ when p is odd and $x \mapsto 5^x$ when $p = 2$. Then we have a product

$$\mathfrak{S}_{\mathbb{Z}_p^\times, \text{fl}} = \mathfrak{S}_{\Delta, \text{fl}} \times \mathfrak{S}_{\mathbb{Z}_p, \text{fl}}.$$

The element $\mathfrak{S}_{\Delta, \text{fl}} = \mathfrak{S}_{\Delta, \text{ét}} \in H^1_{\text{ét}}(X_0(p), (\mathbb{Z}/p\mathbb{Z})^\times)$ gives the mod-p Shimura class previously studied by [Maz77, Mer96, HV19, Mar21, DHRV22, Hor23, Lec23, Zha23a, Zha23b, Zha23c] among others. Here, we focus on the $\mathbb{Z}_p$ Shimura class $\mathfrak{S}_{\mathbb{Z}_p, \text{fl}}$ lying in the cohomology group

$$\widehat{H}^1_{\text{fl}}(X_0(p^\infty), \mathbb{Z}_p) := \varinjlim_n \varprojlim_m H^1_{\text{fl}}(X_0(p^n), \mathbb{Z}/p^m\mathbb{Z}).$$

This group has a natural map to the completed cohomology of coherent sheaves:

$$\varinjlim_n \varprojlim_m H^1\big(X_0(p^n)_{\mathbb{Z}/p^m\mathbb{Z}}, \mathcal{O}\big).$$

Using Serre duality,

$$H^1\big(X_0(p^n)_{\mathbb{Z}/p^m\mathbb{Z}}, \mathcal{O}\big) \cong \mathrm{Hom}\big(H^0(X_0(p^n), \Omega), \mathbb{Z}/p^m\mathbb{Z}\big).$$

Thus for $\widehat{H}^0(X_0(p^\infty), \Omega) := \varinjlim_n H^0(X_0(p^n), \Omega)$, the Shimura class $\mathfrak{S}_{\mathbb{Z}_p}$ can also be viewed as an element in its dual space

$$\mathrm{Hom}_{\mathbb{Z}_p}\Big(\widehat{H}^0(X_0(p^\infty), \Omega), \mathbb{Z}_p\Big),$$

where trace maps define the projective system.

One can perform all of these Shimura class constructions with additional level structure. It will be useful in the p-adic consideration of the Harris–Venkatesh conjecture in the following section to use these constructions for $\Gamma_{0,1}(p^n, N) := \Gamma_0(p^n) \cap \Gamma_1(N)$ and its modular curve $X_{0,1}(p^n, N)$ instead of $\Gamma_0(p^n)$ and $X_0(p^n)$.

5.3. **The p-adic norm.** We apply Theorem 4 to the dual form $f^* = \sum_n \overline{a}_n q^n$ of f. Assuming that $\mathrm{ord}_v(\overline{a}_p) = 0$, we have the element $\widehat{f^*} \in \widehat{H}^0(X_0(p^\infty), \omega(-\mathrm{Cusp}))$ and in particular

$$f \cdot \widehat{f^*} \in \widehat{H}^0\big(X_0(p^\infty), \Omega\big).$$

Recall that $\mathcal{O}_v$ is the completion of $\mathbb{Z}[f]$ at v; given a $\mathbb{Z}_p$-Shimura class $\mathfrak{S}_{\mathbb{Z}_p,\mathrm{fl}} \in \mathrm{Hom}_{\mathbb{Z}_p}(\widehat{H}^0(X_0(p^\infty), \Omega), \mathbb{Z}_p)$, we can define a p-adic period of f,

$$||f||^2_{\mathbb{Z}_p} := \mathfrak{S}_{\mathbb{Z}_p}\big(f \cdot \widehat{f^*}\big) \in \mathcal{O}_v.$$

Concretely, what we have done in Section 5 is the construction of the class $\mathfrak{S}_{\mathbb{Z}/p^n\mathbb{Z},\mathrm{fl}} \in H^1(X_{0,1}(p^{n+1}, N), \mathbb{Z}/p^n\mathbb{Z})$ using the étale cover $X_1(p^n) \longrightarrow X_0(p^n)$, and then the demonstration that this class induces an element $\mathfrak{S}_{\mathbb{Z}/p^n\mathbb{Z}} \in \mathrm{Hom}(H^0(X_{0,1}(p^{n+1}, \Omega), \mathbb{Z}/p^n\mathbb{Z})$ through which we obtain

$$||f||^2_{\mathbb{Z}_p} = \varprojlim_n \mathfrak{S}_{\mathbb{Z}/p^n\mathbb{Z}}\Big(\big(\alpha^{1-n} f^*_n(z) - \alpha^{-n}\omega(p) f^*_{n-1}(z)\big)\, f(z)dz\Big).$$

With these constructions in place, we are now ready to formulate Conjecture 2 in terms of the p-adic norm: we ask whether

$$||f||^2_{\mathbb{Z}_p} = \mathrm{Reg}_{\mathbb{Z}_p}(u)$$

for some element $u \in \mathcal{U}(\mathrm{Ad}(\rho_f)) \otimes \mathbb{Q}$ and all sufficiently large primes p. In other words, the $\mathbb{Z}_p$ Shimura class should act on weight 1 forms compatibly with the $(\mathbb{Z}/p\mathbb{Z})^\times$ Shimura class in the Harris–Venkatesh conjecture, and there is a combined equality for all sufficiently large primes p:

$$||f||^2_{\mathbb{Z}^*_p} = \mathrm{Reg}_{\mathbb{Z}^*_p}(u).$$

5.4. **The p-adic derived Hecke operator.** Define the subspace of p-adic ordinary elements in $H^i(X_1(N), \omega(-\mathrm{Cusp})) \otimes \mathbb{Z}_p$ to be

$$H^i(X_1(N), \omega(-\mathrm{Cusp}))^{\mathrm{ord}} := \bigcap_{n \geq 0} T_p^n\, H^i\big(X_1(N), \omega(-\mathrm{Cusp})\big) \otimes \mathbb{Z}_p.$$

In this space, we have the action of the invertible operator T_p and $\langle p \rangle$. There is a unique invertible operator A on $H^i(X_1(N), \omega(-\mathrm{Cusp}))^{\mathrm{ord}}$ satisfying the equation:

$$A^2 - T_p A + \langle p \rangle p = 0.$$

Such an A can be either defined by the binomial formula,

$$A = \frac{T_p}{2}\Big(1 + \sqrt{1 - 4\langle p \rangle p / T_p^2}\Big) = T_p - \sum_{n \geq 1} \frac{1 \cdot 3 \cdots (2n-3)}{2^{1-n} n!} \left(\frac{\langle p \rangle p}{T_p^2}\right)^n,$$

or by the limit $A = \lim_{n \to \infty} A_n$ with A_n the invertible operator defined by

$$A_0 = T_p,$$

$$A_n = T_p - \frac{\langle p \rangle p}{A_{n-1}}, \qquad n > 0.$$

Serre duality gives a perfect pairing:

$$\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{SD}} : H^0\big(X_1(N), \omega(-\mathrm{Cusp})\big)^{\mathrm{ord}} \otimes H^1\big(X_1(N), \omega(-\mathrm{Cusp})\big)^{\mathrm{ord}} \longrightarrow \mathbb{Z}_p.$$

Using the operator A and $f_n(z) := f(p^n z)$, we have a lifting

$$H^0\big(X_1(N), \omega(-\mathrm{Cusp})\big)^{\mathrm{ord}} \longrightarrow \varprojlim_n H^0\big(X_{0,1}(p^n, N), \omega(-\mathrm{Cusp})\big)$$

$$f \longmapsto \widehat{f} := \varprojlim_n \big(A^{1-n} f_n - A^{-n}\langle p\rangle p f_{n-1}\big).$$

Then we have a derived Hecke operator

$$T_{\mathbb{Z}_p,N} : H^0\big(X_1(N), \omega(-\mathrm{Cusp})\big)^{\mathrm{ord}} \longrightarrow H^1\big(X_1(N), \omega(-\mathrm{Cusp})\big)^{\mathrm{ord}}$$

defined by this lifting and $\cup\mathfrak{S}_{\mathbb{Z}_p}$ such that for any $f_1, f_2 \in H^0(X_1(N), \omega(-\mathrm{Cusp}))^{\mathrm{ord}}$,

$$\mathfrak{S}_{\mathbb{Z}_p}\big(f_1\widehat{f_2}\big) = \big\langle f_2, T_{\mathbb{Z}_p,N}(f_1)\big\rangle_{\mathrm{SD}}.$$

We need only consider new forms $f_1 = f, f_2 = f^*$; in this case, the prediction that

$$||f||^2_{\mathbb{Z}_p} = \mathrm{Reg}_{\mathbb{Z}_p}(u),$$

together with the pairing identity above, yields the derived Hecke operator formulation of Conjecture 2 in terms of $\big\langle f^*, T_{\mathbb{Z}_p,N}(f)\big\rangle_{\mathrm{SD}} = \mathrm{Reg}(u)$.

References

[DHRV22] Henri Darmon, Michael Harris, Victor Rotger, and Akshay Venkatesh. The derived Hecke algebra for dihedral weight one forms. *Michigan Math. J.*, 72:145–207, 2022.

[DS74] Pierre Deligne and Jean-Pierre Serre. Formes modulaires de poids 1. *Ann. Sci. École Norm. Sup. (4)*, 7:507–530 (1975), 1974.

[Hor23] Aleksander Horawa. Motivic action on coherent cohomology of Hilbert modular varieties. *Int. Math. Res. Not. IMRN*, (12):10439–10531, 2023.

[HV19] Michael Harris and Akshay Venkatesh. Derived Hecke algebra for weight one forms. *Exp. Math.*, 28(3):342–361, 2019.

[Lec23] Emmanuel Lecouturier. On triple product L-functions and a conjecture of Harris-Venkatesh. *Int. Math. Res. Not. IMRN*, (22):19476–19506, 2023.

[Mar21] David Marcil. Numerical verification of a conjecture of Harris and Venkatesh. *J. Number Theory*, 221:484–495, 2021.

[Maz77] Barry Mazur. Modular curves and the Eisenstein ideal. *Inst. Hautes Études Sci. Publ. Math.*, (47):33–186 (1978), 1977. With an appendix by Mazur and M. Rapoport.

[Mer96] Loïc Merel. L'accouplement de Weil entre le sous-groupe de Shimura et le sous-groupe cuspidal de $J_0(p)$. *J. Reine Angew. Math.*, 477:71–115, 1996.

[Riv23] Óscar Rivero. Derivatives of Beilinson-Flach classes, Gross-Stark formulas and a p-adic Harris-Venkatesh conjecture. *Doc. Math.*, 28(1):105–131, 2023.

[Shi63] Goro Shimura. On analytic families of polarized abelian varieties and automorphic functions. *Ann. of Math. (2)*, 78:149–192, 1963.

[Shi67] Goro Shimura. Construction of class fields and zeta functions of algebraic curves. *Ann. of Math. (2)*, 85:58–159, 1967.

[Shi94] Goro Shimura. *Introduction to the arithmetic theory of automorphic functions*, volume 11 of *Publications of the Mathematical Society of Japan*. Princeton University Press, Princeton, NJ, 1994. Reprint of the 1971 original, Kanô Memorial Lectures, 1.

[Sta71] Harold M. Stark. Values of L-functions at $s = 1$. I. L-functions for quadratic forms. *Adv. Math.*, 7:301–343 (1971), 1971.

[Sta75] Harold M. Stark. L-functions at $s = 1$. II. Artin L-functions with rational characters. *Adv. Math.*, 17(1):60–92, 1975.

[Sta76] Harold M. Stark. L-functions at $s = 1$. III. Totally real fields and Hilbert's twelfth problem. *Adv. Math.*, 22(1):64–84, 1976.

[Sta80] Harold M. Stark. L-functions at $s = 1$. IV. First derivatives at $s = 0$. *Adv. Math.*, 35(3):197–235, 1980.

[Tat84] John Tate. *Les conjectures de Stark sur les fonctions* L *d'Artin en* $s = 0$, volume 47 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1984. Lecture notes edited by Dominique Bernardi and Norbert Schappacher.

[Zha23a] Robin Zhang. The Harris–Venkatesh conjecture for derived Hecke operators I: imaginary dihedral forms, 2023. arXiv:2301.00570.

[Zha23b] Robin Zhang. The Harris–Venkatesh conjecture for derived Hecke operators II: a unified Stark conjecture, 2023. arXiv:2305.08956.

[Zha23c] Robin Zhang. The Harris–Venkatesh conjecture for derived Hecke operators III: local constants, 2023. arXiv:2301.00612. *Ann. Math. Blaise Pascal*, to appear.

Department of Mathematics, Massachusetts Institute of Technology
Email address: robinz@mit.edu