ARCHIMEDEAN ZETA INTEGRALS FOR FURUSAWA'S PULLBACK FORMULA

ZHENG LIU

Given a positive integer n, define the algebraic group $\mathrm{GSp}(2n)$ over $\mathbb{Z}$ as

$$\mathrm{GSp}(2n,R)=\left\{g\in\mathrm{GL}(2n,R):{}^{\mathrm{t}}g\begin{bmatrix}0&\mathbf{1}_n\\-\mathbf{1}_n&0\end{bmatrix}g=\nu_g\begin{bmatrix}0&\mathbf{1}_n\\-\mathbf{1}_n&0\end{bmatrix},\ \nu_g\in R^\times\right\}$$

for all $\mathbb{Z}$-algebra R. When $n=1$, we write $\mathrm{GL}(2)=\mathrm{GSp}(2)$. Fix an imaginary quadratic field $\mathcal{K}$. Define the algebraic group $\mathrm{GU}(n,n)$ over $\mathbb{Z}$ as

$$\mathrm{GU}(n,n)(R)=\left\{g\in\mathrm{GL}(2n,\mathcal{O}_\mathcal{K}\otimes R):{}^{\mathrm{t}}\bar{g}\begin{bmatrix}0&\mathbf{1}_n\\-\mathbf{1}_n&0\end{bmatrix}g=\nu_g\begin{bmatrix}0&\mathbf{1}_n\\-\mathbf{1}_n&0\end{bmatrix},\ \nu_g\in R^\times\right\},$$

where for $\alpha\in\mathcal{K}$, $\bar{\alpha}$ denotes its image under the nontrivial element in $\mathrm{Gal}(\mathcal{K}/\mathbb{Q})$.

Let $Q_{\mathrm{GU}(3,3)}\subset\mathrm{GU}(3,3))$ be the standard Siegel parabolic subgroup. Let Ξ be a Hecke character of $\mathcal{K}$ and χ be a Dirichlet character, and $I_v(s,\chi,\Xi)$ be the degenerate principal series on $\mathrm{GU}(3,3)(\mathbb{Q}_v)$ consisting of smooth functions $\mathtt{f}_v(s,\chi,\Xi):\mathrm{GU}(3,3)(\mathbb{Q}_v)\to\mathbb{C}$ such that

$$\mathtt{f}_v(s,\chi,\Xi)\left(\begin{bmatrix}\mathfrak{A}&\mathfrak{B}\\0&\mathfrak{D}\end{bmatrix}g\right)=\Xi_v(\det\mathfrak{A})\chi_v(\det\mathfrak{A}\mathfrak{D}^{-1})|\det\mathfrak{A}\mathfrak{D}^{-1}|_v^{s+\frac{3}{2}}\,\mathtt{f}_v(s,\chi,\Xi)(g)$$

for all $g\in\mathrm{GU}(3,3)(\mathbb{Q}_v)$ and $\begin{bmatrix}\mathfrak{A}&\mathfrak{B}\\0&\mathfrak{D}\end{bmatrix}\in Q_{\mathrm{GU}(3,3)}(\mathbb{Q}_v)$. If v is a finite place where χ_v,Ξ_v are unramified, we denote by $\mathtt{f}_v^{\mathrm{sph}}(s,\chi,\Xi)$ the unique section in $I_v(s,\chi,\Xi)$ invariant under the right translation by $\mathrm{GU}(3,3)(\mathbb{Z}_v)$ and taking value 1 on $\mathrm{GU}(3,3)(\mathbb{Z}_v)$. Put $I(s,\chi,\Xi)=\bigotimes_v' I_v(s,\chi,\Xi)$, the restricted tensor product with respect to $\mathtt{f}_v^{\mathrm{sph}}(s,\chi,\Xi)$. Given a section $\mathtt{f}(s,\chi,\Xi)\in I(s,\chi,\Xi)$ analytic in the variable s, the associated Siegel Eisenstein series is defined as

$$E^{\mathrm{Sieg}}(g;\mathtt{f}(s,\chi,\Xi))=\sum_{\gamma\in Q_{\mathrm{GU}(3,3)}(\mathbb{Q})\backslash\mathrm{GU}(3,3)(\mathbb{Q})}\mathtt{f}(s,\chi,\Xi)(\gamma g).$$

Let Π (resp. π) be a cuspidal automorphic representation of $\mathrm{GSp}(4,\mathbb{A}_\mathbb{Q})$ (resp. $\mathrm{GL}(2,\mathbb{A}_\mathbb{Q})$). Given $f\in\pi$, thanks to the isomorphism

$$\begin{aligned}\big(\mathrm{Res}_{\mathcal{K}/\mathbb{Q}}\mathrm{GL}(1)\big)\times\mathrm{GL}(2)/\{(a,a^{-1}\cdot\mathbf{1}_2):a\in\mathrm{GL}(1)\}&\xrightarrow{\sim}\mathrm{GU}(1,1)\\(\mathfrak{a},g)&\longmapsto\mathfrak{a}g,\end{aligned}$$

by choosing a Hecke character $\Upsilon:\mathcal{K}^\times\backslash\mathbb{A}_\mathcal{K}^\times\to\mathbb{C}^\times$ with $\Upsilon_\mathbb{Q}=\Upsilon|_{\mathbb{A}_\mathbb{Q}^\times}=\omega_\pi$, we can define an automrorphic form f^Υ on $\mathrm{GU}(1,1)$ as

$$f^\Upsilon(\mathfrak{a}g)=\Upsilon(\mathfrak{a})f(g),\qquad\qquad \mathfrak{a}\in\mathbb{A}_\mathcal{K}^\times,\ g\in\mathrm{GL}(2,\mathbb{A}_\mathbb{Q}).$$

The space $\pi^\Upsilon=\{f^\Upsilon:f\in\pi\}$ is an irreducible cuspidal automorphic representation of $\mathrm{GU}(1,1)(\mathbb{A}_\mathbb{Q})$.

Put

$$\mathrm{GSp}(4)\times_{\mathrm{GL}(1)}\mathrm{GU}(1,1)=\{(g,h)\in\mathrm{GSp}(4)\times\mathrm{GU}(1,1):\nu_g=\nu_h\},$$

and fix an embedding

$$\imath : \mathrm{GSp}(4) \times_{\mathrm{GL}(1)} \mathrm{GU}(1,1) \longrightarrow \mathrm{GU}(3,3)$$

(1)
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \times \begin{bmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathfrak{c} & \mathfrak{d} \end{bmatrix} \longmapsto \begin{bmatrix} A & & B & \\ & \mathfrak{a} & & \mathfrak{b} \\ C & & D & \\ & \mathfrak{c} & & \mathfrak{d} \end{bmatrix}.$$

We consider the integration of the restriction of $E^{\mathrm{Sieg}}(g; \mathtt{f}(s,\chi,\Xi))$ via $\imath$ to $\mathrm{GSp}(4) \times_{\mathrm{GL}(1)} \mathrm{GU}(1,1)$ against a pair of automorphic forms on $\mathrm{GSp}(4)$ and $\mathrm{GU}(1,1)$. By Furusawa's pullback formula [Fur93], the integral factorizes into a product of local zeta integrals and gives an integral representation for the L-function of $\Pi \times \pi$.

Theorem 1. *Let $\Lambda = (\Xi^c \Upsilon)^{-1}$ and we assume that $\Lambda_{\mathbb{Q}} = \Lambda|_{\mathbb{A}_{\mathbb{Q}}^\times} = \omega_\Pi$. Fix isomorphisms $\Pi \cong \bigotimes_v \Pi_v$, $\pi \cong \bigotimes_v \pi_v$, and suppose that the images of $\varphi \in \Pi$, $f \in \pi$ under these isomorphisms are pure tensors, denoted by $\bigotimes_v \varphi_v$, $\bigotimes_v f_v$.*

$$\begin{aligned} &\int_{[\mathrm{GSp}(4)\times_{\mathrm{GL}(1)}\mathrm{GU}(1,1)]} E^{\mathrm{Sieg}}\big(\imath(g,h); \mathtt{f}(s,\chi,\Xi)\big) \cdot \varphi(g) \cdot f^{\Upsilon}(h)\, \Xi^{-1}(\det h)\, dh\, dg \\ &= B_{\mathbb{S},\Lambda}(\varphi) W_{\mathbb{c}}(f) \cdot \prod_v Z_v\Big(\mathtt{f}_v(s,\chi,\Xi), \mathcal{B}^{\Pi_v}_{\mathbb{S},\Lambda_v}(\varphi_v), \mathcal{W}^{\pi_v,\Upsilon_v}_{\mathbb{c}}(f_v)\Big). \end{aligned}$$

Here

– $\mathbb{S} = \begin{bmatrix} \mathbb{a} & \frac{\mathbb{b}}{2} \\ \frac{\mathbb{b}}{2} & \mathbb{c} \end{bmatrix} \in \mathrm{Sym}_2(\mathbb{Q})$ *such that* $\mathcal{K} = \mathbb{Q}\left(\alpha_{\mathbb{S}} = \frac{\mathbb{b}+\sqrt{\mathbb{b}^2-4\mathbb{a}\mathbb{c}}}{2\mathbb{c}}\right)$,

–

$$B_{\mathbb{S},\Lambda}(\varphi) = \int_{[R_{\mathbb{S}}]} \Lambda_{\mathbb{S}}^{-1}(r) \cdot \varphi(r)\, dr, \qquad W_{\mathbb{c}}(f) = \int_{\mathbb{Q}\backslash\mathbb{A}_{\mathbb{Q}}} f\left(\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}\right) \cdot \psi_{\mathbb{A}_{\mathbb{Q}}}(-\mathbb{c}x)\, dx,$$

is the global Bessel period (resp. Whittater period), where

$$R_{\mathbb{S}} = \left\{ \begin{bmatrix} \imath_{\mathbb{S}}(\mathfrak{z}) & \\ & {}^{\mathrm{t}}\imath_{\mathbb{S}}(\bar{\mathfrak{z}}) \end{bmatrix} \begin{bmatrix} \mathbf{1}_2 & X \\ & \mathbf{1}_2 \end{bmatrix} : \mathfrak{z} \in \mathrm{Res}_{\mathcal{K}/\mathbb{Q}}\, \mathrm{GL}(1),\ X \in \mathrm{Sym}_2 \right\},$$

$$\imath_{\mathbb{S}}(\mathfrak{z}) = \begin{bmatrix} \alpha_{\mathbb{S}} & 1 \\ \bar{\alpha}_{\mathbb{S}} & 1 \end{bmatrix}^{-1} \begin{bmatrix} \mathfrak{z} & \\ & \bar{\mathfrak{z}} \end{bmatrix} \begin{bmatrix} \alpha_{\mathbb{S}} & 1 \\ \bar{\alpha}_{\mathbb{S}} & 1 \end{bmatrix},$$

$$\Lambda_{\mathbb{S}}\left(\begin{bmatrix} \imath_{\mathbb{S}}(\mathfrak{z}) & \\ & {}^{\mathrm{t}}\imath_{\mathbb{S}}(\bar{\mathfrak{z}}) \end{bmatrix} \begin{bmatrix} \mathbf{1}_2 & X \\ & \mathbf{1}_2 \end{bmatrix}\right) = \Lambda(\mathfrak{z})\psi_{\mathbb{A}_{\mathbb{Q}}}(\mathrm{Tr}\,\mathbb{S}X).$$

– *If φ_v, f_v are both spherical, Υ, Ξ, χ are all unramified at v, $\mathbb{S} \in M_2(\mathbb{Z}_v)$, $\mathbb{c} \in \mathbb{Z}_v^\times$, $4\det\mathbb{S}$ generates $\mathrm{disc}(\mathcal{K}_v/\mathbb{Q}_v)$, the local zeta integral $Z_v\Big(\mathtt{f}_v^{\mathrm{sph}}(s,\chi,\Xi), \mathcal{B}^{\Pi_v}_{\mathbb{S},\Lambda_v}(\varphi_v), \mathcal{W}^{\pi_v,\Upsilon_v}_{\mathbb{c}}(f_v)\Big)$ equals*

$$\left(\prod_{1\le j\le 3} L_v\left(2s+j, \Xi_{\mathbb{Q}}\chi^2\eta_{\mathcal{K}/\mathbb{Q}}^{n-j}\right)\right)^{-1} L_v\left(s+\frac{1}{2}, \tilde{\Pi}\times\tilde{\pi}\times\chi\right)$$

with $\eta_{\mathcal{K}/\mathbb{Q}}$ the quadratic character of $\mathbb{Q}^\times\backslash\mathbb{A}_{\mathbb{Q}}^\times$ associated to the extension $\mathcal{K}/\mathbb{Q}$.

– *In general, taking nonzero $B^{\Pi_v}_{\mathbb{S},\Lambda_v} \in \mathrm{Hom}_{R_{\mathbb{S}}(\mathbb{Q}_v)}(\Pi_v, \Lambda_{\mathbb{S},v})$, $W^{\pi_v}_{\mathbb{c}} \in \mathrm{Hom}_{U_{\mathrm{GL}(2)}(\mathbb{Q}_v)}(\pi_v, \psi_{\mathbb{c},v})$ and putting $\mathcal{B}^{\Pi_v}_{\mathbb{S},\Lambda_v}(\varphi_v)$ the function on $\mathrm{GSp}(4,\mathbb{Q}_v)$ sending g_v to $B^{\Pi_v}_{\mathbb{S},\Lambda_v}(g_v\cdot\varphi_v)$, $\mathcal{W}^{\pi_v,\Upsilon_v}_{\mathbb{c}}(f_v)$*

the function on $\mathrm{GU}(1,1)(\mathbb{Q}_v)$ *sending* $\mathfrak{a}g$, $\mathfrak{a}\in\mathcal{K}, g\in\mathrm{GL}(2,\mathbb{Q}_v)$, *to* $\Upsilon_v(\mathfrak{a})\cdot W^{\pi_v}_{\mathbb{c}}(g\cdot f_v)$,

$$
\begin{aligned}
&Z_v\Big(\mathtt{f}_v(s,\chi,\Xi),\mathcal{B}^{\Pi_v}_{\mathbb{S},\Lambda_v}(\varphi_v),\mathcal{W}^{\pi_v,\Upsilon_v}_{\mathbb{c}}(f_v)\Big)\\
(2)\qquad &= B^{\Pi_v}_{\mathbb{S},\Lambda_v}(\varphi_v)^{-1}W^{\pi_v}_{\mathbb{c}}(f_v)^{-1}\int_{\left(R'_{\mathbb{S}}\backslash\,\mathrm{GSp}(4)\times_{\mathrm{GL}(1)}\mathrm{GU}(1,1)\right)(\mathbb{Q}_v)}\mathtt{f}_v(s,\chi,\Xi)\left(\mathcal{S}^{-1}\imath(\eta_{\mathbb{S}}\,g,h)\right)\\
&\qquad\times\mathcal{B}^{\Pi_v}_{\mathbb{S},\Lambda_v}(\varphi_v)(g)\mathcal{W}^{\pi_v,\Upsilon_v}_{\mathbb{c}}(f_v)\left(\begin{bmatrix}0&1\\-1&0\end{bmatrix}h\right)\Xi_v^{-1}(\det h)\,dhdg,
\end{aligned}
$$

with

$$
\eta_{\mathbb{S}}=\begin{bmatrix}1&&&\\ \alpha_{\mathbb{S}}&1&&\\ &&1&-\bar{\alpha}_{\mathbb{S}}\\ &&&1\end{bmatrix},\qquad
\mathcal{S}=\begin{bmatrix}1&&&&&\\ &1&&&&\\ &&1&&&\\ &&&1&&\\ &&1&&1&\\ &1&&&&1\end{bmatrix}.
$$

$$
R'_{\mathbb{S}}=\left\{\left(\begin{bmatrix}\imath_{\mathbb{S}}(\mathfrak{z})&\\&{}^{\mathrm{t}}\imath_{\mathbb{S}}(\bar{\mathfrak{z}})\end{bmatrix}\begin{bmatrix}\mathbf{1}_2&X\\&\mathbf{1}_2\end{bmatrix},\mathfrak{z}\cdot\mathbf{1}_2\right):\mathfrak{z}\in\mathrm{Res}_{\mathcal{K}/\mathbb{Q}}\mathrm{GL}(1),\ X\in\mathrm{Sym}_2\right\}.
$$

We consider the computation of the local zeta integral (2) at the archimedean place in a case that is of arithmetic interest. Suppose that χ_∞,Ξ_∞ are both trivial on $\mathbb{R}_{>0}$ and the ∞-type of Ξ_∞ is $\left(\frac{r}{2},-\frac{r}{2}\right)$. Given a positive integer t, we define the weight-t-classical section $\mathtt{f}^t_\infty(s,\chi,\Xi)$ as

$$
g=\begin{bmatrix}\mathfrak{A}&\mathfrak{B}\\ \mathfrak{C}&\mathfrak{D}\end{bmatrix}\longmapsto\chi_\infty(\nu_g)\nu_g^{\frac{3}{2}(r-t)}|\nu_g|^{\frac{3}{2}(s+\frac{3}{2}-r)}(\det g)^{\frac{r+t}{2}}\det(\mathfrak{C}i+\mathfrak{D})^{-t}|\det(\mathfrak{C}i+\mathfrak{D})|^{-2s-3+t}.
$$

(This section is of scalar weight $(\frac{t+r}{2},\frac{t+r}{2},\frac{t+r}{2};\frac{-t+r}{2},\frac{-t+r}{2},\frac{-t+r}{2})$.) Denote the ∞-type of Λ by $(\frac{r_\Lambda}{2},-\frac{r_\Lambda}{2})$. We consider the evaluation of (2), with $v=\infty$, at $s=k+\frac{l_1+l_2+l-1}{2}$ in case where

- $\Pi_\infty=\mathcal{D}_{l_1,l_2}$ (resp. $\pi_\infty\cong\mathcal{D}_l$), the holomorphic discrete series of weight (l_1,l_2) (resp. weight l) satisfying the condition

$$
(3)\qquad \min\{-l_1+l_2+l,\ l_1+l_2-l\}\geq 3,
$$

  and $\varphi_\infty\in\Pi_\infty$ (resp. $f_\infty\in\pi_\infty$) is a vector of weight (l_1,l_2) (resp. weight l) inside the lowest K_∞-type (where K_∞ is the connected component of the identity inside the maximal compact subgroup of $\mathrm{GSp}(4,\mathbb{R})$ (resp. $\mathrm{GL}(2,\mathbb{R})$)),
- $-l_1+l_2\leq r_\Lambda\leq l_1-l_2$,
- the section $\mathtt{f}_\infty(s,\chi,\Xi)$ is given as

$$
D_{k,l,l_1,l_2,r_\Lambda}\cdot\mathtt{f}_\infty^{|2k+l_1+l_2+l-1|+3}(s,\chi,\Xi)
$$

  with

$$
\begin{aligned}
D_{t,r_\Lambda,l_1,l_2,l}=(2\pi i)^{\frac{3(|2k+l_1+l_2+l-1|+3)-l_1-l_2-l}{2}}&\cdot\left(\mu^+_{\mathrm{GU}(3,3),31}\right)^{\frac{l_1-l_2+r_\Lambda}{2}}\left(\mu^+_{\mathrm{GU}(3,3),13}\right)^{\frac{l_1-l_2-r_\Lambda}{2}}\\
\times\det&\begin{bmatrix}\mu^+_{\mathrm{GU}(3,3),13}&\mu^+_{\mathrm{GU}(3,3),23}\\ \mu^+_{\mathrm{GU}(3,3),31}&\mu^+_{\mathrm{GU}(3,3),32}\end{bmatrix}^{\frac{-l_1+l_2+l-(|2k+l_1+l_2+l-1|+3)}{2}}\\
\times&\left(\frac{\mu^+_{\mathrm{GU}(3,3),21}-\mu^+_{\mathrm{GU}(3,3),12}}{2}\right)^{\frac{l_1+l_2-l-(|2k+l_1+l_2+l-1|+3)}{2}},
\end{aligned}
$$

a differential operator constructed as a polynomial in the Lie algebra operators $\mu^+_{\mathrm{GU}(3,3),ij} \in \mathrm{Lie}\,\mathrm{GU}(3,3)(\mathbb{R})) \otimes_{\mathbb{R}} \mathbb{C}$ defined as

$$\mu^+_{\mathrm{GU}(3,3),ij} = \mu^+_{\mathrm{GU}(3,3),X}, \qquad X = \frac{E_{ij}+E_{ji}}{2} \otimes 1 + \frac{E_{ij}-E_{ji}}{2\sqrt{-1}} \otimes \sqrt{-1}$$
$$\mu^+_{\mathrm{GU}(3,3),X} = \begin{bmatrix} X & \\ & -X \end{bmatrix} \otimes 1 + \begin{bmatrix} & X \\ X & \end{bmatrix} \otimes \sqrt{-1}, \qquad X \in \mathrm{Her}_3(\mathcal{K} \otimes_{\mathbb{Q}} \mathbb{R}).$$

Theorem 2. *In the case described as above,*

$$Z_\infty\Big(D_{k,l,l_1,l_2,r_\Lambda} \cdot \mathtt{f}_\infty^{|2k+l_1+l_2+l-1|+3}(s,\chi,\Xi), \mathcal{B}^{\mathcal{D}_{l_1,l_2}}_{\mathbb{S},\Lambda_\infty}\left(\left[\begin{smallmatrix} & \mathbf{1}_2 \\ \mathbf{1}_2 & \end{smallmatrix}\right]\varphi_\infty\right), \mathcal{W}^{\mathcal{D}_l,\Upsilon_\infty}_{\mathbb{c}}\left(\left[\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right]\right) f_\infty)\Big)\Big|_{s=k+\frac{l+l_1+l_2-1}{2}}$$
$$=\Gamma(k+l_1+l_2+l-2)\Gamma(k+l_1+l_2-1)\Gamma(k+l_2+l-1)\Gamma(k+l_1+l)$$
$$\times\, \mathbb{c}^{k+l_1+1}(\det\mathbb{S})^{-k-l_1-\frac{l_2+l+1}{2}} \begin{cases} \frac{2^{-3l_1-2l_2-2l+10}\pi^{2k+7}i^{2k+l_1+2l_2-2l-r_\Lambda-1}}{\prod_{j=0}^2 \Gamma(2k+l_1+l_2+l+2-j)}, & k+\frac{l_1+l_2+l}{2} \geq \frac{1}{2} \\ 2^{-12k-9l_1-8l_2-8l+15}\pi^{-4k-3l_1-3l_2-3l+10}i^{2k-l_1+l-r_\Lambda}, & k+\frac{l_1+l_2+l}{2} \leq \frac{1}{2} \end{cases}$$

The proof makes use of the existence of a four-variable p-adic L-function with partial interpolation properties. We first introduce some notation that will be used in the proof. We fix an odd prime p and a sufficiently large finite extension F of $\mathbb{Q}_p$. Denote by $\mathcal{O}$ the ring of integers of F. Let

$$T_{\mathrm{GSp}(2n)} = \left\{\mathrm{diag}(a_1,\cdots,a_n,\nu a_1^{-1},\cdots,\nu a_n^{-1}) \in \mathrm{GSp}(2n)\right\},$$
$$T^1_{\mathrm{GSp}(2n)} = T_{\mathrm{GSp}(2n)} \cap \mathrm{Sp}(2n).$$

and

$$\widetilde{\Lambda}_{\mathrm{GSp}(2n)} = \mathcal{O}[\![T^1_{\mathrm{GSp}(2n)}(\mathbb{Z}_p)]\!].$$

Given $m_1 \geq m_2 \geq \cdots \geq m_n \geq 0$, we define the $\mathbb{U}_p$-operator $U_{p,m_1,\ldots,m_n}$ as

$$\int_{U_{\mathrm{GSp}(2n)}(\mathbb{Z}_p)} \text{action by } u\,\mathrm{diag}(p^{m_1},\ldots,p^{m_n},p^{-m_1},\ldots,p^{-m_n})\,du.$$

We define the following p-modified Petersson inner products

$$\mathbf{P}: \mathcal{A}^0_{\mathrm{GSp}(4),\omega,\mathbb{U}_p\neq 0} \times \mathcal{A}_{\mathrm{GSp}(4),\omega} \longrightarrow \mathbb{C}, \qquad \mathbf{P}: \mathcal{A}^0_{\mathrm{GL}(2),\omega,\mathbb{U}_p\neq 0} \times \mathcal{A}_{\mathrm{GL}(2),\omega} \longrightarrow \mathbb{C},$$

on the product of the subspace of cuspidal automorphic forms on GSp(4) (resp. GL(2)) of central character ω on which $\mathbb{U}_p$-operators act invertibly and the whole space of automorphic forms on GSp(4) of central character ω (resp. GL(2)) as

$$\mathbf{P}(\varphi_1,\varphi_2) = \int_{[\mathrm{GSp}(4)]} U^{-1}_{p,m_1,m_2}\varphi_1(g) \times \varphi_2\left(g\begin{bmatrix} & \mathbf{1}_2 \\ \mathbf{1}_2 & \end{bmatrix}_{p\infty}\begin{bmatrix} p^{m_1} & & & \\ & p^{m_2} & & \\ & & p^{-m_1} & \\ & & & p^{-m_2}\end{bmatrix}_p\right) \cdot \omega(\nu_g)^{-1}\,dg, \qquad m_1 \gg m_2 \gg 0$$

$$\mathbf{P}(f_1,f_2) = \int_{[\mathrm{GL}(2)]} U^{-1}_{p,m} f_1(g)\, f_2\left(g\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}_{p\infty}\begin{bmatrix} p^m & \\ & p^{-m}\end{bmatrix}_p\right) \cdot \omega(\det g)^{-1}\,dg, \qquad m \gg 0.$$

One can check that the right hand sides of the above formulas are independent of m_1, m_2, m, and if φ_1, φ_2 (resp. f_1, f_2) both belong to $\mathcal{A}^0_{\mathrm{GSp}(4),\omega,\mathbb{U}_p\neq 0}$ (resp. $\mathcal{A}^0_{\mathrm{GL}(2),\omega,\mathbb{U}_p\neq 0}$, then $\mathbf{P}(\varphi_1,\varphi_2) = \mathbf{P}(\varphi_2,\varphi_1)$, $\mathbf{P}(f_1,f_2) = \mathbf{P}(f_2,f_1)$.

Prood of Theorem 2. Choose $\mathbb{I}_{\mathscr{B}}$-adic and $\mathbb{I}_{\mathscr{B}'}$-adic Hida families $\mathscr{B},\mathscr{B}'$ on GL(2) of tame level $K^p_{\mathrm{GL}(2)}$ such that both admit a Jacquet–Langlands transfer to Hida families of a quaternion algebra D definite at ∞. We also require $\mathscr{B}$ to be primitive. Define

$$\lambda_{\theta(\mathscr{B},\mathscr{B}')}:\mathbb{T}_{\mathrm{GSp}(4),p}\longrightarrow \mathbb{I}_{\theta(\mathscr{B},\mathscr{B}')}=\widetilde{\Lambda}_{\mathrm{GSp}(4)}\otimes_{\widetilde{\Lambda}_{\mathrm{GL}(2)}\hat{\otimes}\widetilde{\Lambda}_{\mathrm{GL}(2)}}(\mathbb{I}_{\mathscr{B}}\widehat{\otimes}\mathbb{I}_{\mathscr{B}'})$$

such that the central character equals $\omega_{\mathscr{B}}\otimes\omega_{\mathscr{B}}$ and

$$\lambda_{\theta(\mathscr{B},\mathscr{B}')}\left(\mathrm{GSp}(4,\mathbb{Z}_v)\begin{bmatrix}\varpi_v&&&\\&\varpi_v&&\\&&1&\\&&&1\end{bmatrix}\mathrm{GSp}(4,\mathbb{Z}_v)\right)=\lambda_{\mathscr{B}}\left(\mathrm{GL}(2,\mathbb{Z}_v)\begin{bmatrix}\varpi_v&\\&1\end{bmatrix}\mathrm{GL}(2,\mathbb{Z}_v)\right),$$

$$\lambda_{\theta(\mathscr{B},\mathscr{B}')}\left(\mathrm{GSp}(4,\mathbb{Z}_v)\begin{bmatrix}\varpi_v&&&\\&1&&\\&&1&\\&&&\varpi_v\end{bmatrix}\mathrm{GSp}(4,\mathbb{Z}_v)\right)=\omega_{\mathscr{B}}^{\frac{1}{2}}\omega_{\mathscr{B}'}^{-\frac{1}{2}}(\varpi_v)\lambda_{\mathscr{B}'}\left(\mathrm{GL}(2,\mathbb{Z}_v)\begin{bmatrix}\varpi_v&\\&1\end{bmatrix}\mathrm{GL}(2,\mathbb{Z}_v)\right),$$

for places v where $K^p_{\mathrm{GL}(2),v}=\mathrm{GL}_2(\mathbb{Z}_v)$ and $D(\mathbb{Q}_v)\cong\mathrm{GL}(2,\mathbb{Q}_v)$, and $\mathbb{T}_{\mathrm{GSp}(4),p}$ is the Hecke algebra containing unramified Hecke operators at such places and $\mathbb{U}_p$-operators at p. Here we view $\widetilde{\Lambda}_{\mathrm{GL}(2)}\hat{\otimes}\widetilde{\Lambda}_{\mathrm{GL}(2)}$ as a subslgebra of $\widetilde{\Lambda}_{\mathrm{GSp}(4)}$ via the embedding induced by

$$\begin{aligned}T^1_{\mathrm{GL}(2)}\times T^1_{\mathrm{GL}(2)}&\longrightarrow T^1_{\mathrm{GSp}(4)},\\ \big(\mathrm{diag}(a_1,a_1^{-1}),\mathrm{diag}(a_2,a_2^{-1})\big)&\longmapsto \mathrm{diag}(a_1a_2,a_1a_2^{-1},a_1^{-1}a_2^{-1},a_1^{-1}a_2).\end{aligned}$$

We can take a tame level $K^p_{\mathrm{GSp}(4)}$ with $\det(K^p_{\mathrm{GSp}(4)})=\widehat{\mathbb{Z}}_p^\times$ such that at a set of classical points of $\mathscr{B},\mathscr{B}'$ that is Zariski dense, there are Yoshida lifts from $D^\times\times_{\det}D^\times$ to GSp(4) belonging to the Hecke eigensystem $\lambda_{\theta(\mathscr{B},\mathscr{B}')}$. Then the spectrum of $\mathbb{T}_{\mathrm{GSp}(4),\mathrm{ord}}$, the Hecke algebra acting on Hida families on GSp(4) of tame level $K^p_{\mathrm{GSp}(4)}$, has an irreducible component $\theta(\mathscr{B},\mathscr{B}')$ with $\lambda_{\theta(\mathscr{B},\mathscr{B}')}$ as the corresponding Hecke eigensystem. Choose another $\mathbb{I}_{\mathscr{C}}$-adic primitive Hida family $\mathscr{C}$ of tame level $K^{p,\prime}_{\mathrm{GL}(2)}$. The convention for p-nebentypus we use for the Hida families is

$$\begin{gathered}\begin{bmatrix}a&*\\&d\end{bmatrix}\longmapsto[\mathrm{diag}(a,a^{-1})]\in\mathcal{O}[\![T^1_{\mathrm{GL}(2)}(\mathbb{Z}_p)]\!],\\ \begin{bmatrix}a_1&*&*&*\\&a_2&*&*\\&&a_1^{-1}\nu&\\&&*&a_2^{-1}\nu\end{bmatrix}\longmapsto[\mathrm{diag}(a_1,a_2,a_1^{-1},a_2^{-1})]\in\mathcal{O}[\![T^1_{\mathrm{GSp}(4)}(\mathbb{Z}_p)]\!].\end{gathered}\tag{4}$$

We fix a finite set S containing p,∞ and all places where there is ramification.

By constructing a p-adic family of Siegel Eisenstein series on GU(3, 3) [Liu24], we can show that there exists

$$\mu^S_{\mathscr{C},\theta(\mathscr{B},\mathscr{B}')}\in\mathscr{M}eas\left(\mathbb{Z}_p^\times,\mathcal{M}'_{\mathrm{GL}(2),\mathrm{ord}}\widehat{\otimes}_{\mathcal{O}}\mathcal{M}_{\mathrm{GSp}(4),\mathrm{ord}}\right)\otimes_{\tilde{\Lambda}_{\mathrm{GL}(2)}\hat{\otimes}_{\mathcal{O}}\tilde{\Lambda}_{\mathrm{GSp}(4)}}(F_{\mathscr{C}}\hat{\otimes}_{\mathcal{O}}F_{\theta(\mathscr{B},\mathscr{B}')}),$$

where $\mathcal{M}'_{\mathrm{GL}(2),\mathrm{ord}}$ (resp. $\mathcal{M}_{\mathrm{GSp}(4),\mathrm{ord}}$ denotes the space of Hida families of tame level $K^{p,\prime}_{\mathrm{GL}(2)}$ (resp. $K^{p}_{\mathrm{GSp}(4)}$) on GL(2) (resp. GSp(4)), and $F_{\mathscr{C}} = \mathrm{Frac}(\mathbb{I}_{\mathscr{C}})$,$F_{\theta(\mathscr{B},\mathscr{B}')} = \mathrm{Frac}(\mathbb{I}_{\theta(\mathscr{B},\mathscr{B}')})$, such that

$$\begin{aligned}(5)\qquad &\left(\int_{\mathbb{Z}_p^\times} \chi(y)y^k \, d\mu^S_{\mathscr{C},\theta(\mathscr{B},\mathscr{B}')}(y)\right)(x)\\ &= \mathbb{c}\sqrt{\det \mathbb{S}}\, 2^{-3l+2} i^l \, \frac{D^S\left(k+\frac{3l}{2}, \sigma_x \times \pi_x \times \chi\right) D^S\left(k+\frac{3l}{2}, \pi_x \times \sigma'_x \times \omega_{\sigma_x}^{\frac{1}{2}}\omega_{\sigma'_x}^{-\frac{1}{2}} \times \chi\right)}{\mathbf{P}(f_x,f_x)}\\ &\times f_{x,\mathbb{c}} f_x \sum_{\varphi\in\mathscr{S}_{\mathrm{GSp}(4),x}} \frac{B^\dagger_{\mathbb{S},\Lambda}(\varphi)\varphi}{\mathbf{P}(\varphi,\varphi)}\end{aligned}$$

where

- $x \in \mathbb{I}_{\theta(\mathscr{B},\mathscr{B}')}(\overline{\mathbb{Q}}_p) \times \mathbb{I}_{\mathscr{C}}(\overline{\mathbb{Q}}_p)$ is a classical point with weight (l,l), l étale over the weight space, where $l \geq 3$ and $-2l+2 \leq k \leq -l-1$
- π_x (resp. σ_x, σ'_x) is an irreducible automorphic representation of $\mathrm{GL}(2,\mathbb{A}_{\mathbb{Q}})$ generated by a form belonging to the eigensystem parameterized by the projection of x in $\mathscr{C}$ (resp. $\mathscr{B},\mathscr{B}'$), and f_x is a nonzero p-ordinary form in $\pi_x^{K^{p,\prime}_{\mathrm{GL}(2)}}$ with $f_{x,\mathbb{c}}$ the $\mathbb{c}$-th coefficient in its q-expansion,
- $\mathscr{S}_{\mathrm{GSp}(4),x}$ is an orthogonal basis, with respect to the above defined modified Petersson inner product $\mathbf{P}(\cdot,\cdot)$, of the space spanned by ordinary cuspidal holomorphic forms on GSp(4) of weight $(l_1,l_2)=(l,l)$ and tame level $K^p_{\mathrm{GSp}(4)}$ with nebentypus at p given by (4) specialized at the character associated to the projection of x inside the weight space, belonging to the Hecke eigenspace parameterized by x,
- $B^\dagger_{\mathbb{S},\Lambda}$ denotes the p-modified Bessel period defined as

$$B^\dagger_{\mathbb{S},\Lambda}(\varphi) = \lambda_{p,m_1,m_2}(\varphi)^{-1} \cdot B_{\mathbb{S},\Lambda}\left(\begin{bmatrix} p^{m_1} & & & \\ & p^{m_2} & & \\ & & p^{-m_1} & \\ & & & p^{-m_2}\end{bmatrix}_p \varphi\right), \qquad m_1 \gg m_2 \gg 0.$$

  One can check that the right hand does not depend on the choice of m_1, m_2 [Liu23, Proposition 2.7.1],
- $D^S(s,\cdot)$ denotes the product of the S-imprimitive L-function and the modified Euler factors at p,∞ for p-adic interpolation (as defined in [CPR89, Coa91]),
- $\mathbb{S} \in \mathrm{Sym}_2(\mathbb{Q}) > 0$ is chosen such that $f_{x,\mathbb{c}}$, $B^\dagger_{\mathbb{S},\Lambda}(\varphi)$ are nonzero at some x, $\varphi \in \mathscr{S}_{\mathrm{GSp}(4),x}$.

For more general classical points $x \in \mathbb{I}_{\theta(\mathscr{B},\mathscr{B}')}(\overline{\mathbb{Q}}_p) \times \mathbb{I}_{\mathscr{C}}(\overline{\mathbb{Q}}_p)$ with weight (l_1, l_2), l étale over the weight space and k satisfying (3), we have

(6)

$$
\begin{aligned}
&\left(\int_{\mathbb{Z}_p^\times} \chi(y) y^k \, d\mu^S_{\mathscr{C},\theta(\mathscr{B},\mathscr{B}')}(y)\right)(x) \\
&= i^{r_\Lambda} \frac{L^S\left(k + \frac{l_1+l_2+l}{2}, \sigma_x \times \pi_x \times \chi\right) L^S\left(k + \frac{l_1+l_2+l}{2}, \pi_x \times \sigma'_x \times \omega_{\sigma_x}^{\frac{1}{2}} \omega_{\sigma'_x}^{-\frac{1}{2}} \times \chi\right)}{\mathbf{P}(f_x, f_x)} \\
&\times E_p\left(k + \frac{l_1+l_2+l}{2}, \sigma_x \times \pi_x \times \chi\right) E_p\left(k + \frac{l_1+l_2+l}{2}, \pi_x \times \sigma'_x \times \omega_{\sigma_x}^{\frac{1}{2}} \omega_{\sigma'_x}^{-\frac{1}{2}} \times \chi\right) \\
&\times \mathbb{c}^{k+l+l_1+l_2} \left(-(\alpha_{\mathbb{S}} - \bar{\alpha}_{\mathbb{S}})^2\right)^{k+l+\frac{l_1+l_2}{2}} \begin{cases} \frac{\prod_{j=0}^2 \Gamma(2k+l_1+l_2+l+2-j)}{2^{3(2k+l_1+l_2+l)} \pi^{3(2k+l_1+l_2+l+1)} i^{2k+l_1+l_2+l+2}}, & k + \frac{l_1+l_2+l}{2} \geq \frac{1}{2} \\ 2^{3(2k+l_1+l_2+l-2)+1} \pi^{-6} i^{2k+l_1+l_2+l}, & k + \frac{l_1+l_2+l}{2} \leq \frac{1}{2} \end{cases} \\
&\times (\alpha_{\mathbb{S}} - \bar{\alpha}_{\mathbb{S}})^{l_1-l} Z_{k,l,l_1,l_2,r_\Lambda} \cdot f_{x,\mathbb{c}} f_x \sum_{\varphi \in \mathscr{S}_{\mathrm{GSp}(4),x}} \frac{B^\dagger_{\mathbb{S},\Lambda}(\varphi)\varphi}{\mathbf{P}(\varphi,\varphi)}
\end{aligned}
$$

with $Z_{k,l,l_1,l_2,r_\Lambda}$ the left hand side of the formula stated in Theorem 2.

On the other hand, by the results on p-adic L-functions for Dirichlet characters, Rankin–Selberg products [Hid88] (Theorem 5.1d.) [CH20, Theorem A] and symplectic groups [Liu20], there exist

$$
\mathcal{L}^{S,*}_{\mathscr{B},\mathscr{C}}, \mathcal{L}^{S,*}_{\mathscr{C},\mathscr{B}'} \in \mathcal{M}eas\left(\mathbb{Z}_p^\times, \Lambda_{\mathrm{GL}(2)} \widehat{\otimes}_{\mathcal{O}} \Lambda_{\mathrm{GSp}(4)}\right) \otimes_{\widetilde{\Lambda}_{\mathrm{GL}(2)} \hat{\otimes}_{\mathcal{O}} \widetilde{\Lambda}_{\mathrm{GSp}(4)}} \left(F_{\mathscr{C}_1} \hat{\otimes}_{\mathcal{O}} F_{\theta(\mathscr{B},\mathscr{B}')}\right)
$$

and

$$
\mathcal{F}_{\theta(\mathscr{B},\mathscr{B}')} \in \left(\mathcal{M}_{\mathrm{GSp}(4),\mathrm{ord}} \otimes_{\widetilde{\Lambda}_{\mathrm{GSp}(4)}} \mathcal{M}_{\mathrm{GSp}(4),\mathrm{ord}}\right) \otimes_{\widetilde{\Lambda}_{\mathrm{GSp}(4)}} F_{\theta(\mathscr{B},\mathscr{B}')})
$$

satisfying the following interpolation properties: for $x \in \mathbb{I}_{\theta(\mathscr{B},\mathscr{B}')}(\overline{\mathbb{Q}}_p) \times \mathbb{I}_{\mathscr{C}}(\overline{\mathbb{Q}}_p)$ at which (6) is proved,

$$
\mathcal{L}^{S,*}_{\mathscr{B},\mathscr{C}}(\kappa, x) = \frac{D^S\left(k + \frac{l+l_1+l_2}{2}, \sigma_x \times \pi_x \times \chi\right)}{(-2i)^{l_1+l_2-1} \mathbf{P}(h_x, h_x)} \tag{7}
$$

$$
\mathcal{L}^{S,*}_{\mathscr{C},\mathscr{B}'}(\kappa, x) = \frac{D^S\left(k + \frac{l+l_1+l_2}{2}, \pi_x \times \sigma'_x \times \omega_{\sigma_x}^{\frac{1}{2}} \omega_{\sigma'_x}^{-\frac{1}{2}} \chi\right)}{(-2i)^{l+1} \mathbf{P}(f_x, f_x)} \tag{8}
$$

$$
\mathcal{F}_{\theta(\mathscr{B},\mathscr{B}')}(x) = 2^{-1} i^{-l_1-l_2+1} \mathbf{P}(h_x, h_x) \sum_{\varphi \in \mathscr{S}_{\mathrm{GSp}(4),x}} \frac{\varphi \boxtimes \varphi}{\mathbf{P}(\varphi,\varphi)}
$$

where $h_x \in \sigma_x$ the unique normalized ordinary form fixed by $K^p_{\mathrm{GL}(2)}$. (Note that the weights of the archimedean components of σ_x, σ'_x are $l_1 + l_2 - 2, l_1 - l_2 + 2$.) By p-adically interpolating the Bessel periods, from $\mathcal{F}_{\theta(\mathscr{B},\mathscr{B}')}$, one can construct

$$
\mathcal{F}_{\theta(\mathscr{B},\mathscr{B}'),\mathbb{S},\mathbf{\Lambda}} \in \mathcal{M}_{\mathrm{GSp}(4),\mathrm{ord}} \otimes_{\widetilde{\Lambda}_{\mathrm{GSp}(4)}} F_{\theta(\mathscr{B},\mathscr{B}')}
$$

such that for almost all x as above,

$$
\mathcal{F}_{\theta(\mathscr{B},\mathscr{B}'),\mathbb{S},\mathbf{\Lambda}}(x) = 2^{-1} i^{-l_1-l_2+1} \mathbf{P}(h_x, h_x) \sum_{\varphi \in \mathscr{S}_{\mathrm{GSp}(4),x}} \frac{B^\dagger_{\mathbb{S},\Lambda}(\varphi)\varphi}{\mathbf{P}(\varphi,\varphi)}. \tag{9}
$$

Denote by $\boldsymbol{f}$ the primitive Hida family associated to $\mathscr{C}$. Combining (5)(7)(8)(9), we see that

$$\mathbb{c}\sqrt{\det \mathbb{S}}2^3 i^{-1} \cdot \mathcal{L}^{S,*}_{\mathscr{B},\mathscr{C}} \mathcal{L}^{S,*}_{\mathscr{C},\mathscr{B}'} \cdot \mathcal{F}_{\theta(\mathscr{B},\mathscr{B}'),\mathbb{S},\boldsymbol{\Lambda}} \cdot \boldsymbol{f}_{\mathbb{c}}\boldsymbol{f}$$
$$\in \mathscr{M}eas\left(\mathbb{Z}_p^\times, \mathcal{M}'_{\mathrm{GL}(2),\mathrm{ord}} \widehat{\otimes}_{\mathcal{O}} \mathcal{M}_{\mathrm{GSp}(4),\mathrm{ord}}\right) \otimes_{\tilde{\Lambda}_{\mathrm{GL}(2)}\hat{\otimes}_{\mathcal{O}}\tilde{\Lambda}_{\mathrm{GSp}(4)}} \left(F_{\mathscr{C}} \hat{\otimes}_{\mathcal{O}} F_{\theta(\mathscr{B},\mathscr{B}')}\right)$$

have the same evaluation as $\mu^S_{\mathscr{C},\theta(\mathscr{B},\mathscr{B}')}$ at points for which (5) are proved. The set of such points is Zariski dense in $\mathbb{I}_{\mathscr{C}} \hat{\otimes} \mathbb{I}_{\theta(\mathscr{B},\mathscr{B}')}[\![\mathbb{Z}_p^\times]\!]$. Thus, we have

$$\mu^S_{\mathscr{C},\theta(\mathscr{B},\mathscr{B}')} = \mathbb{c}\sqrt{\det \mathbb{S}}\, 2^3 i^{-1} \cdot \mathcal{L}^{S,*}_{\mathscr{B},\mathscr{C}} \mathcal{L}^{S,*}_{\mathscr{C},\mathscr{B}'} \cdot \mathcal{F}_{\theta(\mathscr{B},\mathscr{B}'),\mathbb{S},\boldsymbol{\Lambda}} \cdot \boldsymbol{f}_{\mathbb{c}}\boldsymbol{f}.$$

Comparing their valuations at points for which (6) is proved, (noting that for all $k, l, (l_1, l_2)$ considered in the theorem we want to prove here, there exist such points with $k, l, (l_1, l_2)$ as the algebraic part of their projections in the weight space), we get that

$$\begin{aligned} Z_{k,l,l_1,l_2,r_\Lambda} &= \mathbb{c}^{-k-l-l_1-l_2+1} 2^{-l_1-l_2-l+2} i^{-r_\Lambda-l_1+l} \left(-(\alpha_{\mathbb{S}} - \bar{\alpha}_{\mathbb{S}})^2\right)^{-k-l_1-\frac{l_2+l}{2}} \sqrt{\det \mathbb{S}} \\ &\times \begin{cases} \frac{2^{3(2k+l_1+l_2+l)} \pi^{3(2k+l_1+l_2+l+1)} i^{2k+l_1+l_2+l+2}}{\prod_{j=0}^2 \Gamma(2k+l_1+l_2+l+2-j)}, & k + \frac{l_1+l_2+l}{2} \geq \frac{1}{2} \\ 2^{-3(2k+l_1+l_2+l-2)-1} \pi^6 i^{-(2k+l_1+l_2+l)}, & k + \frac{l_1+l_2+l}{2} \leq \frac{1}{2} \end{cases} \\ &\times E_\infty\left(k + \frac{l+l_1+l_2}{2}, \mathcal{D}_{l_1+l_2-2} \times \mathcal{D}_l\right) E_\infty\left(k + \frac{l+l_1+l_2}{2}, \mathcal{D}_l \times \mathcal{D}_{l_1-l_2+2}\right), \end{aligned}$$

and we can rewrite it as

$$\begin{aligned} Z_{k,l,l_1,l_2,r_\Lambda} &= \mathbb{c}^{k+l_1+1} (\det \mathbb{S})^{-k-l_1-\frac{l_2+l+1}{2}} \\ &\times \begin{cases} \frac{2^{4k+l_2+l+2} \pi^{6k+3l_1+3l_2+3l+3} i^{2k+l_2+2l-r_\Lambda+2}}{\prod_{j=0}^2 \Gamma(2k+l_1+l_2+l+2-j)}, & k + \frac{l_1+l_2+l}{2} \geq \frac{1}{2} \\ 2^{-8k-6l_1-5l_2-5l+7} \pi^6 i^{-2k-2l_1-l_2-r_\Lambda}, & k + \frac{l_1+l_2+l}{2} \leq \frac{1}{2} \end{cases} \\ &\times 2^{-4k-3l_1-3l_2-3l+8} \pi^{-4k-3l_1-3l_2-3l+4} i^{l_1+l_2+l} \Gamma(k+l_1+l_2+l-2)\Gamma(k+l_1+l_2-1) \\ &\times \Gamma(k+l_2+l-1)\Gamma(k+l_1+l) \end{aligned}$$

□

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