HIDA FAMILIES OF YOSHIDA LIFTS AND THE RANKIN-SELBERG CONVOLUTION

MING-LUN HSIEH

This short note is a summary of my talk on a joint work with Zheng Liu in the RIMS conference "Arithmetic aspects of automorphic forms and automorphic representations" during January 20-24, 2025. The author thanks the organizers for their hospitality during the conference. The talk was a report on the connection between the endoscopic congruence on GSp(4), two-variable p-adic Rankin-Selberg L-function and the Iwasawa-Greenberg main conjectures for the Rakin-Selberg convolution.

The first example of endoscopic congruence appeared in a series of pioneering works of Hida and Tilouine in 1993-1994, where they proved that the characteristic power series of antityclotomic Selmer group for CM fields is divisible by the anticyclotomic Katz p-adic L-functions by using the CM congruence for Hilbert modular forms. In [AK13] and [BDSP12], the authors made the first attempt to use the Yoshida congruence to study the Bloch-Kato conjecture for the Rankin-Selberg convolution. The goal of our project is to extend these works to the Iwasawa-Greenberg main conjecture for the Rankin-Selberg convolution associated with Hida families.

1. THE IWASAWA-GREENBERG MAIN CONJECTURE FOR RANKIN-SELBERG CONVOLUTION

1.1. Hida families of elliptic modular forms.

Let $\mathcal{O}$ be a ring of integers of a finite extension of $\mathbf{Q}_p$. Let $\mathbf{I}$ be a normal local domain finite flat over the one-variable Iwasawa algebra $\mathcal{O}[\![T]\!]$ with the maximal ideal $\mathfrak{m}_{\mathbf{I}}$. For a primitive cuspidal Hida family $\mathcal{F} = \sum_{n \geq 1} \mathbf{a}(n, \mathcal{F})q^n \in \mathbf{I}[\![q]\!]$ of tame conductor $N_\mathcal{F}$, let $\rho_\mathcal{F} : \Gamma_\mathbf{Q} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\mathrm{Frac}\,\mathbf{I})$ be the associated big Galois representation such that $\mathrm{Tr}\,\rho_\mathcal{F}(\mathrm{Frob}_\ell) = \mathbf{a}(\ell, \mathcal{F})$ for primes $\ell \nmid N_\mathcal{F}$, where Frob_ℓ is the geometric Frobenius at ℓ and let $V_\mathcal{F}$ denote the natural realization of $\rho_\mathcal{F}$ inside the étale cohomology groups of modular curves. Thus, $V_\mathcal{F}$ is a lattice in $(\mathrm{Frac}\,\mathbf{I})^2$ with the continuous Galois action via $\rho_\mathcal{F}$, and the $G_{\mathbf{Q}_p} := \mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$-invariant subspace $\mathrm{Fil}^0 V_\mathcal{F} := V_\mathcal{F}^{I_p}$ fixed by the inertia group I_p at p is free of rank one over $\mathbf{I}$ ([Oht00, Corollary, page 558]). Denote by $\mathbf{T}_1(N_\mathcal{F})$ he Hecke $\mathbf{I}$-algebra acting on the space of ordinary $\mathbf{I}$-adic cusp forms of tame level $N_\mathcal{F}$. Let $\lambda_\mathcal{F} : \mathbf{T}_1(N_\mathcal{F}) \to \mathbf{I}$ be the Hecke eigensystem associated with $\mathcal{F}$ such that $\lambda_\mathcal{F}(T_\ell) = \mathbf{a}(\ell, \mathcal{F})$ for all $\ell \nmid pN_\mathcal{F}$ and let $\mathfrak{m}_\mathcal{F}$ be the maximal ideal of $\mathbf{T}_1(N_\mathcal{F})$ containing $\mathrm{Ker}\,\lambda_\mathcal{F}$. We consider the following

Hypothesis (CR). The residual Galois representation $\bar{\rho}_\mathcal{F} := \rho_\mathcal{F} \pmod{\mathfrak{m}_{\mathbf{I}}} : \Gamma_\mathbf{Q} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is absolutely irreducible and p-distinguished.

Recall that $\rho_\mathcal{F}$ is p-distinguished if the semi-simplication of the restriction of the residual Galois representation $\rho_\mathcal{F} \pmod{\mathfrak{m}_{\mathbf{I}}}$ to the decomposition at p is a sum of two characters $\chi_\mathcal{F}^+ \oplus \chi_\mathcal{F}^-$ with $\chi_\mathcal{F}^+ \not\equiv \chi_\mathcal{F}^- \pmod{\mathfrak{m}_{\mathbf{I}}}$. We collect basic facts about the

Hecke algebra and the Galois representation associated with $\mathcal{F}$ with this hypothesis (CR). The local ring $\mathbf{T}_1(N_\mathcal{F})_{\mathfrak{m}_\mathcal{F}}$ is known to be Gorenstein by [MW86, Prop.2, §9] and [Wil95, Corollary 2, page 482], and with this Gorenstein property, Hida proved in [Hid88a, Theorem 0.1] that the congruence module for $\mathcal{F}$ is isomorphic to $\mathbf{I}/(\eta_\mathcal{F})$ for some non-zero element $\eta_\mathcal{F} \in \mathbf{I}$. Denote by $\varepsilon : \Gamma_\mathbf{Q} \to \mathbf{Z}_p^\times$ be the p-adic cyclotomic character. Put $\mathbf{u} = 1 + 2p \in 1 + 2p\mathbf{Z}_p$. Define the character

$$\varepsilon_T : \Gamma_\mathbf{Q} \to \mathbf{Z}_p[\![T]\!], \quad \varepsilon_T(\sigma) := (1+T)^{\frac{\log_p \varepsilon(\sigma)}{\log_p \mathbf{u}}}.$$

Moreover, we may further assume $\rho_\mathcal{F} : \Gamma_\mathbf{Q} \to \mathrm{GL}_2(\mathrm{Frac}(\mathbf{I}))$ takes values in $\mathrm{GL}_2(\mathbf{I})$ and

$$(1.1) \qquad \rho_\mathcal{F}(\sigma) = \begin{pmatrix} \alpha_{\mathcal{F},p} & * \\ 0 & \alpha_{\mathcal{F},p}^{-1}\varepsilon^{-1}\varepsilon_T^{-1}(\sigma) \end{pmatrix} \text{ for any } \sigma \in G_{\mathbf{Q}_p},$$

where $\alpha_{\mathcal{F},p} : G_{\mathbf{Q}_p} \to \mathbf{I}^\times$ is the unramified character with $\alpha_{\mathcal{F},p}(\mathrm{Frob}_p) = \mathbf{a}(p,\boldsymbol{f})$.

1.2. p-adic Rankin-Selberg L-functions.
For simplicity we shall assume $\mathbf{I} = \mathcal{O}[\![T]\!]$ in this note. Let

$$\boldsymbol{f} = \sum_{n>0} \mathbf{a}(n,\boldsymbol{f})q^n; \quad \boldsymbol{g} = \sum_{n>0} \mathbf{a}(n,\boldsymbol{g})q^n \in \mathcal{O}[\![T]\!][\![q]\!]$$

be primitive Hida families of elliptic modular forms of tame conductor N_1 and N_2 with the trivial tame character defined over the one-variable Iwasawa algebra $\mathcal{O}[\![T]\!]$. We assume $\boldsymbol{f}$ and $\boldsymbol{g}$ both satisfy Hypothesis (CR).

For any $k \in \mathbf{Z}_{\geq 2}$ with $k \equiv 0 \pmod{p-1}$, the specializations

$$\boldsymbol{f}_k := \boldsymbol{f}(\mathbf{u}^k - 1) \in S_{k+2}^{\mathrm{ord}}(\Gamma_0(N_1p));$$
$$\boldsymbol{g}_k := \boldsymbol{g}(\mathbf{u}^k - 1) \in S_{k+2}^{\mathrm{ord}}(\Gamma_0(N_2p)).$$

are p-stabilized newforms. Hida constructed the p-adic Rankin-Selberg L-functions associated to $\boldsymbol{f}$ and $\boldsymbol{g}$.

Theorem 1.1. *There exists $\mathcal{L}_{\boldsymbol{f},\boldsymbol{g}} \in \mathcal{O}[\![T_1,T_2]\!]$ such that for $(k_1,k_2) \in \mathbf{Z}^2$ with $k_1 > k_2 \geq 0$ and $k_1 \equiv k_2 \equiv 0 \pmod{p-1}$, we have*

$$\mathcal{L}_{\boldsymbol{f},\boldsymbol{g}}(k_1,k_2) := L_{\boldsymbol{f},\boldsymbol{g}}(\mathbf{u}^{k_1} - 1, \mathbf{u}^{k_2} - 1)$$
$$= (\sqrt{-1})^{-1-k_1} \frac{L(\frac{k_1+k_2}{2} + 2, \boldsymbol{f}_{k_1} \otimes \boldsymbol{g}_{k_2})}{(-2\sqrt{-1})^{k_1} \|\boldsymbol{f}_{k_1}\|^2} \cdot E_p(1, \boldsymbol{f}_{k_1} \otimes \boldsymbol{g}_{k_2}) \cdot \eta_{\boldsymbol{f}},$$

where $E_p(1, \boldsymbol{f}_{k_1} \otimes \boldsymbol{g}_{k_2})$ is the modified Euler factor at p à la Coates and Perrin-Riou.

This p-adic $\mathcal{L}_{\boldsymbol{f},\boldsymbol{g}}$ is obtained as the specialization of the three-variable p-adic Rakin-Selberg L-function constructed in [Hid88b] (See also [CH20, Theorem A] for the refinement of the interpolation formula).

1.3. Selmer groups.
We consider Selmer groups associated with $\boldsymbol{f}$ and $\boldsymbol{g}$. Let S be the of primes containing the prime factors of pN_1N_2. Let $G_{\mathbf{Q},S}$ be the Galois group of the maximal algebraic extension of $\mathbf{Q}$ that is unramified outside Σ. Let $\rho_{\boldsymbol{f}}, \rho_{\boldsymbol{g}} : G_{\mathbf{Q},S} \to \mathrm{GL}_2(\mathcal{O}[\![T]\!])$ be the associated Galois representations such that the restrictions to $G_{\mathbf{Q}_p}$ satisfy (1.1). Let $\Lambda = \mathcal{O}[\![T_1,T_2]\!]$. Let $i_1, i_2 : \mathcal{O}[\![T]\!] \hookrightarrow \Lambda$

be the $\mathcal{O}$-algebra homomorphisms defined by $i_1(T) = T_1$ and $i_2(T) = T_2$. Define $\chi_{\boldsymbol{f},\boldsymbol{g}} : \Gamma_{\mathbf{Q}} \to \Lambda^\times$ by

$$\chi_{\boldsymbol{f},\boldsymbol{g}} := (i_1(\det \rho_{\boldsymbol{f}}) i_2(\det \rho_{\boldsymbol{g}})^{-1})^{\frac{1}{2}} = \varepsilon_{T_1}^{-\frac{1}{2}} \varepsilon_{T_2}^{\frac{1}{2}}.$$

Define the Galois representations $\rho_i : G_{\mathbf{Q},S} \to \mathrm{GL}_2(\Lambda)$ for $i = 1, 2$ by

$$(1.2) \qquad \rho_1 = i_1 \rho_{\boldsymbol{g}} \otimes \chi_{\boldsymbol{f},\boldsymbol{g}}; \quad \rho_2 = i_2 \rho_{\boldsymbol{f}}$$

Let $T_{\boldsymbol{f},\boldsymbol{g}} = \mathrm{M}_2(\Lambda)$ with $G_{\mathbf{Q},S}$-action given by

$$\rho_{T_{\boldsymbol{f},\boldsymbol{g}}}(g)x = \rho_1(\sigma)x\rho_2(\sigma^{-1}), \quad x \in \mathrm{M}_2(\Lambda).$$

Then

$$\rho_{T_{\boldsymbol{f},\boldsymbol{g}}} \simeq \rho_1 \otimes \rho_2^\vee \simeq \rho_{\boldsymbol{f}}^\vee \otimes \rho_{\boldsymbol{g}} \otimes \chi_{\boldsymbol{f},\boldsymbol{g}}.$$

Let $F^+ T_{\boldsymbol{f},\boldsymbol{g}} \subset T_{\boldsymbol{f},\boldsymbol{g}}$ be the Λ-submodule given by

$$F^+ T_{\boldsymbol{f},\boldsymbol{g}} = \left\{ \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix} \right\}.$$

Then $F^+ T_{\boldsymbol{f},\boldsymbol{g}}$ is a $G_{\mathbf{Q}_p}$-submodule of $T_{\boldsymbol{f},\boldsymbol{g}}$. In addition, $T_{\boldsymbol{f},\boldsymbol{g}}$ along with the critical specializations $\{(k_1, k_2) \in \mathbf{Z}^2 \mid k_1 > k_2 \geq 0\}$ satisfies the Panchishkin condition [Gre94, p. 217]. If A is a locally compact abelian group, denote by $A^* = \mathrm{Hom}_{cts}(A, \mathbf{Q}/\mathbf{Z})$ the Pontryagin dual of A. If A is a Λ-module, then A^* is a Λ-module such that $\lambda \cdot f(a) = f(\lambda a)$ for any $\lambda \in \Lambda$ and $f \in A^*$. Introduce the Greenberg's local conditions:

$$L_\ell = \ker \left(\mathrm{H}^1(\mathbf{Q}_\ell, T_{\boldsymbol{f},\boldsymbol{g}} \otimes_\Lambda \Lambda^*) \to \mathrm{H}^1(\mathbf{Q}_\ell^{\mathrm{ur}}, T_{\boldsymbol{f},\boldsymbol{g}} \otimes_\Lambda \Lambda^*) \right) \text{ if } \ell \neq p;$$
$$L_p = \ker \left(\mathrm{H}^1(\mathbf{Q}_p, T_{\boldsymbol{f},\boldsymbol{g}} \otimes_\Lambda \Lambda^*) \to \mathrm{H}^1(\mathbf{Q}_p^{\mathrm{ur}}, T_{\boldsymbol{f},\boldsymbol{g}}/F^+ T_{\boldsymbol{f},\boldsymbol{g}} \otimes_\Lambda \Lambda^*) \right),$$

and consider the Greenberg Selmer group:

$$\mathrm{Sel}_{\mathbf{Q}}(T_{\boldsymbol{f},\boldsymbol{g}}) = \ker \left\{ \mathrm{H}^1(G_{\mathbf{Q},S}, T_{\boldsymbol{f},\boldsymbol{g}} \otimes_\Lambda \Lambda^*) \to \prod_{\ell < \infty} \frac{\mathrm{H}^1(\mathbf{Q}_\ell, T_{\boldsymbol{f},\boldsymbol{g}} \otimes_\Lambda \Lambda^*)}{L_\ell} \right\}.$$

For any finitely generated Λ-module X, $\mathrm{Char}_\Lambda X$ be the characteristic ideal of X. Now we are in a position to state the Iwasawa-Greeberg main conjecture for the Galois representation $T_{\boldsymbol{f},\boldsymbol{g}}$ following [Gre94].

Conjecture 1. *Suppose that $\boldsymbol{f}$ and $\boldsymbol{g}$ both satisfy Hypothesis (CR). Then we have the following equality of ideals in Λ*

$$\mathrm{Char}_\Lambda(\mathrm{Sel}_{\mathbf{Q}}(T_{\boldsymbol{f},\boldsymbol{g}})^*) = (\mathcal{L}_{\boldsymbol{f},\boldsymbol{g}}).$$

In a joint work with Zheng Liu, we aim to prove the one-sided divisibility

$$(1.3) \qquad \mathrm{Fitt}_\Lambda(\mathrm{Sel}_{\mathbf{Q}}(T_{\boldsymbol{f},\boldsymbol{g}})^*) \subset (\mathcal{L}_{\boldsymbol{f},\boldsymbol{g}}) \iff \mathcal{L}_{\boldsymbol{f},\boldsymbol{g}} \mid \mathrm{Char}_\Lambda(\mathrm{Sel}_{\mathbf{Q}}(T_{\boldsymbol{f},\boldsymbol{g}})^*)$$

by using the Yoshida congruence.

2. Hida theory for Siegel modular forms

2.1. Hermitian domain.
Let GSp_4 be the algebraic group defined by

$$\mathrm{GSp}_4 = \left\{ g \in \mathrm{GL}_4 \mid g \begin{pmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{pmatrix} {}^t g = \nu(g) \begin{pmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{pmatrix} \right\}$$

with the similitude character $\nu : \mathrm{GSp}_4 \to \mathbb{G}_m$. For $(a, b, c) \in \mathbb{G}_m^3$, put

$$[a,\, b;\, c] = \begin{pmatrix} a & & & \\ & b & & \\ & & a^{-1}c & \\ & & & b^{-1}c \end{pmatrix} \in \mathrm{GSp}_4.$$

If N is a positive integer, put

$$K_0(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}_4(\widehat{\mathbf{Z}}) \mid C \equiv 0 \,(\mathrm{mod}\ N) \right\}$$

be an open-compact subgroup of $\mathrm{GSp}_4(\mathbf{A}_f)$. For each non-negative integer r, put

$$\mathcal{I}_r = \left\{ g \in \mathrm{GSp}_4(\mathbf{Z}_p) \mid g \equiv \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & * & * \end{pmatrix} \,(\mathrm{mod}\ p^r) \right\};$$

$$K_r = \left\{ g \in K_0(N) \mid g_p \in \mathcal{I}_r \right\}.$$

2.2. Vector-valued Siegel modular forms.
The Siegel upper half plane of degree 2 is defined by

$$\mathfrak{H}_2 = \left\{ Z \in \mathrm{M}_2(\mathbf{C}) \mid Z = {}^t Z,\ \mathrm{Im}\, Z \text{ is positive definite} \right\}.$$

Then $\mathfrak{H}_2$ is equipped with an action of $\mathrm{Sp}_4(\mathbf{R})$ given by

$$g \cdot Z = (AZ + B)(CZ + D)^{-1}$$

for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $Z \in \mathfrak{H}_2$, and define the automorphy factor

$$J : \mathrm{GSp}_4(\mathbf{R})^+ \times \mathfrak{H}_2 \to \mathrm{GL}_2(\mathbf{C}), \quad J(g, Z) = CZ + D.$$

Let $\kappa = (\kappa_1, \kappa_2) \in \mathbf{Z}^2$ with $\kappa_1 - \kappa_2 \in 2\mathbf{Z}_{\geq 0}$ and let $\kappa^\vee = (-\kappa_2, -\kappa_1)$. For any torsion-free ring R, put

- $\mathcal{L}_\kappa(R) = R[X, Y]_{\kappa_1 - \kappa_2}$ be the space of two-variable homogeneous polynomials of degree $\kappa_1 - \kappa_2$.
- $\rho_\kappa : \mathrm{GL}_2(R) \to \mathrm{Aut}\, \mathcal{L}_\kappa(R)$ is the representation defined by

$$\rho_\kappa \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) P(X, Y) = P(Xa + Yc, Xb + Yd)(ad - bc)^{\kappa_2}.$$

- $\langle\, ,\, \rangle : \mathcal{L}_\kappa(R) \times \mathcal{L}_{\kappa^\vee}(R) \to R \otimes \mathbf{Q}$ is the $\mathrm{GL}_2(R)$-equivariant pairing such that $\langle X^{\kappa_1 - \kappa_2}, Y^{\kappa_1 - \kappa_2} \rangle = 1$.

Let $S_\kappa(K_r, \mathbf{C})$ be the space of cuspidal Siegel modular forms of weight κ and level K_r, consisting of vector-valued holomorphic functions

$$\vec{F} : \mathfrak{H}_2 \times \mathrm{GSp}_4(\mathbf{A}_f) \to \mathcal{L}_\kappa(\mathbf{C})$$

such that for any $\alpha \in \mathrm{GSp}_4(\mathbf{Q})^+$, $u \in K_r$, we have

$$\vec{F}(\alpha \cdot Z, \alpha g_f u) = \rho_\kappa(J(\alpha, Z))\vec{F}(Z, g_f).$$

For any $\ell \mid N$, the Atkin-Lehner involution $\tau_\ell \in \mathrm{End}\, S_\kappa(K_r, \mathbf{C})$ is defined by

$$\tau_N \vec{F}(Z, g_f) := \vec{F}\left(Z, g_f \begin{pmatrix} 0 & \mathbf{1}_2 \\ -\ell \cdot \mathbf{1}_2 & 0 \end{pmatrix}\right).$$

2.3. Integral Siegel modular forms.
Every $\vec{F} \in S_\kappa(K_r, \mathbf{C})$ admits the Fourier expansion:

$$\vec{F}(Z, \mathbf{1}_4) = \sum_{S \in \mathrm{Sym}_2(\mathbf{Z})_+^\vee} a(S, \vec{F})e^{2\pi\sqrt{-1}\,\mathrm{Tr}(SZ)}.$$

Here

$$\mathrm{Sym}_2(\mathbf{Z})_+^\vee = \left\{ S = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} > 0 \mid a, b, c \in \mathbf{Z} \right\}.$$

We call the formal power series

$$\vec{F}(q) := \sum_{S \in \mathrm{Sym}_2(\mathbf{Z})_+^\vee} a(S, \vec{F})q^S \in \mathcal{L}_\kappa(\mathbf{C})[\![q^{\mathrm{Sym}_2(\mathbf{Z})_+^\vee}]\!]$$

the q-expansion of $\vec{F}$. Let $\mathcal{O}_0 := \mathbf{Z}[1/N, \zeta_N] \subset \mathbf{C}$. If R is a $\mathcal{O}_0$-subalgebra of $\mathbf{C}$, define $S_\kappa(K_r, R)$ to be the subspace of cuspidal Siegel modular form defined over R, consisting of $\vec{F} \in S_\kappa(K_r, \mathbf{C})$ such that

$$\vec{F}(q) \in \mathcal{L}_\kappa(R) \otimes R[\![q^{\mathrm{Sym}_2(\mathbf{Z})_+^\vee}]\!].$$

We fix $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$. Let $\mathcal{O}$ be the p-adic completion of $\iota_p(\mathcal{O}_0)$ in $\mathbf{C}_p$. We put

$$S_\kappa(K_r, \mathcal{O}) := S_\kappa(K_r, \mathcal{O}_0) \otimes_{\mathcal{O}_0} \mathcal{O}.$$

2.4. Hecke algebras and the ordinary projector.
The Hecke operators

- $T_\ell = K_0\big[\ell, \ell; \ell\big]K_0$ and $R_\ell = K_0\big[\ell^2, \ell; \ell^2\big]K_0$ for primes $\ell \nmid pN$;
- $U_P := K_1\big[p, p; p\big]K_1$ and $U_Q := K_1\big[p^2, p; p^2\big]K_1$.

act on $S_\kappa(K_1, \mathcal{O})$. Let e_{ord} be Hida's ordinary projector defined by

$$e_{\mathrm{ord}} := \lim_{n \to \infty} (U_P U_Q)^{n!}.$$

Define the space of ordinary Siegel modular forms by

$$S_\kappa^{\mathrm{ord}}(K_1, \mathcal{O}) := e_{\mathrm{ord}} S_\kappa(K_1, \mathcal{O}).$$

Let $T_\kappa^{\mathrm{ord}}(K_1, \mathcal{O})$ be the $\mathcal{O}$-subalgebra in $\mathrm{End}(S_\kappa^{\mathrm{ord}}(K_1, A))$ generated by the above Hecke operators as well as the Atkin-Lehner involutions $\{\tau_\ell\}_{\ell \mid N^-}$.

2.5. Ordinary Λ-adic forms.
Let $\Lambda := \mathcal{O}[\![X_1, X_2]\!]$ and $\mathfrak{m}_\Lambda$ be the maximal ideal of Λ. Let $\mathbf{u} = 1 + 2p$. Put

$$\mathfrak{X}^{\mathrm{cls}} := \left\{ \kappa = (\kappa_1, \kappa_2) \in \mathbf{Z}^2 \mid \kappa_1 \geq \kappa_2 \geq 4,\ \kappa_1 \equiv \kappa_2 \equiv 2 \,(\mathrm{mod}\ p - 1) \right\}.$$

Definition 2.1 (Ordinary Λ-adic forms). Let $\mathbf{S}^{\mathrm{ord}}(N)$ be the space of the q-expansions

$$\mathbf{F} = \sum_S \mathbf{a}(S, F)q^S \in \Lambda[\![q^{\mathrm{Sym}_2(\mathbf{Z})_+^\vee}]\!].$$

such that for any $\kappa \in \mathfrak{X}^{\mathrm{cls}}$

$$\mathbf{F}(\kappa) := \mathbf{F}(\mathbf{u}^{\kappa_1} - 1, \mathbf{u}^{\kappa_2} - 1) = \langle \vec{\mathbf{F}}_\kappa, Y^{\kappa_1 - \kappa_2} \rangle = \vec{\mathbf{F}}_\kappa|_{X=1, Y=0}.$$

for some unique $\vec{\mathbf{F}}_\kappa \in S_\kappa^{\mathrm{ord}}(K_1, \mathcal{O})$.

2.6. **Hida theory for Siegel modular forms.** Let $\mathbb{T}^{\mathrm{ord}}(N)$ be the Λ-subalgebra of $\mathrm{End}_\Lambda(\mathbf{S}^{\mathrm{ord}}(N))$ generated by Hecke operators. Let $P_\kappa := (X_1 - \mathbf{u}^{\kappa_1} + 1, X_2 - \mathbf{u}^{\kappa_2} + 1)$ be the height one ideal of Λ.

Theorem 2.2 (Hida-Pilloni). *(1) $\mathbf{S}^{\mathrm{ord}}(N)$ is a free Λ-module with finite rank,*
(2) For every weight $\kappa \in \mathfrak{X}^{\mathrm{cls}}$, we have the Hecke module isomorphism

$$\mathbf{S}^{\mathrm{ord}}(N) \otimes_\Lambda \Lambda/P_\kappa \simeq S_\kappa^{\mathrm{ord}}(K_1, \mathcal{O}), \quad \mathbf{F} \mapsto \vec{\mathbf{F}}_\kappa.$$

(3) The specialization map $\mathbf{T}^{\mathrm{ord}}(N)/P_\kappa \to T_\kappa^{\mathrm{ord}}(N)$ is surjective and its kernel is contained in the kernel of $\mathbf{T}^{\mathrm{ord}}(N)/P_\kappa$.

3. Congruence ideal for Yoshida lifts

3.1. **Λ-adic Yoshida lifts.** We shall impose the following running hypotheses $(\star)$

Hypothesis ($\star$). Let $N^- := \gcd(N_1, N_2)$.
  (1) The tame conductors N_1 and N_2 are square-free.
  (2) $\boldsymbol{f}$ and $\boldsymbol{g}$ both satisfy the Hypothesis (CR), and

$$\bar\rho_{\boldsymbol{f}} \not\simeq \bar\rho_{\boldsymbol{g}}.$$

  (3) There exists a prime $q \mid N^-$ such that $\bar\rho_{\boldsymbol{f}}$ and $\bar\rho_{\boldsymbol{g}}$ are both ramified at q.
  (4) The Atkin-Lehner eigenvalues of $\boldsymbol{f}$ and $\boldsymbol{g}$ at all primes $\ell \mid N^-$ have the same sign $\epsilon_\ell \in \{\pm 1\}$.
  (5) p-distinction: $\mathbf{a}(p, \boldsymbol{f}) \not\equiv \mathbf{a}(p, \boldsymbol{g}) \pmod{\mathfrak{m}_\Lambda}$.

Let $N = \mathrm{lcm}(N_1, N_2)$. Write $\mathbf{T}_{\boldsymbol{f}} = \mathbf{T}_1(N_1)_{\mathfrak{m}_{\boldsymbol{f}}}$ and $\mathbf{T}_{\boldsymbol{g}} = \mathbf{T}_1(N_2)_{\mathfrak{m}_{\boldsymbol{g}}}$. Let $\eta : \Lambda \simeq \Lambda$ be the $\mathcal{O}$-algebra isomorphism defined by

$$\eta(T_1) = \mathbf{u}^{-4}(1 + T_1)(1 + T_2) - 1, \ \eta(T_2) = (1 + T_1)(1 + T_2)^{-1} - 1.$$

Proposition 3.1 (Zheng Liu-H.). *With the Hypothesis $(\star)$, there exists a Λ-algebra homomorphism*

$$\theta : \mathbf{T}_2^{\mathrm{ord}}(N) \otimes_{\Lambda, \eta} \Lambda \to \mathbf{T}_{\boldsymbol{f}} \otimes_{\mathcal{O}[\![T]\!]} \mathbf{T}_{\boldsymbol{g}}$$

determined by

$$\theta(T_\ell) = T_\ell \otimes 1 + \langle \ell \rangle_{T_1}^{\frac{1}{2}} \langle \ell \rangle_{T_2}^{-\frac{1}{2}} 1 \otimes T_\ell,$$
$$\theta(U_P) = U_p \otimes 1, \quad \theta(U_Q) = U_p \otimes U_p$$
$$\theta(\tau_\ell) = \begin{cases} \epsilon_\ell & \text{if } \ell \mid N^-/q \\ -\epsilon_q & \text{if } \ell = q. \end{cases}$$

Moreover, there exists a Hida family $\theta_{\boldsymbol{f},\boldsymbol{g}} \in \mathbf{S}^{\mathrm{ord}}(N)$ such that
  (1) $\mathbf{a}(S, \theta_{\boldsymbol{f},\boldsymbol{g}}) \not\equiv 0 \pmod{\mathfrak{m}_\Lambda}$ for infinitely $S \in \mathrm{Sym}_2(\mathbf{Z})_+^\vee$.
  (2) The homomorphism $\lambda_{\boldsymbol{f},\boldsymbol{g}} := (\lambda_{\boldsymbol{f}} \otimes \lambda_{\boldsymbol{g}}) \circ \theta$ is the Hecke eigensystem associated with $\theta_{\boldsymbol{f},\boldsymbol{g}}$.

Remark 3.2. For any $\kappa = (\frac{k_1+k_2}{2} + 2, \frac{k_1-k_2}{2} + 2) \in \mathfrak{X}^{\mathrm{cls}}$, we have

$$L(s, \theta_{\boldsymbol{f},\boldsymbol{g}}(\kappa)) = L(s, \boldsymbol{f}_{k_1})L(s + \frac{k_2 - k_1}{2}, \boldsymbol{g}_{k_2}).$$

We call the Hecke eigenform $\theta_{\boldsymbol{f},\boldsymbol{g}}$ the Λ-adic Yoshida lift.

The construction of $\theta_{\boldsymbol{f},\boldsymbol{g}}$ relies on the p-adic interpolation of explicit theta liftings from $\mathrm{O}(D)$ to $\mathrm{Sp}(4)$, where D is the definite quaternion algebra ramified at ∞ and the prime q in the Hypothesis $(\star)$.

The non-vanishing mod p of $\mathbf{a}(S, \theta_{\boldsymbol{f},\boldsymbol{g}})$ is obtained by proving the non-vanishing mod p of the Bessel period $B_\xi(\theta_{\boldsymbol{f}_{k_1},\boldsymbol{g}_{k_2}})$ for infinitely many anticyclotomic characters ξ in [HN17].

3.2. Congruence ideals.

Let $F_\Lambda = \mathrm{Frac}(\Lambda)$ and $\mathfrak{m}$ be the maximal ideal of $\mathbf{T} := \mathbf{T}_2^{\mathrm{ord}}(N)$ containing $\mathrm{Ker}\,\lambda_{\boldsymbol{f},\boldsymbol{g}}$. There is a decomposition of Hecke modules

$$\mathbf{S}^{\mathrm{ord}}(N)_{\mathfrak{m}} \otimes F_\Lambda = \mathbf{S}^{\mathrm{end}} \oplus \mathbf{S}^{\mathrm{st}},$$

where $\mathbf{S}^{\mathrm{end}}$ is the subspace spanned by Hida families of Yoshida lifts. Namely, the subspace $\mathbf{S}^{\mathrm{end}}$ is annihilated by $\mathrm{Ker}\,\theta$ for θ in Proposition 3.1. Using Arthur's endoscopic classifications of representations of $\mathrm{GSp}(4)$, we can show that the subspace $\mathbf{S}^{\mathrm{end}}$ only contains Hida families of Yoshida lifts arising from $\mathrm{O}(D)$ by an analysis of the Atkin-Lehner eigenvalues. This implies that we have a Hecke module decomposition

$$\mathbf{S}^{\mathrm{end}} = F_\Lambda \theta_{\boldsymbol{f},\boldsymbol{g}} \oplus \mathbf{S}'$$

such that $\lambda_{\boldsymbol{f},\boldsymbol{g}}$ does not appear in any Hecke eigensystems of $\mathbf{S}'$. Put

$$\mathbf{S}^\perp := \mathbf{S}' \oplus \mathbf{S}^{\mathrm{st}}.$$

For $? = \perp$ or st, define $\mathbf{T}^?$ to be the Λ-subalgebra of $\mathrm{End}\,\mathbf{S}^?$ generated by Hecke operators $\{T_\ell, R_\ell\}_{\ell \nmid pN}$ and $\{U_P, U_Q\}$. Let $\mathfrak{a}_{\boldsymbol{f},\boldsymbol{g}}^?$ be the ideal of $\mathbf{T}^?$ generated by the relations satisfied by the Hecke eigensystem $\lambda_{\boldsymbol{f},\boldsymbol{g}}$:

$$T_\ell - \mathbf{a}(\ell, \boldsymbol{f}) - \langle \ell \rangle_{T_1}^{\frac{1}{2}} \langle \ell \rangle_{T_2}^{-\frac{1}{2}} \mathbf{a}(\ell, \boldsymbol{g}),\ U_P - \mathbf{a}(p, \boldsymbol{f});\ U_Q - \mathbf{a}(p, \boldsymbol{f})\mathbf{a}(p, \boldsymbol{g}).$$

The congruence ideal $\mathcal{C}_{\boldsymbol{f},\boldsymbol{g}}^?$ of Λ is defined by the kernel of the structure morphism $\Lambda \to \mathbf{T}^?/\mathfrak{a}_{\boldsymbol{f},\boldsymbol{g}}^?$. By definition, we have

$$\Lambda/\mathcal{C}_{\boldsymbol{f},\boldsymbol{g}}^? \xrightarrow{\sim} \mathbf{T}^?/\mathfrak{a}_{\boldsymbol{f},\boldsymbol{g}}^?.$$

Moreover, we can show that

$$\mathbf{T}_{\mathfrak{m}} \otimes F_\Lambda \simeq \mathbf{T}_{\boldsymbol{f},\boldsymbol{g}} \otimes F_\Lambda \oplus \mathbf{T}^{\mathrm{st}} \otimes F_\Lambda$$

is an isomorphism as F_Λ-algebras. Therefore, applying [TU22, Corollary 2.7] to our situation $T = \mathbf{T}_{\mathfrak{m}}$ and $T' = \mathbf{T}_{\boldsymbol{f},\boldsymbol{g}}$, we obtain that

$$(3.1) \qquad \eta_{\boldsymbol{f}} \eta_{\boldsymbol{g}} \cdot \mathcal{C}_{\boldsymbol{f},\boldsymbol{g}}^{\mathrm{st}} \subset \mathcal{C}_{\boldsymbol{f},\boldsymbol{g}}^\perp.$$

4. The algebraic side

In this section, we explain briefly the proof of the divisibility

Theorem 4.1. *With the hypotheses $(\star)$, we have*

$$\mathrm{Fitt}_\Lambda(\mathrm{Char}_\Lambda \mathrm{Sel}_{\mathbf{Q}}(T_{\boldsymbol{f},\boldsymbol{g}})^*) \subset \mathcal{C}_{\boldsymbol{f},\boldsymbol{g}}^{\mathrm{st}}.$$

following closely the approach in the proof of [Urb01, Theorem 3.7]. Let $R = \mathbf{T}_{\mathfrak{m}}^{\mathrm{st}}$ be the localization of $\mathbf{T}^{\mathrm{st}}$ at the maximal ideal $\mathfrak{m}$ containing $\mathfrak{a}_{\boldsymbol{f},\boldsymbol{g}}$ and let $\mathfrak{a} = \mathfrak{a}_{\boldsymbol{f},\boldsymbol{g}} R$. Let $\mathscr{T}_R : G_{\mathbf{Q},S} \to R$ be the pseudo character characterized by

$$\mathscr{T}_R(\mathrm{Frob}_\ell) = T_\ell.$$

By the very definition,

$$\mathscr{T}_R \equiv \operatorname{Tr}\rho_1 + \operatorname{Tr}\rho_2 \ (\operatorname{mod} \mathfrak{a}),$$

where ρ_1, ρ_2 are Galois representations introduced in (1.2). According to [BC09, Theorem 1.4.4], we can associate to $\mathscr{T}_R$ a pseudo representation

$$\rho_R : G_{\mathbf{Q},S} \to \operatorname{GL}_4(\operatorname{Frac}(R)).$$

Let $\mathscr{R} := R[\rho_R(G_{\mathbf{Q},S})] \subset \operatorname{M}_4(\operatorname{Frac}(R))$. Then $\mathscr{R}$ is the GMA (generalized matrix algebra) of the form

$$\mathscr{R} = \begin{pmatrix} \operatorname{M}_2(R) & \operatorname{M}_2(R_{12}) \\ \operatorname{M}_2(R_{21}) & \operatorname{M}_2(R) \end{pmatrix}$$

with R_{12} and R_{12} fractional ideals in $\operatorname{Frac}(R)$ and $R_{12}R_{21} \subset \mathfrak{a}$. For $\sigma \in G_{\mathbf{Q},S}$, we can write

$$\rho_R(\sigma) = \begin{pmatrix} A_\sigma & B_\sigma \\ C_\sigma & D_\sigma \end{pmatrix}, \quad A_\sigma, D_\sigma \in \operatorname{M}_2(R),\ B_\sigma \in \operatorname{M}_2(R_{12}),\ C_\sigma \in \operatorname{M}_2(R_{21})$$

with

$$A_\sigma \equiv \rho_1(\sigma)\ (\operatorname{mod} \mathfrak{a}); \quad D_\sigma \equiv \rho_2(\sigma)\ (\operatorname{mod} \mathfrak{a}).$$

Let e_i be the standard basis in $(\operatorname{Frac}R)^4$ and let $\mathscr{L} = \mathscr{R}e_4 = R_{12}e_1 \oplus R_{12}e_2 \oplus Re_3 \oplus Re_4$. Then we have exact sequence of $G_{\mathbf{Q},S}$-modules

$$(4.1) \qquad 0 \longrightarrow \rho_1 \otimes_\Lambda R_{12}/\mathfrak{a}R_{12} \longrightarrow \mathscr{L}/\mathfrak{a}\mathscr{L} \longrightarrow \rho_2 \otimes_\Lambda R/\mathfrak{a} \longrightarrow 0.$$

Lemma 4.2. *The $G_{\mathbf{Q},S}$-module $\mathscr{L}$ has no quotient isomorphic to $\rho_1 \otimes R/\mathfrak{m}$.*

Proof. Let $k = R/\mathfrak{m}$ be the residue field. Suppose that there exists a surjective $G_{\mathbf{Q},S}$-module homomorphism $\mathscr{L} \twoheadrightarrow \rho_1 \otimes k$ and let $\mathscr{N}$ be the kernel. Then we must have $\mathscr{N} + \rho_1 \otimes R_{12} = \mathscr{L}$. Otherwise, $\mathscr{L}/(\mathscr{N} + \rho_1 \otimes R_{12})$ would be a non-trivial quotient of the irreducible module $\rho_1 \otimes k$, and hence $\rho_1 \otimes R_{12} \subset \mathscr{N}$, which in turns implies the existence of a surjective homomorphism $\rho_2 \otimes k = \mathscr{L}/(\rho_1 \otimes R_{12} + \mathfrak{m}\mathscr{L}) \to \rho_1 \otimes k = \mathscr{L}/\mathscr{N}$. This contradicts to $\rho_1 \otimes k \not\simeq \rho_2 \otimes k$. On the other hand, the element $E_{11} = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} \in \mathscr{R}$, so we have

$$e_4 \in E_{11}\mathscr{L} = E_{11}\mathscr{N} \subset \mathscr{N} \Rightarrow \mathscr{L} \subset \mathscr{N},$$

which is a contradiction. $\square$

The sequence (4.1) induces the exact sequence of discrete $G_{\mathbf{Q},S}$-modules

$$(4.2) \quad 0 \longrightarrow (\rho_2 \otimes \rho_1^\vee \otimes R/\mathfrak{a})^* \longrightarrow (\mathscr{L} \otimes \rho_1^\vee \otimes R/\mathfrak{a})^* \longrightarrow (\rho_1 \otimes \rho_1^\vee \otimes R_{12}/\mathfrak{a}R_{12})^* \longrightarrow 0.$$

Taking Galois cohomology, we obtain the connecting homomorphism:

$$\delta : ((\rho_1 \otimes \rho_1^\vee \otimes R_{12}/\mathfrak{a}R_{12})^*)^{G_{\mathbf{Q},S}} \longrightarrow \operatorname{H}^1(\mathbf{Q}, (\rho_2 \otimes \rho_1^\vee \otimes R/\mathfrak{a})^*).$$

Let $\operatorname{Tr} : \rho_1 \otimes \rho_1^\vee \to R$ be the trace morphism. Since ρ_1 is residually irreducible, Tr induces isomorphism

$$\operatorname{Tr}^* : (R_{12}/\mathfrak{a}R_{12})^* \simeq ((\rho_1 \otimes \rho_1^\vee \otimes R_{12}/\mathfrak{a}R_{12})^*)^{G_{\mathbf{Q},S}}.$$

Define the R-module homomorphism

$$\mathbf{c} := \delta \circ \operatorname{Tr}^* : (R_{12}/\mathfrak{a}R_{12})^* \longrightarrow \operatorname{H}^1(\mathbf{Q}, (\rho_2 \otimes \rho_1^\vee \otimes R/\mathfrak{a})^*) = \operatorname{H}^1(\mathbf{Q}, \rho_1 \otimes \rho_2^\vee \otimes (R/\mathfrak{a})^*).$$

For each $f \in (R_{12}/\mathfrak{a}R_{12})^*$, the class $\Delta(f)$ is represented by the cocycle $\mathbf{c}(f)$: $G_{\mathbf{Q},S} \to (\rho_2 \otimes \rho_1^\vee \otimes R/\mathfrak{a})^*$ given by

$$(4.3) \qquad \langle \mathbf{c}(f)(\sigma), y \rangle := f\left(\mathrm{Tr}(B_\sigma \rho_2(\sigma^{-1})y)\right) \quad y \in \rho_2 \otimes \rho_1^\vee \otimes R/\mathfrak{a}.$$

Let $\mathrm{ht}_1(\Lambda)$ be the set of height one primes of Λ.

Proposition 4.3. *(1) The map Δ is injective;*
   (2) For each $P \in \mathrm{ht}_1(\Lambda)$, there exists $t_P \in \Lambda - P$ such that

$$\mathrm{loc}_p([t_P \cdot \mathbf{c}_f]) = 0 \in \mathrm{H}^1(\mathbf{Q}_p^{\mathrm{ur}}, , T_{\boldsymbol{f},\boldsymbol{g}}/F^+ T_{\boldsymbol{f},\boldsymbol{g}} \otimes_\Lambda (\Lambda/\mathcal{C}_{\boldsymbol{f},\boldsymbol{g}}^{\mathrm{st}})^*)$$

   for any $f \in (R_{12}/\mathfrak{a}R_{12})^*$.

Proof. By Lemma 4.2, we have

$$((\mathscr{L} \otimes \rho_1^\vee \otimes R/\mathfrak{m})^*)^{G_{\mathbf{Q},S}} = \{0\},$$

which implies that

$$((\mathscr{L} \otimes \rho_1^\vee \otimes R/\mathfrak{a})^*)^{G_{\mathbf{Q},S}} = \{0\},$$

so the connection homomorphism δ is injective; hence $\Delta = \delta \circ \mathrm{Tr}^*$ is injective.

To see (2), consider the specialization $\rho_R(Q)$ of ρ_R at $Q \in \mathfrak{X}$ with $k_1 = k_{Q_1} > k_2 = k_{Q_2}$ and $k_1 \equiv k_2 \equiv 2 \,(\mathrm{mod}\ p - 1)$. The local Galois representation at p is of the form

$$\rho_R(k_1, k_2)|_{I_p} \sim \begin{pmatrix} 1 & * & * & * \\ & \varepsilon^{\frac{k_2 - k_1}{2}} & * & * \\ & & \varepsilon^{1 - \frac{k_1 + k_2}{2}} & * \\ & & & \varepsilon^{1 - k_1} \end{pmatrix}.$$

By [Urb01, Theorem 3.4], we thus have

$$\rho_R|_{G_{\mathbf{Q}_p}} \sim \begin{pmatrix} \xi_1 & * & * & * \\ & \xi_2 & * & * \\ & & \xi_3 & * \\ & & & \xi_4 \end{pmatrix},$$

where

$$\xi_1 = \alpha_1, \quad \xi_2 = \alpha_2 \langle \varepsilon \rangle^{\frac{1}{2}}_{(1+T_2)(1+T_1)^{-1}-1},$$
$$\xi_3 = \alpha_3 \varepsilon^{-1} \langle \varepsilon \rangle^{-\frac{1}{2}}_{\mathbf{u}^{-4}(1+T_1)(1+T_2)-1}, \quad \xi_4 = \alpha_4 \varepsilon^{-1} \langle \varepsilon \rangle^{-1}_{\mathbf{u}^{-2}(1+T_1)-1},$$

and α_i are unramified characters such that

$$\alpha_1 \equiv \alpha_4^{-1} \equiv \alpha_{\boldsymbol{f},p} \,(\mathrm{mod}\ \mathfrak{a}); \quad \alpha_2 \equiv \alpha_3^{-1} \equiv \alpha_{\boldsymbol{g},p} \,(\mathrm{mod}\ \mathfrak{a}).$$

There exists non-zero $v \in \mathscr{L}_P$ fixed by I_p and $\rho_R(\mathrm{Frob}_p)v = \mathbf{a}(p, \boldsymbol{f})v$. By the p-distinction assumption $\mathbf{a}(p, \boldsymbol{f}) \not\equiv \mathbf{a}(p, \boldsymbol{g}) \,(\mathrm{mod}\ \mathfrak{m})$, the residual characters $\xi_i \,(\mathrm{mod}\ \mathfrak{m})$ are distinct. Therefore, we may take we write $v = ae_1 + ce_2 + t_P e_3 + de_4$ with $t_P \in \Lambda - P$, $a, c \in R_{12}$ and $d \in \mathfrak{a}$. The equation $\rho_R(I_p)v = v$ yields

$$A_\sigma \begin{pmatrix} a \\ c \end{pmatrix} + B_\sigma \begin{pmatrix} t_P \\ d \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \text{ for } \sigma \in I_p.$$

For $\sigma \in I_p$, $\rho_2(\sigma) \equiv \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$ (mod $\mathfrak{a}$), so we find that

$$t_P \cdot B_\sigma \rho_2(\sigma^{-1}) \, (\text{mod } T_{\boldsymbol{f},\boldsymbol{g}} \otimes_\Lambda \mathfrak{a} R_{12} + F^+ T_{\boldsymbol{f},\boldsymbol{g}} \otimes_R R_{12})$$

$$\equiv B_\sigma \begin{pmatrix} t_P & 0 \\ d & 0 \end{pmatrix} \rho_2(\sigma^{-1}) \equiv \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \rho_2(\sigma^{-1}) - A_\sigma \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \rho_2(\sigma^{-1})$$

$$\equiv \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} - \rho_1(\sigma) \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \rho_2(\sigma^{-1}).$$

This shows that the class

$$\mathrm{loc}_p(t_P[\mathbf{c}_f]) = 0 \text{ in } \mathrm{H}^1(I_p, T_{\boldsymbol{f},\boldsymbol{g}}/F^+ T_{\boldsymbol{f},\boldsymbol{g}} \otimes_\Lambda (\Lambda/C_{\boldsymbol{f},\boldsymbol{g}})^*).$$

This completes the proof. $\square$

Proposition 4.3 and the exact sequence (4.2) produce a surjective map

(4.4) $\mathrm{Fitt}_\Lambda(\mathrm{Sel}_{\mathbf{Q}}(T_{\boldsymbol{f},\boldsymbol{g}})^*) \twoheadrightarrow R_{12}/\mathfrak{a} R_{12}.$

A crucial point is that the fractional ideal R_{12} is a faithful R-module. Otherwise we can apply the modularity results for GL(2) to show that there exists a Λ-adic Yoshida lift in $\mathbf{S}^{\mathrm{st}}$. We thus conclude Theorem 4.1 from (4.4) and the faithfulness of R_{12}.

5. The analytic side: work in progress

In an ongoing work with Zheng Liu, we aim to the inclusion

(5.1) $\mathcal{C}^\perp_{\boldsymbol{f},\boldsymbol{g}} \subset (\mathcal{L}_{\boldsymbol{f},\boldsymbol{g}})\eta_{\boldsymbol{f}}\eta_{\boldsymbol{g}}.$

It is clear that (1.3) is a consequence of Theorem 4.1, (3.1) and (5.1). The strategy to prove (5.1) is as follows.

Step 1. We begin with constructing the Λ-adic form $\mathcal{E} \in \mathbf{S}^{\mathrm{ord}}(N)$ obtained by the pull-back of a nice Λ-adic Kligen-Eisenstein series on $U(2,2)$ attached to a CM form θ_χ.

Step 2. Write down the spectral decomposition

$$\eta_{\boldsymbol{f}}\eta_{\boldsymbol{g}}\mathcal{L}_{\boldsymbol{f},\boldsymbol{g}} \cdot \mathcal{E} = A \cdot \theta_{\boldsymbol{f},\boldsymbol{g}} + \mathcal{H}, \quad \mathcal{G} \in \mathbf{S}^\perp.$$

The element $A \in F_\Lambda$ enjoys the interpolation formula:

$$A(\kappa) = \frac{\langle \mathcal{E}(\kappa), \theta_{\boldsymbol{f},\boldsymbol{g}}(\kappa) \rangle_{\mathrm{Pet}}}{\langle \theta_{\boldsymbol{f},\boldsymbol{g}}(\kappa), \theta_{\boldsymbol{f},\boldsymbol{g}}(\kappa) \rangle_{\mathrm{Pet}}} \cdot \mathcal{L}_{\boldsymbol{f},\boldsymbol{g}}(k_1, k_2) \cdot \eta_{\boldsymbol{f}}(k_1)\eta_{\boldsymbol{g}}(k_2)$$

for almost $\kappa = (\frac{k_1+k_2}{2}+2, \frac{k_1-k_2}{2}+2) \in \mathfrak{X}^{\mathrm{cls}}$. We prove that

(NV) $A \in \Lambda^\times.$

This would imply that $\mathcal{H} \in \mathbf{S}^\perp \cap \mathbf{S}^{\mathrm{ord}}(N)_{\mathfrak{m}}$ and the existence of a non-trivial congruence

(5.2) $\mathcal{H} \equiv \theta_{\boldsymbol{f},\boldsymbol{g}} \, (\text{mod } \eta_{\boldsymbol{f}}\eta_{\boldsymbol{g}}\mathcal{L}_{\boldsymbol{f},\boldsymbol{g}}).$

Step 3. We may assume $L_p(\boldsymbol{f},\boldsymbol{g}) \in \mathfrak{m}_\Lambda$. Then $\mathbf{a}(S, \mathcal{H}) \in \Lambda^\times$ for some S. The Λ-linear map

$$\mathbf{T}^\perp \twoheadrightarrow \Lambda, \quad t \mapsto \frac{\mathbf{a}(S, t\mathcal{H})}{\mathbf{a}(S, \mathcal{H})}$$

becomes a Λ-algebra homomorphism modulo $\eta_{\boldsymbol{f}}\eta_{\boldsymbol{g}}\mathcal{L}_{\boldsymbol{f},\boldsymbol{g}}$. It follows from (5.2) that

$$\Lambda/\mathcal{C}^{\perp}_{\boldsymbol{f},\boldsymbol{g}} \simeq \mathbf{T}^{\perp}/\mathfrak{a}_{\boldsymbol{f},\boldsymbol{g}} \twoheadrightarrow \Lambda/(\eta_{\boldsymbol{f}}\eta_{\boldsymbol{g}}\mathcal{L}_{\boldsymbol{f},\boldsymbol{g}}).$$

This shows that $\mathcal{C}^{\perp}_{\boldsymbol{f},\boldsymbol{g}} \subset (\eta_{\boldsymbol{f}}\eta_{\boldsymbol{g}}\mathcal{L}_{\boldsymbol{f},\boldsymbol{g}})$.

Remark 5.1. Proving (NV) is the most challenging part of the project. We can also understand that (NV) is about creating the non-trivial Yoshida congruence. The idea is to use Zheng Liu's construction of 4-variable p-adic L-functions for $\mathrm{GSp}(4) \times \mathrm{GL}(2)$ via Furusawa's pull-back formula. By the explicit Petersson norm formula of p-ordinary Yoshida lifts

$$\frac{\langle \theta_{\boldsymbol{f},\boldsymbol{g}}(\kappa), \theta_{\boldsymbol{f},\boldsymbol{g}}(\kappa)\rangle_{\mathrm{Pet}}}{\|\boldsymbol{f}_{k_1}\|^2} = \eta_{\boldsymbol{g}}(k_2) \cdot L_p(\boldsymbol{f},\boldsymbol{g})(k_1, k_2),$$

Zheng Liu's four-variable p-adic L-functions yields the congruence relation

$$A(\kappa) \, (\mathrm{mod} \, \mathfrak{m}_{\Lambda}) = \eta_{\boldsymbol{f}}(k_1) \frac{\langle \mathcal{E}(\kappa), \theta_{\boldsymbol{f},\boldsymbol{g}}(\kappa)\rangle_{\mathrm{Pet}}}{\|\boldsymbol{f}_{k_1}\|^2} \, (\mathrm{mod} \, \mathfrak{m}_{\Lambda})$$

$$\approx B_{\xi}(\theta_{\boldsymbol{f}_{k_1},\boldsymbol{g}_{k_2}}(\kappa)) \cdot \frac{L(\frac{k_1}{2}+1, \boldsymbol{f}_{k_1} \otimes \theta_{\chi})}{\|\boldsymbol{f}_{k_1}\|^2} \cdot \frac{L(\frac{k_2}{2}+1, \boldsymbol{g}_{k_2} \otimes \theta_{\chi})}{\|\theta_{\chi}\|^2},$$

where χ is a Hecke character of an imaginary quadratic field with the infinity type $(m, -m)$ and $1 + k_2/2 \le m \le k_1/2$. Therefore, we can then apply known results on the non-vanishing mod p of L-values with anticyclotomic twists (Cornut-Vatsal, Hida, Chida-H. and H.) to prove that $A \in \Lambda$ and $A(\kappa)$ is a p-adic unit.

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DEPARTMENT OF MATHEMATICS, NATIONAL TAIWAN UNIVERSITY, AND NATIONAL CENTER
FOR THEORETICAL SCIENCES, NO. 1, SEC. 4, ROOSEVELT ROAD, TAIPEI 10617, TAIWAN
   E-mail address: mlhsieh@math.ntu.edu.tw