ANTICYCLOTOMIC EULER SYSTEMS FOR CM FIELDS

YU-SHENG LEE
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR

Abstract. The article surveys the author's preprint [Lee25] on a construction of an anticyclotomic Euler systems for characters of CM fields and its application toward the Iwasawa main conjecture.

Introduction

Let $\mathcal{K}$ be a CM field with maximal totally real subfield $\mathcal{F}$. Fix an odd prime $p > 0$ which is unramified in $\mathcal{F}$. We assume that $\mathcal{K}/\mathcal{F}$ is ordinary at p in the sense that every prime of $\mathcal{F}$ above p splits in $\mathcal{K}$. Under this assumption we can then fix a CM type, which is a subset $\Sigma \subset \mathrm{Hom}(\mathcal{K}, \mathbf{C})$ such that if $S_p^{\mathcal{K}}$ denote the set of primes of $\mathcal{K}$ above p and $\Sigma_p \subset S_p^{\mathcal{K}}$ is the subset of primes induced by embeddings in Σ, through a fixed choice of embeddings $\iota_\infty \colon \mathcal{K} \to \mathbf{C}$ and $\iota_p \colon \mathcal{K} \to \mathbf{C}_p$, then we have

$$\mathrm{Hom}(\mathcal{K}, \mathbf{C}) = \Sigma \sqcup \Sigma^c, \quad S_p^{\mathcal{K}} = \Sigma_p \sqcup \Sigma_p^c$$

where $\Sigma^c = c \circ \Sigma$ and $\Sigma_p^c = c \circ \Sigma_p^c$ for the complex multiplication $c \in Gal(\mathcal{K}/\mathcal{F})$.

Given a finite order character ψ of the absolute Galois group $\mathcal{G}_{\mathcal{K}}$ of $\mathcal{K}$. The Iwasawa-Greenberg main conjecture relates the following two objects associated to ψ. On the one hand, by class field theory, the Galois group of the maximal pro-p abelian extension which is unramified away p has a free part W of $\mathbf{Z}_p$-rank $d + 1 - \delta$, where $d = [\mathcal{F} : \mathbf{Q}]$ and δ is the Leopoldt defect. We let $\widetilde{K}/\mathcal{K}$ denote the extension such that $Gal(\widetilde{K}/\mathcal{K}) \cong W$. Define $\Lambda_W = \mathcal{O}[\![W]\!]$, where $\mathcal{O}$ is the ring of integers of a sufficiently large field extension E over $\mathbf{Q}_p$, and let $\mathcal{G}_{\mathcal{K}}$ acts on the Pontryagin dual Λ_W^* of which through $\psi_\Lambda = \psi\langle * \rangle$, where $\langle * \rangle \colon \mathcal{G}_{\mathcal{K}} \to W \to \Lambda^\times$ is the tautological character. Consider the Selmer group

$$\mathrm{Sel}(\psi_\Lambda, \Sigma_p) = \ker \left\{ H^1(\mathcal{K}, \Lambda_W^*(\psi_\Lambda)) \to \prod_{w \notin \Sigma_p} H^1(I_w, \Lambda_W^*(\psi_\Lambda)) \right\}$$

where w ranges through all finite primes of $\mathcal{K}$ outside Σ_p and I_w is the inertia group at w. Then the Pontryagin dual $X(\Lambda_W) \coloneqq \mathrm{Sel}(\psi, \Sigma_p)^\vee$ is a finitely generated torsion Λ-module, for which we can consider the characteristic ideal $F(\Lambda_W) \coloneqq \mathrm{char}_{\Lambda_W}(X(\Lambda_W))$.

On the other hand, fix a CM type Σ as above. By the work Katz [Kat78] and Hida-Tilouine [HT93] there exists a p-adic L-function $L_p(\psi, \Sigma_p) \in \Lambda_W$ that interpolates the algebraic part of the L-value $L(0, \psi\alpha)$, when α is a character of W such that the L-value is critical in the sense of Deligne's conjecture.

Conjecture. *The characteristic ideal $F(\Lambda_W)$ is generated by $L(\psi, \Sigma_p)$ in Λ_W.*

The conjecture can also be formulated for subextensions in $\widetilde{K}/\mathcal{K}$. But unless the subextension contains the cyclotomic $\mathbf{Z}_p$-extension, the Pontryagin dual of the Selmer group defined above could fail to be torsion.

In particular, the case when ψ is anticyclotomic and $\widetilde{\mathcal{K}}^a \subset \widetilde{K}$ is the maximal anticyclotomic subextension is considered in [HT93] and [HT94]. Here $\widetilde{\mathcal{K}}^a$ is anticyclotomic in the sense that $c \in Gal(\mathcal{K}/\mathcal{F})$ acts by the inversion on $W^- = \mathcal{G}(\widetilde{K}^a/\mathcal{K})$. In this case W^- has the $\mathbf{Z}_p$-rank $d = [\mathcal{F} : \mathbf{Q}]$. And we say ψ is anticyclotomic if $\psi(\gamma^c) = \psi^{-1}(\gamma)$ for $\gamma \in \mathcal{G}_{\mathcal{K}}$. Using the CM congruences between modulars forms, it is shown in *loc.cit.*

that the characteristic ideal $F(\Lambda_W^-)$ belongs to the ideal generated by $L_p(\psi, \Sigma_p)$, the projection of the Katz-Hida-Tilouine p-adic L-function to the anticyclotomic space $\mathcal{O}[\![W^-]\!]$. Furthermore, assume that

- The order of ψ is coprime to p.
- The prime-to-p part of the conductor of ψ is a product of primes that splits in $\mathcal{K}$.
- The restriction $\psi|_{D_w}$ is nontrivial at the decomposition group D_w of each $w \in \Sigma_p$.
- The restriction of ψ to the absolute Galois group of $\mathcal{K}(\sqrt{(-1)^{(p-1)/2}p})$ is nontrivial.

Then the crucial torsion property is obtained in [Hid06b, Thm.5.33] by relating the Selmer group to Galois deformation problems and the full main conjecture is proved in [Hid06a].

In [Lee25], we consider the same anticyclotomic main conjecture under a different set of assumptions on ψ. One of the main result is as follows.

Theorem 0.1. *Assume there exists a place $w_0 \in \Sigma_p$ of degree one and that*

($\mathcal{K}$) *$\mathcal{K}/\mathcal{F}$ is generic in the sense of [Roh82] and every prime of $\mathcal{F}$ above 2 splits in $\mathcal{K}$.*

Identify the decomposition group D_{w_0} with the absolute Galois group $\mathcal{G}_{\mathbf{Q}_p}$ of $\mathbf{Q}_p$ and let $\omega \colon \mathcal{G}_{\mathbf{Q}_p} \to (\mathbf{Z}/p\mathbf{Z})^\times$ be the Teichmuller character. Suppose ψ is ramified only at primes that split in $\mathcal{K}$ and

(ψ1) The restriction $\psi|_{\mathcal{G}_{\mathbf{Q}_p}}$ is not congruent to the trivial character $\mathbf{1}$ or $\omega^{\pm 1}$.

(ψ2) The restriction $\psi|_{D_w}$ is nontrivial at the decomposition group D_w of each $w \in \Sigma_p$.

Then $X(\Lambda^-)$ is torsion and we have $(L_p(\psi, \Sigma_p)) \subset F(\Lambda_W^-)$ in $E[\![W]\!]$.

Our proof does not rely on global Galois deformation problems and instead follows from the construction of an anticyclotomic Euler system associated to ψ. Then the general machinery from [Rub00] allows us to bound the size of the Selmer group from above and obtain the theorem. This is the reason that the inclusion relation we obtained is opposite to that of [HT94],

Congruences and Euler systems. Our construction of the Euler system follows from Urban's approach in [Urb20], which uses Eisenstein congruences to reproduce the Euler system of cyclotomic units. The idea of using congruences relations to construct cohomology classes dates back to the proof of the converse of Herbrand-Ribet theorem in [Rib76] and is also the main ingredient of [HT93].

The usual strategy goes as follows. Let $\lambda \coloneqq \mathbb{T}^{\text{ord}} \to \Lambda_W$ be a Hecke eigensystem of the big ordinary Hecke algebra $\mathbb{T}^{\text{ord}}$ such that the associated Galois representation is reducible. In favorable cases, one can find a different eigensystem λ' with irreducible Galois representation and such that λ is congruent to λ' modulo some ideal J. Then one can construct a nontrivial cohomology class over the quotient Λ_W/J.

While the strategy has succeeded in many cases, we cannot construct Euler systems, which are integral classes that value in Λ_W. The insight of Urban's method is that we should add more variables to the Hecke algebra by consider a larger space of p-adic modular forms. In our case, this is done by allowing modular forms that are not ordinary at w_0. Then under our assumptions the resulting Hecke algebra has an extra variable coming from the reducibility ideal of the local deformation at w_0.

0.1. **Outlines of the article.** In the first section we will review the theory of Hecke algebras and the associated Galois representations for the space of modular forms over a definite unitary group. In particular, we introduce the larger Hecke algebra $\mathbb{T}$ acting on modular forms that are possibly non-ordinary at w_0.

In the second section, we explain that the Hecke algebra $\mathbb{T}$ has the expected dimension by borrowing techniques from p-adic local Langlands. We then formulate an exact sequence that will be fundamental when we apply a generalization of Ribet's lemma in our setting.

The actual construction is done in the last section, using the eigensystem constructed by the author in previous work. We then explain the properties of the resulting Euler system and sketch the proof of our main theorem.

0.2. **Notations.** Let $\mathcal{K}$, $\mathcal{F}$, and p be as in the introduction and let $\epsilon = \epsilon_{cyc}$ denote the p-th cyclotomic character. We assume that $\mathcal{K}/\mathcal{F}$ is ordinary and fix a CM type Σ. Let $\mathbf{a}$ and $\mathbf{h}$ denote respectively the set of archimedean and finite places of $\mathcal{F}$. Write $I_{\mathcal{K}} = \mathrm{Hom}(\mathcal{K}, \bar{\mathbf{Q}})$ and let $c \in \mathcal{G}(\mathcal{K}/\mathcal{F})$ be the complex conjugation. For $\sigma \in I_{\mathcal{K}}$ we define $\sigma_p \coloneqq \iota_p \circ \sigma$, which determines a place of $\mathcal{K}$ above p. Conversely, if $w \in S_p^{\mathcal{K}}$, we let I_w denote the set of $\sigma \in I_{\mathcal{K}}$ such that σ_p induces w. If L is a non-archimedean local field, we normalize the Artin reciprocity map Art_L so that a uniformizer is sent to the geometric Frobenius.

We let $\mathbf{A} = \mathbf{A}_{\mathcal{F}}$ and $\mathbf{A}_{\mathcal{K}}$ be the ring of adeles and let $|\cdot|_{\mathcal{F}}$ and $|\cdot|_{\mathcal{K}}$ denote the norm on which. If $\kappa = \sum_{\sigma\in\Sigma} a_\sigma \sigma + b_\sigma \sigma c \in \mathbf{Z}[\mathcal{I}]$, for $\alpha_\infty = (\alpha_\sigma)_{\sigma\in\Sigma} \in \mathbf{A}_{\mathcal{K},\infty}^\times$ and $\alpha_p = (\alpha_w, \alpha_{\overline{w}})_{w\in\Sigma_p} \in \prod_{w|p} \mathcal{K}_w^\times$ we define

$$\alpha_\infty^\kappa = \prod_{\sigma\in\Sigma} (\alpha_\sigma)^{a_\sigma} (\bar{\alpha}_\sigma)^{b_\sigma} \in \mathbf{C}^\times, \quad \alpha_p^\kappa = \prod_{w\in\Sigma_p} \prod_{\sigma\in I_w} \sigma_p(\alpha_w)^{a_\sigma} \sigma_p(\alpha_{\overline{w}})^{b_\sigma} \in \mathbf{C}_p^\times.$$

where we identity $\sigma \in \mathcal{I}$ with embeddings into $\mathbf{C}$ and $\mathbf{C}_p^\times$ by composition with the fixed choice of embeddings ι_∞ and ι_p. We will sometimes let Σ denote the formal sum $\sum_{\sigma\in\Sigma} \sigma$ and similarly for Σ^c

We say a Hecke character χ of $\mathcal{K}^\times$ is algebraic of the infinity type κ if $\chi_\infty(\alpha) = \alpha^\kappa$. When this is the case we define the p-adic avatar $\hat{\chi}\colon \mathbf{A}_{\mathcal{K}}^\times \to \bar{\mathbf{Z}}_p^\times$ of χ by $\hat{\chi}(\alpha) = \chi(\alpha)\alpha_\infty^{-\kappa} \cdot \alpha_p^\kappa$.

When R is an $\mathcal{F}$-algebra and $m = (m_{ij}) \in \mathrm{M}_{r,s}(\mathcal{K} \otimes_{\mathcal{F}} R)$, we denote by $m^\intercal = (m_{ji}), m^c = (m_{ij}^c)$, and $m^* = (m_{ji}^c)$ respectively the transpose, conjugate, and conjugate-transpose of m.

1. Modular forms on definite unitary groups

Let G be the definite unitary group over $\mathcal{F}$ such that for any $\mathcal{F}$-algebra R

$$G(R) = \{g \in \mathrm{GL}_2(\mathcal{K} \otimes_{\mathcal{F}} R) \mid gg^* = \mathbf{1}_n\}.$$

In this section we recall the definition of algebraic modular forms on G and the big Hecke algebra acting on which. We also recall the properties of the Galois pseudo-representation of $\mathcal{G}_{\mathcal{K}}$ to the big Hecke algebra.

1.1. **Algebraic modular forms.** Let $B = TN \subset \mathrm{GL}_2$ be the subgroup of upper triangular matrices and its Levi decomposition. Following [Ger18, Def 2.3], we identify $X^*(T)$ with $\mathbf{Z}^2$ and $k = (k_1, k_2) \in X^*(T)$ is said to be dominant if $k_1 \geq k_2$. Then $\xi_k(R) \coloneqq \mathrm{Sym}^{k_1-k_2} R^2 \otimes \det^{k_2}$ defines an algebraic representation of GL_2 of highest weight k.

Suppose $v \in \mathbf{h}$ and $v = w\overline{w}$ splits in $\mathcal{K}$, then $\mathrm{GL}_n(\mathcal{F}_v \otimes_{\mathcal{F}} \mathcal{K}) \cong \mathrm{GL}_n(\mathcal{K}_w) \times \mathrm{GL}_n(\mathcal{K}_{\overline{w}})$. Let ι_w denote the projection to the component at w, then $\iota_w\colon G(\mathcal{F}_v) \to \mathrm{GL}_n(\mathcal{K}_w)$ is an isomorphism and $\iota_w(g_v) = \iota_{\overline{w}}(g_v)^{-\intercal}$ if we identify $\mathcal{K}_w = \mathcal{F}_v = \mathcal{K}_{\overline{w}}$. Fix a choice of $w \mid v$ for each $v \in \mathbf{h}$ that splits, we will identify $\mathrm{GL}_2(\mathcal{K}_w)$ with $G(\mathcal{F}_v)$ via ι_w, so $B(\mathcal{K}_w), N(\mathcal{K}_w)$ and $T(\mathcal{K}_w)$ are viewed as subgroups of $G(\mathcal{F}_v)$. In particular, pick $w \in \Sigma_p$ when $v \in S_p$, then we view $K_p \coloneqq \prod_{w\in\Sigma_p} \mathrm{GL}_2(\mathcal{O}_w)$ as an open compact subgroup of $\prod_{v|p} G(\mathcal{F}_v)$.

Fix a finite extension E over $\mathbf{Q}_p$ that contains $\iota_p(\sigma(\mathcal{K}))$ for all $\sigma \in I_{\mathcal{K}}$. Let $\mathcal{O} = \mathcal{O}_E$ be the ring of integers in E and $\varpi \in \mathcal{O}$ be a uniformizer. When $\underline{k} = (k_\sigma) \in (\mathbf{Z}^2)^\Sigma$ is dominant in the sense that $k_\sigma = (k_{\sigma,1}, k_{\sigma,2})$ is dominant for each $\sigma \in \Sigma$, let $\xi_{\underline{k}}$ be the algebraic K_p-representation over $\mathcal{O}$ given by

$$\xi_{\underline{k}} = \bigotimes_{\sigma\in\Sigma} \xi_{k_\sigma}, \quad \xi_{\underline{k}}(g) = \otimes_{w\in\Sigma_p} \otimes_{\sigma\in I_w} \xi_{k_\sigma}(g_w) \text{ for } g = (g_w) \in K_p.$$

Definition 1.1. Let A be an $\mathcal{O}$-module, $\underline{k}$ be a dominant weight, and $U \subset G(\mathbf{A}_f^p) \times K_p$ be an open compact subgroup. We define the space of algebraic modular forms of weight $\underline{k}$, coefficients in A, and level U by

$$S_{\underline{k}}(U, A) = \left\{ f : G(\mathcal{F})\backslash G(\mathbf{A}_f)/ \to A \otimes_{\mathcal{O}} \xi_{\underline{k}}(\mathcal{O}) \mid f(gu) = \xi_{\underline{k}}(u_p)^{-1} \cdot f(g), u \in U \right\}.$$

Let $U^p = \prod_{v\in\mathbf{h}\backslash S_p} U_v$ be an open compact subgroup of $G(\mathbf{A}_f^p)$. For integers $c \geq b \geq 0$ and $c > 0$ we define the open subgroup $\mathrm{Iw}(w^{b,c}) = \{k \in \mathrm{GL}_2(\mathcal{O}_w) \mid k \bmod \varpi_w^c \in B(\mathcal{O}/\varpi_w^c) \text{ and } k \bmod \varpi_w^b \in N(\mathcal{O}/\varpi_w^b)\}$

for each $w \in \Sigma_p$ and put $\mathrm{Iw}(p^{b,c}) = \prod_{w \in \Sigma_p} \mathrm{Iw}(w^{b,c}) \subset K_p$. In [Ger18], the Hecke operators acting on $S_{\underline{k}}(U^p \mathrm{Iw}(p^{b,c}), A)$ are defined by the following double cosets operators.

- If $v = w\overline{w}$ splits in $\mathcal{K}$ and $U_v = \mathrm{GL}_2(\mathcal{O}_w)$, we define
$$T_w^{(1)} = \left[\mathrm{GL}_2(\mathcal{O}_w)\begin{pmatrix} \varpi_w & \\ & 1 \end{pmatrix}\mathrm{GL}_2(\mathcal{O}_w)\right], \quad T_w^{(2)} = \left[\mathrm{GL}_2(\mathcal{O}_w)\begin{pmatrix} \varpi_w & \\ & \varpi_w \end{pmatrix}\mathrm{GL}_2(\mathcal{O}_w)\right],$$
We can similarly define $T_{\overline{w}}^{(j)}$, but then $T_{\overline{w}}^{(1)} = (T_w^{(2)})^{-1} T_w^{(1)}$ and $T_{\overline{w}}^{(2)} = (T_w^{(2)})^{-1}$.
- If $w \in \Sigma_p$, for $\alpha_w^{(1)} = \left(\begin{smallmatrix} \varpi_w & \\ & 1 \end{smallmatrix}\right)$ and $\alpha_w^{(2)} = \left(\begin{smallmatrix} \varpi_w & \\ & \varpi_w \end{smallmatrix}\right) \in T(\mathcal{K}_w)$ and $u \in T(\mathcal{O}_w)$ we define
$$U_{\underline{k},w}^{(j)} = (w_0\underline{k})^{-1}(\alpha_w^{(j)}) \cdot [\mathrm{Iw}(w^{b,c})\alpha_w^{(j)}\mathrm{Iw}(w^{b,c})] \quad \langle u \rangle_{\underline{k}} = (w_0\underline{k})^{-1}(u) \cdot [\mathrm{Iw}(w^{b,c})u\mathrm{Iw}(w^{b,c})].$$
Here $w_0\underline{k} := (k_{2,\sigma}, k_{1,\sigma})_\sigma$ defines a character of $T(\mathcal{K}_w)$ by the same recipe defining $\xi_{\underline{k}}$.

The Hecke operators are compatible with the inclusion $S_{\underline{k}}(U^p \mathrm{Iw}(p^{b,c}), A) \subset S_{\underline{k}}(U^p \mathrm{Iw}(p^{b',c'}), A)$ when $b' \geq b$ and $c' \geq c$. We define the ordinary projector $e := \lim_{n \to \infty} \prod_{w \in \Sigma_p} (U_{\underline{k},w}^{(1)} U_{\underline{k},w}^{(2)})^{n!}$ when A is equal to $\mathcal{O}$ or $E/\mathcal{O}$. Omit the subscript $\underline{k}$ when $\xi_{\underline{k}}$ is the trivial representation, we then define
$$S^{\mathrm{ord}}(U^p, E/\mathcal{O}) = \varinjlim_{b \geq 1} eS(U^p \mathrm{Iw}(p^{b,b}), E/\mathcal{O})$$
and let $\mathbb{T}^{\mathrm{ord}}(U^p)$ be the Hecke algebra acting on which generated by the Hecke operators defined above.

Put $\mathrm{Iw}'(p^{b,c}) = \prod_{w \neq w_0} \mathrm{Iw}(w^{b,c})$ and let $U_0 \subset \mathrm{GL}_2(\mathcal{O}_{w_0})$ be an open subgroup. We can also consider the projector $e' := \lim_{n \to \infty} \prod_{w \neq w_0} (U_{\underline{k},w}^{(1)} U_{\underline{k},w}^{(2)})^{n!}$ on $S_{\underline{k}}(U^p U_0 \mathrm{Iw}'(p), A)$ and define
$$S(U^p, E/\mathcal{O}) = \varinjlim_{U_0} \varinjlim_{b \geq 1} e' S(U^p U_0 \mathrm{Iw}'(p^{b,b}), E/\mathcal{O}),$$
where U_0 goes through open compact subgroups of $\mathrm{GL}_2(\mathcal{O}_{w_0})$. Let $\mathbb{T}(U^p)$ denote the Hecke algebra generated by the Hecke operators except for $U_{\underline{k},w_0}^{(1)}$.

For $b \geq 0$ let $T(w^b) = \{u \in T(\mathcal{O}_w) \mid u \equiv \mathbf{1} \mod \varpi_w^b\}$ for each $w \in \Sigma_p$ and put $T(p^b) = \prod_{w \in \Sigma_p} T(w^b)$. Define $\Lambda^+ := \Lambda[\![T(p^0)]\!]$. Then the diamond operators $\langle u \rangle$ define structures of Λ^+-algebras on the Hecke algebras and the clear inclusion $S^{\mathrm{ord}}(U^p, E/\mathcal{O}) \subset S(U^p, E/\mathcal{O})$ induces an Λ^+-algebra homomorphism
$$\mathbb{T}(U^p) \to \mathbb{T}^{\mathrm{ord}}(U^p).$$

1.2. **Completed homology and cohomology.** Let $N^\vee := \mathrm{Hom}_{\mathcal{O}}(N, E/\mathcal{O})$ denote the Pontryagin dual of a locally compact $\mathcal{O}$-module N. We then define the (ordinary-)completed homology of tame level U^p by
$$M(U^p) = S(U^p, E/\mathcal{O})^\vee \text{ and } M^{\mathrm{ord}}(U^p) = S^{\mathrm{ord}}(U^p, E/\mathcal{O})^\vee.$$
Note that $S^{\mathrm{ord}}(U^p, E/\mathcal{O}) \subset S(U^p, E/\mathcal{O})$ induces a Hecke equivariant homomorphism $M(U^p) \to M^{\mathrm{ord}}(U^p)$. Furthermore, if we define $S(U^p) = \mathrm{Hom}_{\mathcal{O}}(E/\mathcal{O}, S(U^p, E/\mathcal{O}))$, it can be verified that
$$M(U^p) \cong \mathrm{Hom}_{\mathcal{O}}(S(U^p), \mathcal{O}), \qquad S(U^p) \cong \mathrm{Hom}_{\mathcal{O}}^{\mathrm{cts}}(M(U^p), \mathcal{O})$$
and similar statement holds for $S^{\mathrm{ord}}(U^p) = \mathrm{Hom}_{\mathcal{O}}(E/\mathcal{O}, S^{\mathrm{ord}}(U^p, E/\mathcal{O}))$ as well. Therefore it makes sense to call $S(U^p)$ and $S^{\mathrm{ord}}(U^p)$ the (ordinary-)completed cohomology of tame level U^p. We also note that the space $S(U^p)_E := S(U^p) \otimes_{\mathcal{O}} E$ is equipped with a structure of E-Banach space for which $S(U^p)$ is open.

Let $T_p' = \prod_{w \neq w_0} T(\mathcal{K}_w)$ and identify $\mathcal{K}_{w_0} = \mathbf{Q}_p$, we can define a $\mathrm{GL}_2(\mathbf{Q}_p) \times T_p'$-representation on $S(U^p, E/\mathcal{O})$ by right translation. Furthermore, let $T'(p^b) = \prod_{w \neq w_0} T(w^b)$, then the restriction of the representation to $\mathrm{GL}_2(\mathbf{Z}_p) \times T'(p^0)$ is an injective object following the same argument in [Pan22, §3]. Write

$\underline{k}' = (k_\sigma)_{w\neq w_0}$ and let $\pi_{\underline{k}} = \xi_{k_{\sigma_0}} \otimes \underline{k}'$ be the $\mathrm{GL}_2(\mathbf{Z}_p)\times T'(p^0)$ representation defined by the same recipe defining $\xi_{\underline{k}}$. The injectivity then yields the following proposition.

Proposition 1.2. *The subspace $S(U^p)_E^{\mathrm{alg}}$ of algebraic vectors, defined as*

$$\mathrm{Im}\left(ev\colon \bigoplus_{\underline{k}} \mathrm{Hom}_{E[\mathrm{GL}_2(\mathbf{Z}_p)\times T'(p^0)]}(\pi_{\underline{k}}^*(\mathcal{O}), S(U^p)_E)\otimes_E \pi_{\underline{k}}^*(E) \to S(U^p)_E \right)$$

where $\underline{k}$ ranges through all dominant weights, is dense in the E-Banach space $S(U^p)_E$.

We remark that a generalization of the control theorem for ordinary modular forms implies that the space $\mathrm{Hom}_{\mathcal{O}[\mathrm{GL}_2(\mathbf{Z}_p)\times T'(p^0)]}(\pi_{\underline{k}}^*(\mathcal{O}), S(U^p)_E)$ is Hecke-equivariantly isomorphic to $e' S_{\underline{k}}(U^p\,\mathrm{GL}_2(\mathbf{Q}_p)\mathrm{Iw}'(p^{0,1}), E)$. Therefore the proposition is a modified version of the density of crystalline points from [Eme10] and [Pan22].

1.3. **Galois representations.** Thanks to the work of [Ger18], we can associated a Galois representation r_π of $\mathcal{G}_{\mathcal{K}}$ to an irreducible $G(\mathbf{A}_f)$-representation π inside the space of algebraic modular form. Moreover, if $\pi\cap S_{\underline{k}}(U^p\,\mathrm{GL}_2(\mathcal{K}_{w_0})\mathrm{Iw}(p^{0,1}), E)\neq 0$, then r_π is unramified at the places where U_v is hyperspecial and ordinary at $w\in\Sigma_p\setminus\{w_0\}$. As the Hecke operators all acts semi-simply, the Hecke algebra $\mathbb{T}(U^p)$ is reduced and we can glue the Galois representations from finite levels into a big Galois pseudo-representation

$$T(U^p)\colon\, \mathcal{G}_{\mathcal{K}} \to \mathbb{T}(U^p)$$

with the properties listed below.

(1) Let $S\supset S_p$ be a finite set of places $v\in\mathbf{h}$ such that U_v is hyperspecial if $v\notin S$. Denote the maximal Galois extension that is unramified outside S by $\mathcal{K}^S/\mathcal{K}$, then $T(U^p)$ factors through $Gal(\mathcal{K}^S/\mathcal{K})$.

(2) Let $v\notin S$ be a split place and Fr_w be the geometric Frobenius at $w\mid v$, then

$$T(U^p)(\mathrm{Fr}_w) = T_w^{(1)}, \quad (\epsilon\det T(U^p))(\mathrm{Fr}_w) = T_w^{(2)}.$$

(3) Let D_w be the decomposition group at $w\in S_p\setminus\{w_0\}$, then $T(U^p)|_{D_w} = \Psi_{w,1}+\epsilon^{-1}\Psi_{w,2}$, where $\Psi_{w,1},\Psi_{w,2}\colon D_w\to\mathbb{T}(U^p)$ are characters and satisfy

$$\Psi_{w,1}\circ\mathrm{Art}_w(\varpi_w) = U_w^{(1)} \qquad \Psi_{w,1}\circ\mathrm{Art}_w(x) = \langle\left(\begin{smallmatrix} x & \\ & 1\end{smallmatrix}\right)\rangle \text{ for } x\in\mathcal{O}_w^\times$$
$$\Psi_{w,2}\circ\mathrm{Art}_w(\varpi_w) = U_w^{(2)}/U_w^{(1)} \qquad \Psi_{w,2}\circ\mathrm{Art}_w(x) = \langle\left(\begin{smallmatrix} 1 & \\ & x\end{smallmatrix}\right)\rangle \text{ for } x\in\mathcal{O}_w^\times$$

Moreover, $T(U^p)$ is $\Psi_{w,1}$-ordinary in the sense that for all $\sigma,\tau\in D_w$ and $\eta\in\mathcal{G}_{\mathcal{K}}$ we have

$$T(U^p)(\sigma\tau\eta)-\Psi_{w,1}(\sigma)T(U^p)(\tau\eta)-\Psi_{w,2}(\tau)T(U^p)(\sigma\eta)+\Psi_{w,1}(\sigma)\Psi_{w',2}(\tau)T(U^p)(\eta)=0.$$

(4) Let D_{w_0} be the decomposition group at w_0, then

$$(\epsilon\det T(U^p))\circ\mathrm{Art}_{w_0}(\varpi_{w_0}) = U_{w_0}^{(2)} \quad (\epsilon\det T(U^p))\circ\mathrm{Art}_{w_0}(x) = \langle\left(\begin{smallmatrix} x & \\ & x\end{smallmatrix}\right)\rangle \text{ for } x\in\mathcal{O}_{w_0}^\times$$

We remark that a pseudo-representation $T^{\mathrm{ord}}(U^p)\colon\,\mathcal{G}_{\mathcal{K}}\to\mathbb{T}^{\mathrm{ord}}(U^p)$ with the same properties also exists and is in fact isomorphic to the pushforward of $T(U^p)$ via the homomorphism $\mathbb{T}(U^p)\to\mathbb{T}^{\mathrm{ord}}(U^p)$. Moreover, there exists characters $\Psi_{w_0,1},\Psi_{w_0,2}\colon D_{w_0}\to\mathbb{T}^{\mathrm{ord}}(U^p)$ such that the third property above holds for w_0.

1.4. **Hida family.** Let $\mathcal{I}$ be a local complete Noetherian ring that is finite over Λ^+ and flat over $\mathcal{O}$. We define the space of $\mathcal{I}$-adic Hida families of tame level U^p by

$$S^{\mathrm{ord}}(U^p,\mathcal{I}) \coloneqq \{\mathcal{F}\in S^{\mathrm{ord}}(U^p)\widehat{\otimes}_{\mathcal{O}}\mathcal{I} \mid (\langle u\rangle\otimes\mathbf{1})\mathcal{F} = (\mathbf{1}\otimes\langle u\rangle)\mathcal{F} \text{ for } u\in\Lambda^+\}.$$

Here $(\langle u\rangle\otimes\mathbf{1})$ denotes the action through the Hecke algebra and $(\mathbf{1}\otimes\langle u\rangle)$ denotes the action through $\mathcal{I}$.

Define $M^{\mathrm{ord}}(U^p,\mathcal{I}) = M^{\mathrm{ord}}(U^p)\otimes_{\Lambda^+}\mathcal{I}$. From the duality between the completed homology and cohomology, it is easy to check that we have a Hecke-equivariant isomorphism $S^{\mathrm{ord}}(U^p,\mathcal{I})\cong\mathrm{Hom}_{\mathcal{I}}(M^{\mathrm{ord}}(U^p,\mathcal{I}),\mathcal{I})$.

Moreover, suppose U^p is sufficiently small in the sense that $G(\mathcal{F}) \cap g_i(U^pK_p)g_i^{-1} = \{1\}$ for all $i \in I$ in a decomposition $G(\mathbf{A}_f) = \bigsqcup_{i\in I} G(\mathcal{F})t_i(U^pK_p)$, then we can show that $M^{\mathrm{ord}}(U^p, \mathcal{I})$ and $S^{\mathrm{ord}}(U^p, \mathcal{I})$ are finite free over $\mathcal{I}$ and there exists a Hecke-equivariant isomorphism

$$M^{\mathrm{ord}}(U^p, \mathcal{I}) \cong \mathrm{Hom}_{\mathcal{I}}(S^{\mathrm{ord}}(U^p, \mathcal{I}), \mathcal{I}).$$

2. Results from p-adic local Langlands

Fix a maximal ideal $\mathfrak{m} \subset \mathbb{T}(U^p, \mathcal{O})$. Enlarge E if necessary, we assume that $\mathbb{T}(U^p, \mathcal{O})/\mathfrak{m}$ coincides with the residue field $\mathbb{F}$ of $\mathcal{O}$. Write $\mathbb{T}_{\mathfrak{m}} = \mathbb{T}(U^p)_{\mathfrak{m}}$ and let $T_{\mathfrak{m}} \colon \mathcal{G}_{\mathcal{K}} \to \mathbb{T}_{\mathfrak{m}}$ denote the localization of the pseudo-representation $T(U^p)$. If ω denotes the Teichmuller character, we say $T_{\mathfrak{m}}$ is residually reducible and locally generic at w_0 if there exists characters $\bar{\delta}_1, \bar{\delta}_2 \colon \mathcal{G}_{\mathcal{K}} \to \mathbb{F}^\times$ such that

(red.gen) $$T_{\mathfrak{m}} \equiv \bar{\delta}_1 + \bar{\delta}_2 \mod \mathfrak{m}, \quad \text{and} \quad \bar{\delta}_1\bar{\delta}_2^{-1}|_{D_{w_0}} \neq \mathbf{1}, \omega^{\pm}.$$

From now on we identify D_{w_0} with $\mathcal{G}_{\mathbf{Q}_p}$, the absolute Galois group of $\mathbf{Q}_p$. We shall apply techniques from p-adic local Langlands to study $S(U^p, E/\mathcal{O})_{\mathfrak{m}}$ as a $\mathrm{GL}_2(\mathbf{Q}_p) \times T'_p$-representation. For a technical reason, it is necessary to make a twist as follows so that the $\mathrm{GL}_2(\mathbf{Q}_p)$-action has a central character.

Fix a continuous character $\zeta \colon \mathcal{G}_{\mathbf{Q}_p} \to \mathcal{O}^\times$ such that $\zeta \equiv \omega\delta_1\delta_2|_{\mathcal{G}_{\mathbf{Q}_p}} \mod \varpi$. Then $\xi_{\mathfrak{m}} := \zeta(\epsilon \det T_{\mathfrak{m}})^{-1}|_{\mathcal{G}_{\mathbf{Q}_p}}$ is trivial modulo $\mathfrak{m}$ and there exists a well-defined square-root character $\xi_{\mathfrak{m}}^{1/2} \colon \mathcal{G}_{\mathbf{Q}_p} \to \mathbb{T}_{\mathfrak{m}}$ since p is odd. Identify $\mathbf{Q}_p^\times$ with the central torus of $\mathrm{GL}_2(\mathbf{Q}_p)$. We let $S(U^p, E/\mathcal{O})_{\mathfrak{m}}$ be a Q-representation for

$$Q := \mathrm{GL}_2(\mathbf{Q}_p) \times \mathbf{Q}_p^\times \times T'_p,$$

on which $\mathbf{Q}_p^\times \times T'_p$ acts by the usual translation, but $\mathrm{GL}_2(\mathbf{Q}_p)$ acts with the translation twisted by $\xi_{\mathfrak{m}}^{1/2} \circ \mathrm{Art}$ and has a central character $\epsilon\zeta$. Take the localization $S^{\mathrm{ord}}(U^p, E/\mathcal{O})_{\mathfrak{m}}$ via $\mathbb{T}(U^p) \to \mathbb{T}^{\mathrm{ord}}(U^p)$. We make a similar twist of the translation by $T(\mathbf{Q}_p) \times T'_p$ on $S^{\mathrm{ord}}(U^p, E/\mathcal{O})_{\mathfrak{m}}$ and define an M-representation for

$$M := T(\mathbf{Q}_p) \times \mathbf{Q}_p^\times \times T'_p.$$

2.1. **Local-global compatibility.** We very briefly review Păskūnas' theory of blocks of a p-adic analytic group H and refer the details to [Pas13]. Let $\mathrm{Mod}_H^{\mathrm{sm}}(\mathcal{O})$ be the category of $\mathcal{O}[H]$-modules V such that

- Each $v \in V$ is fixed by some open compact subgroup of H.
- Each $v \in V$ is annihilated by ϖ^h for some $h \geq 0$.

We call $V \in \mathrm{Mod}_H^{\mathrm{sm}}(\mathcal{O})$ admissible if $V^K[\varpi^h]$ is $\mathcal{O}$-finite for any h and any compact open subgroup K of H.

Let $\mathrm{Mod}_H^{\mathrm{lfin}}(\mathcal{O})$ be the full subcategory of $V \in \mathrm{Mod}_H^{\mathrm{sm}}(\mathcal{O})$ such that each $v \in V$ generates a $\mathcal{O}[H]$-submodule of finite length. A general formalism ensures that an irreducible $\pi \in \mathrm{Mod}_H^{\mathrm{lfin}}(\mathcal{O})$ admits an injective envelope $\tilde{J}_\pi$, or dually, the Pontryagin dual $\pi^\vee$ admits a projective envelope $\tilde{P}_{\pi^\vee}$.

We say two irreducible representations π, π' are equivalent if there exists a sequence $\pi = \pi_1, \cdots, \pi_n = \pi'$ of irreducible representations such that all successive π_i, π_{i+1} have nontrival extensions. A block $\mathfrak{B}$ is an equivalence class of irreducible representations. Given a block $\mathfrak{B}$, we let $\mathfrak{C}_H^{\mathfrak{B}}(\mathcal{O})$ denote the subcategory of all $S^\vee$, where $S \in \mathrm{Mod}_H^{\mathrm{lfin}}(\mathcal{O})$ and all the irreducible subquotients of S belong to $\mathfrak{B}$. We have

(P1) $\mathrm{Hom}_H(\tilde{P}_\pi, \tilde{P}_{\pi'}) = 0$ if π, π' are not equivalent and $\tilde{P}_\pi \in \mathfrak{C}_H^{\mathfrak{B}}(\mathcal{O})$ if $\pi \in \mathfrak{B}$.

(P2) Let $\tilde{P}_{\mathfrak{B}} = \oplus_{\pi\in\mathfrak{B}}\tilde{P}_\pi$ and $\tilde{E}_{\mathfrak{B}} = \mathrm{End}_H(\tilde{P}_{\mathfrak{B}})$, then the evaluation map $\mathrm{Hom}_H(\tilde{P}_{\mathfrak{B}}, N) \otimes_{\tilde{E}_{\mathfrak{B}}} \tilde{P}_{\mathfrak{B}} \to N$ is an isomorphism if $N \in \mathfrak{C}_H^{\mathfrak{B}}(\mathcal{O})$. In fact $\mathfrak{m}(N) := \mathrm{Hom}_H(\tilde{P}_{\mathfrak{B}}, N)$ defines an equivalence between $\mathfrak{C}_H^{\mathfrak{B}}(\mathcal{O})$ and the category of pseudo-compact $\tilde{E}_{\mathfrak{B}}$-modules.

We now apply the results from [Pas13, §7] to our settings. Let $\chi_i\colon \mathcal{G}_{\mathbf{Q}_p} \to \mathbb{F}^\times$ denote the character $\omega\bar{\delta}_i$. If we identify χ_i with the $\mathbf{Q}^\times$-characters obtained through composition with Art, then under the generic assumption in (red.gen) the set $\mathfrak{B} = \{\pi_1, \pi_2\}$ is a block of irreducible $\mathrm{GL}_2(\mathbf{Q}_p)$-representations for

$$\pi_1 := \mathrm{Ind}_B^{\mathrm{GL}_2} \chi_1 \otimes \chi_2\omega^{-1} \text{ and } \pi_2 := \mathrm{Ind}_B^{\mathrm{GL}_2} \chi_2 \otimes \chi_1\omega^{-1}.$$

For the same technical reason, we actually use the subcategories $\mathrm{Mod}^{\mathrm{lfin}}_{\mathrm{GL}_2(\mathbf{Q}_p),\epsilon\zeta}(\mathcal{O})$ and $\mathrm{Mod}^{\mathrm{lfin}}_{T(\mathbf{Q}_p),\epsilon\zeta}(\mathcal{O})$ of representations on which the central $\mathbf{Q}_p^\times$ acts by $\epsilon\zeta$, and we identify the latter category with $\mathrm{Mod}^{\mathrm{lfin}}_{\mathbf{Q}_p^\times}(\mathcal{O})$ via $\mathbf{Q}_p^\times \cong \left(\begin{smallmatrix} 1 & \\ & * \end{smallmatrix}\right) \subset T(\mathbf{Q}_p)$. Let $\tilde{P}_j$ denote the projective envelope of $\pi^\vee$ and let $R^{\epsilon\zeta}$ represent the universal pseudo-deformation of $\chi_1 + \chi_2$ with fixed determinant $\epsilon\zeta$, then we have the following results.

(1) $R^{\epsilon\zeta}$ is a regular local ring of relative dimension 3 over $\mathcal{O}$ and the reducibility ideal of which is generated by a regular element x_{red}, we write $R^{\epsilon\zeta}_{\mathrm{red}} := R^{\epsilon\zeta}/(x_{\mathrm{red}})$.
(2) $\mathrm{End}_{\mathrm{GL}_2(\mathbf{Q}_p)}(\tilde{P}_j)$ is isomorphic to $R^{\epsilon\zeta}$ as a ring and $\mathrm{Hom}_{\mathrm{GL}_2(\mathbf{Q}_p)}(\tilde{P}_i, \tilde{P}_j)$ is free of rank one over $R^{\epsilon\zeta}$ if $i \neq j$. Moreover, let $\Phi_{ij} \in \mathrm{Hom}_{\mathrm{GL}_2(\mathbf{Q}_p)}(\tilde{P}_j, \tilde{P}_i)$ be a generator, then $\Phi_{ji} \circ \Phi_{ij}$ generates the reducibility ideal under the isomoprhism $\mathrm{End}_{\mathrm{GL}_2(\mathbf{Q}_p)}(\tilde{P}_j) \cong R^{\epsilon\zeta}$.
(3) Let $\tilde{P}_{\mathfrak{B}} = \tilde{P}_2 \oplus \tilde{P}_2$, then $R^{\epsilon\zeta}$ is isomorphic to the center of $\tilde{E}_{\mathfrak{B}} := \mathrm{End}_{\mathrm{GL}_2(\mathbf{Q}_p)}(\tilde{P}_{\mathfrak{B}})$.
(4) Let $\mathrm{Ord}\colon \mathrm{Mod}_{\mathrm{GL}_2(\mathbf{Q}_p)}(\mathcal{O}) \to \mathrm{Mod}_{T(\mathbf{Q}_p)}(\mathcal{O})$ denote Emerton's functor [Eme10] with respect to B. Then $\mathrm{Ord}(\tilde{P}_j) = \tilde{P}_{\chi_j^\vee}$ and $\mathrm{Ord}\colon \mathrm{End}_{\mathrm{GL}_2(\mathbf{Q}_p)}(\tilde{P}_j) \to \mathrm{End}_{T(\mathbf{Q}_p)}(\tilde{P}_{\chi_j^\vee})$ is surjective with the reducibility ideal as the kernel. Therefore we can identify $\mathrm{End}_{T(\mathbf{Q}_p)}(\tilde{P}_{\chi_j^\vee})$ with $R^{\epsilon\zeta}_{\mathrm{red}}$.

On the other hand, consider the $\mathbb{T}_{\mathfrak{m}}^\times$-valued character of $\mathbf{Q}_p^\times \times T'_p$ defined by $(\epsilon \det T(U^p))$ at w_0 and by $\Psi_{w,1} \otimes \Psi_{w,2}$ at $w \neq w_0$ after composition with the Artin map. Let $\upsilon\colon \mathbf{Q}_p^\times \times T'_p \to \mathbb{F}^\times$ denote the residual character. Then $\tilde{P} = \tilde{P}_{\upsilon^\vee}$ is free of rank one over $\tilde{E} = \mathrm{End}_{\mathbf{Q}_p^\times \times T'_p}(\tilde{P})$, which is isomorphic to the universal deformation ring of υ. Now $\tilde{P}_{i,\mathfrak{m}} := \tilde{P}_i \otimes \tilde{P}$ is the projective envelope of $(\pi \otimes \upsilon)^\vee$. Put $\tilde{P}_{\mathfrak{B},\mathfrak{m}} = \tilde{P}_{\mathfrak{B}} \otimes \tilde{P}$ and $\tilde{E}_{\mathfrak{m}} = \mathrm{End}_Q(\tilde{P}_{\mathfrak{B},\mathfrak{m}}) \cong \tilde{E}_{\mathfrak{B}} \otimes \tilde{E}$, then $R_{\mathfrak{m}} := R^{\epsilon\zeta} \otimes \tilde{E}$ acts on $\mathrm{Hom}_Q(\tilde{P}_{\mathfrak{m}}, M(U^p)_{\mathfrak{m}})$ in two ways:

- $R_{\mathfrak{m}}$ acts on $\tilde{P}_{\mathfrak{m}}$ as the center of $\tilde{E}_{\mathfrak{m}} = \mathrm{End}_Q(\tilde{P}_{\mathfrak{B},\mathfrak{m}})$.
- $R_{\mathfrak{m}}$ acts on $M(U^p)_{\mathfrak{m}}$ through the homomorphism $R_{\mathfrak{m}} := R^{\epsilon\zeta} \otimes \tilde{E} \to \mathbb{T}_{\mathfrak{m}}$, which is induced respectively from $\xi_{\mathfrak{m}}^{1/2}\epsilon T_{\mathfrak{m}} \mid_{\mathcal{G}_{\mathbf{Q}_p}}$ and υ by the universal property.

We remark that since we don't know yet whether $\mathbb{T}_{\mathfrak{m}}$ is Noetherian, the homomorphism from $R^{\epsilon\zeta}$ comes from first applying the universal property to the Hecke algebras at finite levels then take the inverse limit.

The following proposition is a modified version of the local-global compatibility from [Pan22].

Proposition 2.1. *All the irreducible subquotients of $S(U^p, E/\mathcal{O})_{\mathfrak{m}}$ are isomorphic to either $\pi_1 \otimes \upsilon$ or $\pi_2 \otimes \upsilon$. As a consequence the two $R_{\mathfrak{m}}$-actions on $\mathrm{Hom}_Q(\tilde{P}_{\mathfrak{m}}, M(U^p)_{\mathfrak{m}})$ coincide.*

2.2. **Reducible part of the completed homology.** We can deduce from the local-global compatibility that there is a projective resolution

$$0 \to \tilde{P}_{\mathfrak{B},\mathfrak{m}}^{\oplus r} \to \tilde{P}_{\mathfrak{B},\mathfrak{m}}^{\oplus r} \to M(U^p)_{\mathfrak{m}} \to 0$$

The first morphism above can be represented by a matrix $A = \left(\begin{smallmatrix} A_{11} & A_{12}\Phi_{12} \\ A_{21}\Phi_{21} & A_{22} \end{smallmatrix}\right)$ for $A_{ij} \in M_r(R_{\mathfrak{m}})$. Now apply the right exact Ord to the $\mathrm{GL}_2(\mathbf{Q}_p)$-part of the sequence. We have $\mathrm{Ord}(\tilde{P}_{i,\mathfrak{m}}) = \tilde{P}_{\chi_i^\vee,\mathfrak{m}} := \tilde{P}_{\chi_i^\vee} \otimes \tilde{P}$ and it can be checked that $\mathrm{Ord}(M(U^p)_{\mathfrak{m}}) = M^{\mathrm{ord}}(U^p)_{\mathfrak{m}}$. Thus we obtain the right exact sequence

$$\tilde{P}^{\oplus r}_{\chi_1^\vee,\mathfrak{m}} \oplus \tilde{P}^{\oplus r}_{\chi_2^\vee,\mathfrak{m}} \xrightarrow{\overline{A}_{11} \oplus \overline{A}_{22}} \tilde{P}^{\oplus r}_{\chi_1^\vee,\mathfrak{m}} \oplus \tilde{P}^{\oplus r}_{\chi_2^\vee,\mathfrak{m}} \to M^{\mathrm{ord}}(U^p)_{\mathfrak{m}} \to 0 \tag{1}$$

where $\overline{A}_{ii}$ denote the reduction in $R^{\mathrm{red}}_{\mathfrak{m}} := R^{\epsilon\zeta}_{\mathrm{red}} \otimes \tilde{E}$. This shows that $M^{\mathrm{ord}}(U^p)_{\mathfrak{m}}$ is the direct sum of

$$M^{\mathrm{ord}}(U^p)_{\mathfrak{m},i} := \mathrm{Hom}_M(\tilde{P}_{\chi_i^\vee,\mathfrak{m}}, M^{\mathrm{ord}}(U^p)_{\mathfrak{m}}) \otimes_{R^{\mathrm{red}}_{\mathfrak{m}}} \tilde{P}_{\chi_i^\vee,\mathfrak{m}} \cong \mathrm{Hom}_M(\tilde{P}_{\chi_i^\vee,\mathfrak{m}}, M^{\mathrm{ord}}(U^p)_{\mathfrak{m}})$$

where the isomorphism follows since $\tilde{P}_{\chi_i^\vee,\mathfrak{m}}$ is free of rank one over $R^{\mathrm{red}}_{\mathfrak{m}} \cong \mathrm{End}_M(\tilde{P}_{\chi_i^\vee,\mathfrak{m}})$. In fact, this implies that there are exactly two maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2$ in $\mathbb{T}^{\mathrm{ord}}(U^p)_{\mathfrak{m}}$, which are characterized by $\Psi_{2,w_0} \bmod \mathfrak{m}_i = \chi_i$, and that $M^{\mathrm{ord}}(U^p)_{\mathfrak{m},i} = M^{\mathrm{ord}}(U^p)_{\mathfrak{m}_i}$. Another conclusion we can draw, using that $M^{\mathrm{ord}}(U^p)$ is finite over Λ^+ and $\dim \Lambda^+ < \dim R^{\mathrm{red}}_{\mathfrak{m}}$, is that both $\overline{A}_{11}$ and $\overline{A}_{22}$ are injective.

This last fact has two importance consequences for us. First, apply the snake lemma to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{P}^{\oplus r}_{\mathfrak{B},\mathfrak{m}} & \xrightarrow{A} & \tilde{P}^{\oplus r}_{\mathfrak{B},\mathfrak{m}} & \longrightarrow & M(U^p)_{\mathfrak{m}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle x_{\mathrm{red}}} & & \downarrow{\scriptstyle x_{\mathrm{red}}} & & \downarrow{\scriptstyle x_{\mathrm{red}}} & & \\ 0 & \longrightarrow & \tilde{P}^{\oplus r}_{\mathfrak{B},\mathfrak{m}} & \xrightarrow{A} & \tilde{P}^{\oplus r}_{\mathfrak{B},\mathfrak{m}} & \longrightarrow & M(U^p)_{\mathfrak{m}} & \longrightarrow & 0 \end{array}$$

Write $M^{\mathrm{red}} := M/x_{\mathrm{red}}M$ if M is an $R_{\mathfrak{m}}$-module. We can show that x_{red} is injective on $\tilde{P}_{\mathfrak{B},\mathfrak{m}}$. Thus for both $j = 1, 2$, the space $\mathrm{Hom}_Q(\tilde{P}_{j,\mathfrak{m}}, M(U^p)_{\mathfrak{m}}[x_{\mathrm{red}}])$ is isomorphic to the kernel of the homomorphism

$$A' = \begin{pmatrix} \bar{A}_{ii} & \bar{A}_{ij}\Phi_{ij} \\ & \bar{A}_{jj} \end{pmatrix} : \mathrm{Hom}_{Q'}(\tilde{P}_{j,\mathfrak{m}}, \tilde{P}^{\oplus r}_{\mathfrak{B},\mathfrak{m}})^{\mathrm{red}} \to \mathrm{Hom}_{Q'}(\tilde{P}_{j,\mathfrak{m}}, \tilde{P}^{\oplus r}_{\mathfrak{B},\mathfrak{m}})^{\mathrm{red}}$$

which we know to be injective. Therefore $\mathrm{Hom}_Q(\tilde{P}_{j,\mathfrak{m}}, M(U^p)_{\mathfrak{m}}[x_{\mathrm{red}}]) = 0$ for $j = 1, 2$.

Second, we can observe that the same homomorphism above also appears in the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_Q(\tilde{P}_{2,\mathfrak{m}}, \tilde{P}^{\oplus r}_{1,\mathfrak{m}})^{\mathrm{red}} & \xrightarrow{\bar{A}_{11}} & \mathrm{Hom}_Q(\tilde{P}_{2,\mathfrak{m}}, \tilde{P}^{\oplus r}_{1,\mathfrak{m}})^{\mathrm{red}} & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & \mathrm{Hom}_Q(\tilde{P}_{2,\mathfrak{m}}, \tilde{P}^{\oplus r}_{\mathfrak{B},\mathfrak{m}})^{\mathrm{red}} & \xrightarrow{A'} & \mathrm{Hom}_Q(\tilde{P}_{2,\mathfrak{m}}, \tilde{P}^{\oplus r}_{\mathfrak{B},\mathfrak{m}})^{\mathrm{red}} & \longrightarrow & \mathrm{Hom}_Q(\tilde{P}_{2,\mathfrak{m}}, M(U^p)_{\mathfrak{m}})^{\mathrm{red}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{Ord}} & & \downarrow{\scriptstyle \mathrm{Ord}} & & \downarrow{\scriptstyle \mathrm{Ord}} & & \\ 0 & \longrightarrow & \mathrm{End}_M(\tilde{P}_{\chi_2^\vee,\mathfrak{m}})^r & \xrightarrow{\bar{A}_{22}} & \mathrm{End}_M(\tilde{P}_{\chi_2^\vee,\mathfrak{m}})^r & \longrightarrow & \mathrm{Hom}_M(\tilde{P}_{\chi_2^\vee,\mathfrak{m}}, M^{\mathrm{ord}}(U^p)_{\mathfrak{m}}) & \longrightarrow & 0 \end{array}$$

where the exactness of the columns follows from $\mathrm{Hom}_Q(\tilde{P}_{2,\mathfrak{m}}, \tilde{P}_{2,\mathfrak{m}})^{\mathrm{red}} \cong \mathrm{End}_M(\tilde{P}_{\chi_2^\vee,\mathfrak{m}})$. Since taking Ord is Hecke-equivariant, apply the snake lemma and use the presentation (1) gives the result below.

Proposition 2.2. *We have the following exact sequences of* $\mathbb{T}(U^p, \mathcal{O})_{\mathfrak{m}} \times R^{\mathrm{red}}_{\mathfrak{m}}$*-modules*

$$0 \to M^{\mathrm{ord}}(U^p)_{\mathfrak{m},1} \to \mathrm{Hom}_Q(\tilde{P}_{2,\mathfrak{m}}, M(U^p)_{\mathfrak{m}})^{\mathrm{red}} \to M^{\mathrm{ord}}(U^p)_{\mathfrak{m},2} \to 0$$
$$0 \to M^{\mathrm{ord}}(U^p)_{\mathfrak{m},2} \to \mathrm{Hom}_Q(\tilde{P}_{1,\mathfrak{m}}, M(U^p)_{\mathfrak{m}})^{\mathrm{red}} \to M^{\mathrm{ord}}(U^p)_{\mathfrak{m},1} \to 0$$

Let $\Lambda := [\![T(p^1)]\!]$, which is a subalgebra of $\Lambda^+ = [\![T(p^0)]\!]$. If U^p is sufficiently small, then $M^{\mathrm{ord}}(U^p)$ is finite free over Λ and therefore so is $\mathrm{Hom}_Q(\tilde{P}_{i,\mathfrak{m}}, M(U^p)_{\mathfrak{m}})^{\mathrm{red}}$ by the proposition above. Write $\mathbf{m}_i = \mathrm{Hom}_Q(\tilde{P}_{i,\mathfrak{m}}, M(U^p)_{\mathfrak{m}})$ and suppose $s = \mathrm{rank}_\Lambda \mathbf{m}_i^{\mathrm{red}}$. By the topological Nakayama lemma there exists a surjection $\Lambda[\![x_{\mathrm{red}}]\!]^{\oplus s} \to \mathbf{m}_i$. Let N denote the kerne and apply the snake lemma to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & \Lambda[\![x_{\mathrm{red}}]\!]^{\oplus s} & \longrightarrow & \mathbf{m}_i & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle x_{\mathrm{red}}} & & \downarrow{\scriptstyle x_{\mathrm{red}}} & & \downarrow{\scriptstyle x_{\mathrm{red}}} & & \\ 0 & \longrightarrow & N & \longrightarrow & \Lambda[\![x_{\mathrm{red}}]\!]^{\oplus s} & \longrightarrow & \mathbf{m}_i & \longrightarrow & 0 \end{array}$$

shows that $N^{\mathrm{red}} \cong \mathfrak{m}_i[x_{\mathrm{red}}]$, which is trivial by the first consequence above. Therefore $N = 0$ by Nakayama's lemma, as a result $\mathfrak{m}_1, \mathfrak{m}_2$, and $\mathrm{Hom}_Q(\tilde{P}_{\mathfrak{B},\mathfrak{m}}, M(U^p)_{\mathfrak{m}}) = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ are finite free over $\Lambda[\![x_{\mathrm{red}}]\!]$.

From $x_{\mathrm{red}} \in R_{\mathfrak{m}} \to \mathbb{T}_{\mathfrak{m}}$ we can view $\mathbb{T}_{\mathfrak{m}}$ as a $\Lambda[\![x_{\mathrm{red}}]\!]$-algebra. The local-global compatibility then implies the injectivity of the $\Lambda[\![x_{\mathrm{red}}]\!]$-algebra homomorphism

$$\mathbb{T}_{\mathfrak{m}} \hookrightarrow \mathrm{End}_{\Lambda[\![x_{\mathrm{red}}]\!]}(\mathrm{Hom}_Q(\tilde{P}_{\mathfrak{B},\mathfrak{m}}, M(U^p)_{\mathfrak{m}})).$$

We can now conclude that the right hand side is finite free and obtain the corollary below.

Corollary 2.3. *Assume that U^p is sufficiently small, then $\mathbb{T}_{\mathfrak{m}}$ is a finite free $\Lambda[\![x_{\mathrm{red}}]\!]$-algebra.*

2.3. **Fundamental exact sequence.** Let $\mathfrak{G}$ be an abelian profinite group. The algebra $\mathcal{I} := \mathcal{O}[\![\mathfrak{G}]\!]$ is local complete Noetherian and flat over $\mathcal{O}$. We assume $\mathfrak{G} = W \times \Delta$, where Δ is a finite and $W \cong \mathbf{Z}_p^n$.

We fix a surjective homomorphism $\lambda^{\mathrm{ord}} \colon \mathbb{T}^{\mathrm{ord}}(U^p, \mathcal{I}) \to \mathcal{I}$ of Λ^+-algebras and let

$$\lambda \colon \mathbb{T}(U^p, \mathcal{O}) \to \mathbb{T}^{\mathrm{ord}}(U^p, \mathcal{O}) \to \mathcal{I}$$

denote the composition. Let $\mathfrak{m}_{\mathcal{I}} \subset \mathcal{I}$ be the maximal ideal, $\mathfrak{m} = \lambda^{-1}(\mathfrak{m}_{\mathcal{I}})$, and $\mathfrak{m}_\circ = (\lambda^{\mathrm{ord}})^{-1}(\mathfrak{m}_{\mathcal{I}})$. We assume the maximal ideal $\mathfrak{m}$ satisfies (red.gen), then we know $\mathfrak{m}_\circ$ is one of the only two maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2 \subset \mathbb{T}^{\mathrm{ord}}_{\mathfrak{m}} := \mathbb{T}^{\mathrm{ord}}(U^p)_{\mathfrak{m}}$, say $\mathfrak{m}_\circ = \mathfrak{m}_1$. We now make the following assumptions on the space of Hida families $M^{\mathrm{ord}}(U^p, \mathcal{I})_{\mathfrak{m}_i} := M^{\mathrm{ord}}(U^p)_{\mathfrak{m}_i} \otimes_{\Lambda^+} \mathcal{I}$ for $i = 1, 2$.

(C1) There exists a nonzero $\mathbb{T}_{\mathfrak{m}}$-linear map $\Theta \colon M^{\mathrm{ord}}(U^p, \mathcal{I})_{\mathfrak{m}_1} \to \mathcal{I}$, where $\mathbb{T}_{\mathfrak{m}}$ acts on $\mathcal{I}$ via λ.
(C2) There exists an ideal $\mathfrak{q} \subset \mathbb{T}_{\mathfrak{m}}$ containing x_{red} such that $\mathfrak{q}M^{\mathrm{ord}}(U^p, \mathcal{I})_{\mathfrak{m}_2} = 0$.
(C3) There exists $F \in M^{\mathrm{ord}}(U^p, \mathcal{I})_{\mathfrak{m}_1}$ such that $\Theta(F) \neq 0$ and $\mathfrak{q}F = 0$.

To simplify the notation, let $M = \mathrm{Hom}_Q(\tilde{P}_{1,\mathfrak{m}}, M(U^p)_{\mathfrak{m}}) \otimes_{\Lambda^+} \mathcal{I}$, $M_i^{\mathrm{ord}} = M^{\mathrm{ord}}(U^p, \mathcal{I})_{\mathfrak{m}_i}$ for $i = 1, 2$, and define $S_1^{\mathrm{ord}} = \ker(\Theta) \subset M_1^{\mathrm{ord}}$. By Proposition 2.2, we have a right exact sequence of $\mathbb{T}_{\mathfrak{m}}$-modules

$$M_2^{\mathrm{ord}} \to M^{\mathrm{red}} \to M_1^{\mathrm{ord}} \to 0$$

Let $S \subset M$ be the preimage of S_1^{ord} via $M \to M^{\mathrm{red}} \to M_1^{\mathrm{ord}}$. By (C2) we have the commutative diagram

$$\begin{array}{ccccccc} S & \xrightarrow{x_{\mathrm{red}}} & \mathfrak{q}S & \longrightarrow & \mathfrak{q}S^{\mathrm{red}} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ M & \xrightarrow{x_{\mathrm{red}}} & \mathfrak{q}M & \longrightarrow & \mathfrak{q}M^{\mathrm{red}} & \longrightarrow & 0 \end{array}$$

and apply the snake lemma gives a right exact sequence of $\mathbb{T}_{\mathfrak{m}}$-modules

$$M/S \xrightarrow{x_{\mathrm{red}}} \mathfrak{q}M/\mathfrak{q}S \to \mathfrak{q}M_1^{\mathrm{red}}/\mathfrak{q}S_1^{\mathrm{red}} \to 0$$

Note that by definition we have $M/S \cong M_1^{\mathrm{ord}}/S_1^{\mathrm{ord}}$ and $x_{\mathrm{red}}M \subset S$. We can then use (C2) to show that $\mathfrak{q}M^{\mathrm{red}}/\mathfrak{q}S^{\mathrm{red}} = \mathfrak{q}M_1^{\mathrm{ord}}/\mathfrak{q}S_1^{\mathrm{ord}}$. And with more work we can deduce the left-exactness of the sequence.

Proposition 2.4. *Under the assumptions (C1), (C3), and (C2), we have the following commutative diagram of $\mathbb{T}_{\mathfrak{m}}$-modules where the bottom row is exact*

$$\begin{array}{ccccccccc} & & M/S & \xrightarrow{x_{\mathrm{red}}} & (M/S) \otimes_{\mathbb{T}_{\mathfrak{m}}} \mathfrak{q} & \longrightarrow & (M/S) \otimes_{\mathbb{T}_{\mathfrak{m}}} \mathfrak{q}^{\mathrm{red}} & \longrightarrow & 0 \\ & & \downarrow{\cong} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_1^{\mathrm{ord}}/S_1^{\mathrm{ord}} & \xrightarrow{x_{\mathrm{red}}} & \mathfrak{q}M/\mathfrak{q}S & \longrightarrow & \mathfrak{q}M_1^{\mathrm{ord}}/\mathfrak{q}S_1^{\mathrm{ord}} & \longrightarrow & 0 \end{array}$$

The proof of the proposition is a long commutative algebra argument and uses Proposition 2.3 in an essential way. One first uses (C3) and reduces the proposition to showing that a suitable localization of $\mathfrak{q}M/\mathfrak{q}S$, which becomes a $\mathrm{Frac}\Lambda[\![x_{\mathrm{red}}]\!]$-module, is nontrivial. But Proposition 2.3 implies that the suitable localization of $\mathbb{T}_\mathfrak{m}$ is a discrete valuation ring. Therefore the localization of $\mathfrak{q}M/\mathfrak{q}S$ coincides with that of M/S, which we can show to be nontrivial by (C3).

Remark 2.5. Aside from some modifications, the formulation and the proof of the proposition are entirely from [Urb20, Prop. 6.3.5], which deals with the case of Eisenstein ideals on modular curves. We also refer to [Urb20, Thm 3.4.1] for a different proof which relies on the geometry of eigencurves.

3. Construction of Euler systems

Let ψ be the finite order anticyclotomic Hecke character as in the introduction and let W^- be the Galois group of the maximal anticyclotomic $\mathbf{Z}_p^d$ extension $\tilde{\mathcal{K}}^a$, where $d = [\mathcal{F} : \mathbf{Q}]$. Write $\mathcal{I} = \mathcal{O}[\![W^-]\!]$. We would first like to have an eigensystem

$$\lambda^{\mathrm{ord}} \colon \mathbb{T}^{\mathrm{ord}}(U^p) \to \mathcal{I}$$

that involes the deformation of ψ and satisfies the assumptions from the previous section.

3.1. Hida family of theta lifts.

Let $\Psi := \psi^{-1}\langle * \rangle$, where $\langle * \rangle \colon \mathcal{G}_\mathcal{K} \to \mathcal{I}$ is the tautological character. In the previous work [Lee24] of the author, after enlarging E to a finite extension of $\bar{\mathbf{Q}}_p^{un}$, we can interpolate theta lifts of Hecke characters and obtain

- a Hida family $\mathcal{F} \in S^{\mathrm{ord}}(U^p, \mathcal{I})$ which induces the eigensystem $\lambda^{\mathrm{ord}} \colon \mathbb{T}^{\mathrm{ord}}(U^p) \to \mathcal{I}$ with

$$\lambda^{\mathrm{ord}}(T_w^{(1)}) = \epsilon^{-1}\hat{\chi}_\circ(\varpi_w) + \epsilon^{-1}\hat{\chi}_\circ\Psi(\varpi_w), \quad \lambda^{\mathrm{ord}}(U_w^{(1)}) = \epsilon^{-1}\hat{\chi}_\circ\Psi(\varpi_w). \tag{2}$$

- an element $L = \Omega_p^{-2\Sigma}\mathbf{B}(\mathcal{F}, \mathcal{F}) \in \mathcal{I}$, where $\mathbf{B} \colon S^{\mathrm{ord}}(U^p, \mathcal{I}) \times S^{\mathrm{ord}}(U^p, \mathcal{I}) \to \mathcal{I}$ is a certain Hecke-equivariant symmetric pairing, which induces a homomorphism $S^{\mathrm{ord}}(U^p, \mathcal{I}) \to M^{\mathrm{ord}}(U^p, \mathcal{I})$.

Here $\chi_\circ$ is an auxiliary Hecke character of $\mathcal{K}$ that defines the theta lift and $(\Omega_p, \Omega_\infty)$ are the CM periods associated to $\mathcal{K}$. Let α be Hecke character such that the p-adic avatar $\widehat{\alpha}$ defines a character of W^-. We have shown in [Lee25] that if α has the infinity type $(k+1)\Sigma$ for $k \geq 0$ and $\tilde{\alpha} := \alpha/\alpha^c$, then

$$\frac{1}{\Omega_p^{(2k+2)\Sigma}} \int_{W^-} \widehat{\alpha}\, L = C(\mathcal{K}, \chi) \left(\frac{2\pi}{\Omega_\infty}\right)^{(2k+2)\Sigma} \mathrm{Im}(\delta)^k (\psi^{-1}\tilde{\alpha})(z_\delta^{-1}) \frac{\Gamma((k+2)\Sigma)}{(2\pi)^{(k+2)\Sigma}} L(1, \psi^{-1}\tilde{\alpha}) \prod_{w \in \Sigma_p} E_w(\psi^{-1}\tilde{\alpha}),$$

where $\delta = -\delta^c \in \mathcal{K}$, $z_\delta \in \mathbf{A}_{\mathcal{K},f}^\times$, and $C(\chi, \mathcal{K}) \in \mathcal{O}$ are constant and $E_w(\psi^{-1}\tilde{\alpha})$ is the modified Euler factor

$$\frac{(1 - \psi^{-1}\tilde{\alpha}(\varpi_{\overline{w}})q_{\overline{w}}^{-1})(1 - \psi\tilde{\alpha}^{-1}(\varpi_w))}{\varepsilon(1, (\psi^{-1}\tilde{\alpha})_w, \psi_w)}$$

introduced by Coates and Perrin-Rious [Coa89]. From the interpolation formula we see that L coincides with the anticyclotomic p-adic L-function of Katz and Hida-Tilouine up to a units and the constant $C(\chi, \mathcal{K})$.

Remark 3.1. We have $C(\chi, \mathcal{K}) \neq 0$ if the central L-value of $\chi_\circ$ is nonzero. To the limited knowledge of the author, the most general result towards the existence of such an character $\chi_\circ$ with the properties

(χ1) $\chi_\circ$ restricts to $\epsilon_{\mathcal{K}/\mathcal{F}}|\cdot|_\mathcal{F}$, where $\epsilon_{\mathcal{K}/\mathcal{F}}$ is the quadratic character associated to $\mathcal{K}/\mathcal{F}$.
(χ2) $\chi_\circ$ is unramified at places above p and has the infinity type Σ^c.
(χ3) The central L-value of $\chi_\circ$ is nonzero.

is to combine the results of [Roh82] and [Hsi12], which actually produces the stronger result that the central value is nonzero modulo p. This is the main reason we impose the condition ($\mathcal{K}$).

3.2. **Main construction.** Let $\lambda\colon \mathbb{T}(U^p) \to \mathcal{I}$ denote the composition like before, From (2) we have

$$\lambda \circ T(U^p) = \epsilon^{-1}\hat{\chi}_\circ + \epsilon^{-1}\hat{\chi}_\circ \Psi.$$

Thus λ is surjective by Chebotarev's density and $\mathfrak{m}$ satisfies (red.gen) because of (ψ1). Moreover,

- (C1) holds if we let Θ be the evaluation map of $M^{\mathrm{ord}}(U^p, \mathcal{I})_{\mathfrak{m}_1} \cong \mathrm{Hom}_{\mathcal{I}}(S^{\mathrm{ord}}(U^p, \mathcal{I})_{\mathfrak{m}_1}, \mathcal{I})$ at $\mathcal{F}$.
- the first half of (C3) holds if we let $F = \Omega_p^{-2\Sigma}\mathbf{B}(*, \mathcal{F}) \in M^{\mathrm{ord}}(U^p, \mathcal{I})_{\mathfrak{m}_1}$, since the p-adic L-function $\Theta(F) = L$ is nonzero by [Hid10].

To verify the next half and (C2), we first recall the general process of constructing cohomology classes from a generically irreducible pseudo-representation. Suppose $T\colon \mathcal{G} \to R$ is a two-dimensional pseudo-representation into a Henselian local ring R with maximal ideal $\mathfrak{m}_R$ and of odd residual characteristic. Assume there exists distinct characters $\bar{\delta}_i\colon \mathcal{G} \to R/\mathfrak{m}_R$ of $\mathcal{G}$ for $i = 1, 2$ such that $T \equiv \bar{\delta}_1 + \bar{\delta}_2 \bmod \mathfrak{m}_R$. Fix a choice of $z \in \mathcal{G}$ with $\bar{\delta}_1(z) \neq \bar{\delta}_2(z)$ We then apply the Hensel lemma to the characteristic polynomial

$$P(z, X) \coloneqq X^2 - T(z)X + \det(T)(z) \equiv (X - \bar{\delta}_1(z))(X - \bar{\delta}_2(z)) \mod \mathfrak{m}_R$$

and lift $\bar{\delta}_1(z)$ and $\bar{\delta}_2(z)$ to roots α, β of $P(z, X)$ such that $\alpha - \beta \in R^\times$. We then define the functions

$$A(\sigma) = \frac{T(\sigma z) - \beta T(\sigma)}{\alpha - \beta} \quad D(\sigma) = \frac{T(\sigma z) - \alpha T(\sigma)}{\beta - \alpha} \quad x(\sigma, \tau) = a(\sigma\tau) - a(\sigma)a(\tau). \tag{3}$$

Then $A(\sigma) \equiv \bar{\delta}_1(\sigma)$, $D(\sigma) \equiv \bar{\delta}_2(\sigma) \bmod \mathfrak{m}_R$, and $x(\sigma, \tau)$ generate the reducibility ideal of T.

Moreover, when there exists σ_0, τ_0 such that $x = x(\sigma_0, \tau_0) \in R$ is not a zero divisor, the map

$$\sigma \mapsto \begin{pmatrix} A(\sigma) & B(\sigma) \\ C(\sigma) & D(\sigma) \end{pmatrix} \coloneqq \begin{pmatrix} A(\sigma) & x(\sigma, \tau_0)/x(\sigma_0, \tau_0) \\ x(\sigma_0, \sigma) & D(\sigma) \end{pmatrix} \in \mathrm{GL}_2(R[1/x]) \tag{4}$$

defines a group representation and satisfies $x(\sigma, \tau) = B(\sigma)C(\tau)$. Let $B \subset R[1/x]$ and $C \subset R$ be the R-submodules generated respectively by $B(\sigma)$ and $C(\sigma)$ for $\sigma \in \mathcal{G}$. The following proposition is a subcase of [BC09, Thm 1.5.5], which generalizes Ribet's lemma.

Proposition 3.2. *Given a homomorphism $\phi\colon R \to S$ and suppose $\phi \circ T$ is reducible, so that $\delta_1 \coloneqq \phi \circ A(\sigma)$ and $\delta_2 \coloneqq \phi \circ D(\sigma)$ become characters. Write $\delta = \delta_1\delta_2^{-1}$, then there exists injections*

$$\mathrm{Hom}_R(B, S) \hookrightarrow H^1(\mathcal{G}, S(\delta)) \quad f \mapsto \delta_2(\sigma)^{-1} f(B(\sigma))$$
$$\mathrm{Hom}_R(C, S) \hookrightarrow H^1(\mathcal{G}, S(\delta^{-1})) \quad f \mapsto \delta_1(\tau)^{-1} f(C(\tau))$$

Now, let $\delta_1 = \epsilon^{-1}\hat{\chi}_\circ$, $\delta_2 = \epsilon^{-1}\hat{\chi}_\circ\Psi$ and let $\bar{\delta}_i$ be the residual characters. As in previous section, we have two distinct characters $\chi_i = \omega\bar{\delta}_i \mid_{\mathcal{G}_{\mathbf{Q}_p}}$ and let $R^{\epsilon\zeta}$ denote the universal deformation ring of $\chi_1 + \chi_2$.

Fix $z \in \mathcal{G}_{\mathbf{Q}_p}$ such that $\chi_1(z) \neq \chi_2(z)$. We let α_0, β_0 be the roots of $P(z, X)$ and define the $R^{\epsilon\zeta}$-valued functions $A_0(\sigma)$, $D_0(\sigma)$, and $x_0(\sigma, \tau)$ on $\mathcal{G}_{\mathbf{Q}_p}$ by (3). Then pick $\sigma_0, \tau_0 \in \mathcal{G}_{\mathbf{Q}_p}$ such that $x_{\mathrm{red}} = x(\sigma_0, \tau_0)$ and let $B_0(\sigma)$, $C_0(\sigma)$ be defined by (4). Note that by definition $R^{\epsilon\zeta} = B_0 \coloneqq (B_0(\sigma), \sigma \in \mathcal{G}_{\mathbf{Q}_p})$.

Let $\alpha, \beta \in \mathbb{T}_{\mathfrak{m}}$ denote the images of α_0, β_0 under $R^{\zeta\epsilon} \to \mathbb{T}_{\mathfrak{m}}$. We similarly define the $\mathbb{T}_{\mathfrak{m}}$-valued functions $A(\sigma), D(\sigma)$, and $x(\sigma, \tau)$ on $\mathcal{G}_{\mathcal{K}}$ by (3), then the restrictions of the functions to $\mathcal{G}_{\mathbf{Q}_p}$ coincide with the images of $A_0(\sigma), D_0(\sigma)$, and $x_0(\sigma, \tau)$. In particular $x_{\mathrm{red}} = x(\sigma_0, \tau_0)$ is not a zero divisor by Proposition 2.3. We can then define $B(\sigma)$ and $C(\sigma)$, which also coincide with the images of $B_0(\sigma)$ and $C_0(\sigma)$ when restricted to $\mathcal{G}_{\mathbf{Q}_p}$.

Lemma 3.3. *Let $x_i(\sigma, \tau)$ denote the image of $x(\sigma, \tau)$ in $\mathbb{T}^{\mathrm{ord}}_{\mathfrak{m}_i}$ for $i = 1, 2$, then*

- *$x_1(\sigma, \tau) = 0 = x_2(\tau, \sigma)$ for $\sigma \in D_{w_0}$ and $\tau \in \mathcal{G}_{\mathcal{K}}$.*
- *$x_1(\sigma, \tau) = 0 = x_2(\tau, \sigma)$ for $\sigma \in \mathcal{G}_{\mathcal{K}}$ and $\tau \in D_{\overline{w}_0}$.*

The lemma essentially follows from that the ordinary Galois representation to $\mathbb{T}^{\mathrm{ord}}_{\mathfrak{m}_i}$ has $\epsilon^{-1}\hat{\chi}_\circ\Psi$ as a subrepresentation when $i=1$ and has $\epsilon^{-1}\hat{\chi}_\circ$ as a subrepresentation when $i=2$. We now define the ideal

$$\mathfrak{q} = (x(\sigma,\tau_0) = B(\sigma)C(\tau_0) = B(\sigma)x_{\mathrm{red}} \mid \sigma \in \mathcal{G}_{\mathcal{K}}) \subset \mathbb{T}_{\mathfrak{m}}.$$

Then $x_{\mathrm{red}} \in \mathfrak{q}$ be definition; $\mathfrak{q}F = 0$ since the action factors through λ and $\lambda \circ T(U^p)$ is reducible; and $\mathfrak{q}M^{\mathrm{ord}}(U^p,\mathcal{I})_{\mathfrak{m}_2} = 0$ since $x_2(\sigma,\tau_0) = 0$ by the lemma. Thus we have verified (C2) and (C3).

We may now apply Proposition 2.4 and get the commutative diagram

$$\begin{array}{ccccccccc}
 & & (M/S) & \longrightarrow & (M/S)\otimes_{\mathbb{T}_{\mathfrak{m}}}\mathfrak{q} & \xrightarrow{\ q\ } & (M/S)\otimes_{\mathbb{T}_{\mathfrak{m}}}\mathfrak{q}^{\mathrm{red}} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle s} & & \\
0 & \longrightarrow & M_1^{\mathrm{ord}}/S_1^{\mathrm{ord}} & \longrightarrow & \mathfrak{q}M/\mathfrak{q}S & \xrightarrow{\ r\ } & \mathfrak{q}M_1^{\mathrm{ord}}/\mathfrak{q}S_1^{\mathrm{ord}} & \longrightarrow & 0
\end{array}$$

Definition 3.4. We define $f \in \mathrm{Hom}_{\mathbb{T}_{\mathfrak{m}}}(B,\mathcal{I})$ for $B = (B(\sigma), \sigma \in \mathcal{G}_{\mathcal{K}}) \subset \mathbb{T}_{\mathfrak{m}}[1/x_{\mathrm{red}}]$ as follows. Let $\tilde{F} \in M$ be a preimage of F mod S_1^{ord}, which is unique modulo S. For $b \in B$, let $q(b) = bx_{\mathrm{red}} = bC(\tau_0) \in \mathfrak{q}$ and define

$$y'(b) = (\tilde{F} \bmod S) \otimes q(b) \in (M/S)\otimes_{\mathbb{T}_{\mathfrak{m}}}\mathfrak{q}$$
$$y(b) = p(y'(b)) = q(b)\tilde{F} \bmod \mathfrak{q}S \in \mathfrak{q}M/\mathfrak{q}S$$

On the other hand, observe that $(M/S)\otimes_{\mathbb{T}_{\mathfrak{m}}}\mathfrak{q}^{\mathrm{red}} = (M_1^{\mathrm{ord}}/S_1^{\mathrm{ord}})\otimes_{\mathbb{T}_{\mathfrak{m}}}\mathfrak{q}^{\mathrm{red}}$ and therefore

$$q(y'(b)) = (F \bmod S_1^{\mathrm{ord}}) \otimes q(b) \in (M_1^{\mathrm{ord}}/S_1^{\mathrm{ord}})\otimes_{\mathbb{T}_{\mathfrak{m}}}\mathfrak{q}^{\mathrm{red}}$$
$$r(y(b)) = (s\circ q)(y'(b)) = q(b)F \bmod \mathfrak{q}S_1^{\mathrm{ord}} \in \mathfrak{q}M^{\mathrm{ord}}/\mathfrak{q}S^{\mathrm{ord}}$$

But $q(b)F = 0$ by (C2), so there exists $F(b) \in M_1^{\mathrm{ord}}/S_1^{\mathrm{ord}}$ that maps to $y(b)$. We define $f(b) = \Theta(F(b)) \in \mathcal{I}$.

Observe that if $b \in \mathbb{T}_{\mathfrak{m}}$, then we can take $F(b) = qF$ and $f(b) = qL$. In particular this is the case if $b = B(\sigma)$ for $\sigma \in \mathcal{G}_{\mathbf{Q}_p}$. On the other hand, if $w \in \Sigma_p \setminus \{w_0\}$ and $\sigma_1,\sigma_2 \in D_w$ we can show that

$$(A(\sigma_1) - D(\sigma_1))B(\sigma_2) = (A(\sigma_2) - D(\sigma_2))B(\sigma_1).$$

Moreover, let $\mathfrak{f}$ be a square-free product of split primes such that $\mathfrak{f}$ is prime to p and $\mathfrak{f}+\mathfrak{f}^c = \mathcal{O}_{\mathcal{K}}$. We let $\mathcal{R}$ be the set of $\mathfrak{s} := \mathfrak{f}\mathfrak{f}^c$ such that $\iota_w^{-1}(\mathrm{GL}_2(\mathcal{O}_w)) \subset U^p$ for all $w \mid \mathfrak{f}$. We then define $U^p_{\mathfrak{s}} \subset U^p$ by replacing U_v with $\mathrm{Iw}(w^{1,1})$ when $w \mid \mathfrak{f}$ and $w \mid v$, and define $\mathcal{I}_{\mathfrak{s}} = \mathcal{O}[\![\mathfrak{G}^a_{\mathfrak{s}}]\!]$, where $\mathfrak{G}^a_{\mathfrak{s}}$ is the Galois group of the maximal anticyclotomic pro-p abelain extension $\widetilde{\mathcal{K}}^a_{\mathfrak{s}}$ that is unramified away $p\mathfrak{s}$.

For $\mathfrak{s} \in \mathcal{R}$, let $\Psi_{\mathfrak{s}} := \psi^{-1}\langle * \rangle_{\mathfrak{s}}$, where $\langle * \rangle_{\mathfrak{s}} \colon \mathcal{G}_{\mathcal{K}} \to \mathcal{I}_{\mathfrak{s}}$ is the tautological character. Then the construction in [Lee24] also gives a Hida family $\mathcal{F}_{\mathfrak{s}} \in S^{\mathrm{ord}}(U^p_{\mathfrak{s}},\mathcal{I}_{\mathfrak{s}})$ and an eigensystem $\lambda_{\mathfrak{s}} := \mathbb{T}^{\mathrm{ord}}(U^p_{\mathfrak{s}}) \to \mathcal{I}_{\mathfrak{s}}$. satisfying (2). Furthermore, when $\mathfrak{s}, \ell\mathfrak{s} \in \mathcal{R}$ we have the commutative diagram

$$\begin{array}{ccccc}
 & \nearrow & \mathbb{T}(U^p_{\ell\mathfrak{s}})_{\mathfrak{m}} & \xrightarrow{\ \lambda_{\ell\mathfrak{s}}\ } & \mathcal{I}_{\ell\mathfrak{s}} \\
 & & \downarrow{\scriptstyle \varphi^{\ell\mathfrak{s}}_{\mathfrak{s}}} & & \downarrow{\scriptstyle \phi^{\ell\mathfrak{s}}_{\mathfrak{s}}} \\
R^{\zeta\epsilon} & \longrightarrow & \mathbb{T}(U^p_{\mathfrak{s}})_{\mathfrak{m}} & \xrightarrow{\ \lambda_{\mathfrak{s}}\ } & \mathcal{I}_{\mathfrak{s}}
\end{array}$$

where $\varphi^{\ell\mathfrak{s}}_{\mathfrak{s}}$ is induced by the obvious inclusions between algebraic modular forms and $\phi^{\ell\mathfrak{s}}_{\mathfrak{s}}$ is induced by the homomorphis $\mathfrak{G}^a_{\ell\mathfrak{s}} \to \mathfrak{G}^a_{\mathfrak{s}}$. Then if $\ell = \mathfrak{l}\mathfrak{l}^c$ for a prime $\mathfrak{l}$, we can show that

$$\varphi^{\ell\mathfrak{s}}_{\mathfrak{s}}(B_{\ell\mathfrak{s}}(\sigma)) = B_{\mathfrak{s}}(\sigma) \text{ if } \sigma \in \mathcal{G}_{\mathcal{K}} \text{ and } \phi^{\ell\mathfrak{s}}_{\mathfrak{s}} \circ f_{\ell\mathfrak{s}} = (1 - \epsilon\Psi_{\mathfrak{s}}(\varpi_{\bar{\mathfrak{l}}}))(1 - \Psi_{\mathfrak{s}}(\varpi_{\bar{\mathfrak{l}}})) \cdot (f_{\mathfrak{s}} \circ \varphi^{\ell\mathfrak{s}}_{\mathfrak{s}}).$$

We now apply Proposition 3.2 and let $Z_{\mathfrak{s}} \in H^1(\mathcal{K}, \mathcal{I}_{\mathfrak{s}}(\Psi_{\mathfrak{s}}^{-1}))$ be the class constructed from $B_{\mathfrak{s}}(\sigma)$ and $f_{\mathfrak{s}}$; and $Z_{\mathfrak{s},p} \in H^1(\mathcal{G}_{\mathbf{Q}_p}, \mathcal{I}_{\mathfrak{s}}(\Psi_{\mathfrak{s}}^{-1}))$ be the class constructed from $B_{\mathfrak{s}}(\sigma)$ and $B_0 = R^{\epsilon\zeta} \to \mathbb{T}(U^p_{\mathfrak{s}}) \to \mathcal{I}_{\mathfrak{s}}$. We omit the subscript when $\mathfrak{s} = \mathcal{O}_{\mathcal{K}}$. Then by the discussion above we obtain the following theorem.

Theorem 3.5. *The cohomology classes* $\{\mathcal{Z}_{\mathfrak{s}} \in H^1(\mathcal{K}, \mathcal{I}_{\mathfrak{s}}(\Psi_{\mathfrak{s}}^{-1}))\}_{\mathfrak{s}\in\mathcal{R}}$ *has the following properties.*

1. *There exists a finite set* S *of primes of* $\mathcal{K}$ *such that* $\mathcal{Z}_{\mathfrak{s}}$ *is unramified at* w *if* $w \notin S$ *and* $w \nmid \mathfrak{s}$.
2. *If* $w \in \Sigma_p \setminus \{w_0\}$, $\mathrm{loc}_w(\mathcal{Z}_{\mathfrak{s}}) \in H^1(\mathcal{K}_w, \mathcal{I}_{\mathfrak{s}}(\Psi_{\mathfrak{s}}^{-1}))$ *is annihilated by* $(\Psi_{\mathfrak{s}}^{-1}(\sigma) - 1)$ *for any* $\sigma \in D_w$.
3. *At* w_0, $\mathrm{loc}_{w_0}(\mathcal{Z}) \in H^1(\mathbf{Q}_p, \mathcal{I}(\Psi^{-1}))$ *coincides with* $L \cdot \mathcal{Z}_p$.
4. *If* $\ell\mathfrak{s}, \mathfrak{s} \in \mathcal{R}$ *and* $\ell = \mathfrak{l}\mathfrak{l}^c$ *for a prime* $\mathfrak{l}$, *then* $\phi_{\mathfrak{s}}^{\ell\mathfrak{s}}(\mathcal{Z}_{\ell\mathfrak{s}}) = (1 - \epsilon\Psi_{\mathfrak{s}}(\varpi_{\bar{\mathfrak{l}}}))(1 - \Psi_{\mathfrak{s}}(\varpi_{\bar{\mathfrak{l}}})) \cdot \mathcal{Z}_{\mathfrak{s}}$

We call $\{\mathcal{Z}_{\mathfrak{s}} \in H^1(\mathcal{K}, \mathcal{I}_{\mathfrak{s}}(\Psi_{\mathfrak{s}}^{-1}))\}_{\mathfrak{s}\in\mathcal{R}}$ an anticyclotomic Euler system because of the last property. We also remark that the third property should be viewed as a form of explicit reciprocity law.

3.3. **Application to the main conjecture.** Write $\Lambda_W^- = \mathcal{I}$ and let $\psi_\Lambda := \psi\langle *\rangle$. To comform to the previous notations, we replace Σ_p with Σ_p^c and consider instead the main conjecture for

$$\mathrm{Sel}(\psi_\Lambda, \Sigma_p^c) = \ker\left\{H^1(\mathcal{K}, \Lambda_W^{-*}(\psi_\Lambda)) \to \prod_{w\notin\Sigma_p^c} H^1(I_w, \Lambda_W^{-*}(\psi_\Lambda))\right\},$$

where $\Lambda_W^{-*} = \mathrm{Hom}^{\mathrm{cts}}(\Lambda_W^-, \mathbf{Q}_p/\mathbf{Z}_p)$. From the algebraic functional equation and the technique from [Hsi10], we can first reduce the theorem to showing that $(L_p(\psi_\Lambda, \Sigma_p^c))$ is contained in the characteristic ideal of

$$\mathrm{Sel}^{str}(\epsilon\psi_\Lambda^{-1}, \Sigma_p) = \ker\left\{H^1(\mathcal{K}, \Lambda_W^{-*}(\epsilon\psi_\Lambda^{-1})) \to \prod_{w\nmid p} H^1(I_w, \Lambda_W^{-*}(\epsilon\psi_\Lambda^{-1})) \times \prod_{w\in\Sigma_p^c} H^1(\mathcal{K}_w, \Lambda_W^{-*}(\epsilon\psi^{-1}\langle *\rangle^{-1}))\right\}$$

in $E[\![W^-]\!]$. In fact, let $L^\iota \in \Lambda_W^-$ denote the image of L under the map $\langle\gamma\rangle \mapsto \langle\gamma\rangle^{-1}$. We can replace $L_p(\psi_\Lambda, \Sigma_p^c)$ with (L^ι) since the interpolation formula implies that the two generates the same ideal in $E[\![W^-]\!]$. We then apply the specialization principle [Och05] and further reduce the theorem to the proposition below.

Proposition 3.6. *For almost all characters* $\alpha\colon W^- \to \mathcal{O}$, *the length of*

$$\ker\left\{H^1(\mathcal{K}, (E/\mathcal{O})(\epsilon(\psi\alpha)^{-1})) \to \prod_{w\nmid p} H^1(I_w, (E/\mathcal{O})(\epsilon(\psi\alpha)^{-1})) \prod_{w\in\Sigma_p^c} H^1(\mathcal{K}_w, (E/\mathcal{O})(\epsilon(\psi\alpha)^{-1}))\right\}$$

is finite and bounded by the length of the $\mathcal{O}/(\alpha(L^\iota))$.

Let $\alpha^\iota\colon \mathcal{I}_{\mathfrak{s}} \to \mathcal{I}_{\mathfrak{s}}$ be the homomorphism induced by $\langle\gamma\rangle_{\mathfrak{s}} \mapsto \alpha^{-1}(\gamma)\langle\gamma^{-1}\rangle_{\mathfrak{s}}$ Then α^ι defines an isomorphism between Galois modules $\alpha^\iota\colon \mathcal{I}_{\mathfrak{s}}(\Psi_{\mathfrak{s}}^{-1}) = \mathcal{I}_{\mathfrak{s}}(\psi\langle *\rangle_{\mathfrak{s}}^{-1}) \to \mathcal{I}_{\mathfrak{s}}(\psi\alpha\langle *\rangle_{\mathfrak{s}})$ and by Shapiro lemma (cf [SV16, Lem 5.8] for example), we then have an isomorphism between Galois modules

$$\alpha^\iota(\mathcal{Z}_{\mathfrak{s}}) \in H^1(\mathcal{K}, \mathcal{I}_{\mathfrak{s}}(\psi\alpha\langle *\rangle_{\mathfrak{s}})) \cong \varprojlim_{L\subset\mathcal{K}_{\mathfrak{s}}^a} H^1(L, \mathcal{O}(\psi\alpha))$$

where $\mathcal{G}_{\mathcal{K}}$ acts on the left by $\langle *\rangle$ and on the right by the usual Galois actions on cohomology groups. Translating the result from Theorem 3.5, we have the collection of classes $\{z_L \in H^1(L, \mathcal{O}(\psi\alpha))\}_{L\subset\widetilde{\mathcal{K}}_{\mathfrak{s}}^a,\, \mathfrak{s}\in\mathcal{R}}$ with the following properties.

1. If $L \subset \widetilde{\mathcal{K}}_{\mathfrak{s}}^a$, then z_L is unramified at w if $w \notin S$ and $w \nmid \mathfrak{s}$.
2. If $w \in \Sigma_p \setminus \{w_0\}$, $\mathrm{loc}_w(z_L) \in H^1(L_w, \mathcal{O}(\psi\alpha))$ is annihilated by $((\psi\alpha)(\sigma) - 1)$ for any $\sigma \in D_w$.
3. Let $z \in H^1(\mathcal{K}, \mathcal{O}(\psi\alpha))$ and $z_p \in H^1(\mathbf{Q}_p, \mathcal{O}(\psi\alpha))$ be the bottom class of $\alpha^\iota(\mathcal{Z})$ and $\alpha^\iota(\mathcal{Z}_p)$, then
$$\mathrm{loc}_{w_0}(z) = \alpha(L^\iota) \cdot z_p \in H^1(\mathbf{Q}_p, \mathcal{O}(\psi\alpha)).$$
4. Let $\ell\mathfrak{s}, \mathfrak{s} \in \mathcal{R}$ with $\ell = \mathfrak{l}\mathfrak{l}^c$ for a prime $\mathfrak{l}$ and define $P_{\mathfrak{l}}(X) = (1 - \epsilon(\psi\alpha)^{-1}(\varpi_{\bar{\mathfrak{l}}})X)(1 - (\psi\alpha)^{-1}(\varpi_{\bar{\mathfrak{l}}})X)$. Suppose $L_{\mathfrak{s}} \subset \widetilde{\mathcal{K}}_{\mathfrak{s}}^a$ and $L_{\mathfrak{s}\ell} \subset \widetilde{\mathcal{K}}_{\mathfrak{s}\ell}^a$ and the inclusions do not hold for larger ideals, then
$$\mathrm{Cor}_{L_{\mathfrak{s}}}^{L_{\mathfrak{s}\ell}}(z_{L_{\mathfrak{s}\ell}}) = P_{\mathfrak{l}}(\mathrm{Fr}_{\mathfrak{l}}) \cdot z_{L_{\mathfrak{s}}}.$$

Now the proposition can be proved with the general machinery in [Rub00] after a slight modification.

Acknowledgement

The author would like to thank Professor Kenichi Namikawa for the opportunity to present and his hospitality during the conference. The author would also like to thank Professor Eric Urban and Professor Ming-Lun Hsieh for their support and encouragement during the period of the project.

References

[BC09] Joël Bellaïche and Gaëtan Chenevier, *Families of Galois representations and Selmer groups*, Astérisque, no. 324, Société mathématique de France, 2009 (en). MR 2656025

[Coa89] John Coates, *On p-adic L-functions attached to motives over Q II*, Boletim da Sociedade Brasileira de Matemática **20** (1989), no. 1, 101–112.

[Eme10] Matthew Emerton, *Ordinary parts of admissible representations of p-adic reductive groups I. Definition and first properties*, Représentations p-adiques de groupes p-adiques III : méthodes globales et géométriques, Astérisque, no. 331, Société mathématique de France, 2010.

[Ger18] David Geraghty, *Modularity lifting theorems for ordinary galois representations*, Mathematische Annalen **373** (2018), no. 3–4, 1341–1427.

[Hid06a] Haruzo Hida, *Anticyclotomic main conjectures*, Doc. Math. (2006), 465–532. MR 2290595

[Hid06b] ———, *Hilbert modular forms and Iwasawa theory*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, Oxford, 2006. MR 2243770

[Hid10] ———, *The Iwasawa μ-invariant of p-adic Hecke L-functions*, Annals of Mathematics **172** (2010), no. 1, 41–137.

[Hsi10] Ming-Lun Hsieh, *The algebraic functional equation of Selmer groups for CM fields*, Journal of Number Theory **130** (2010), no. 9, 1914–1924.

[Hsi12] ———, *On the non-vanishing of Hecke L-values modulo p*, American Journal of Mathematics **134** (2012), no. 6, 1503–1539.

[HT93] H. Hida and J. Tilouine, *Anti-cyclotomic Katz p-adic L-functions and congruence modules*, Ann. Sci. École Norm. Sup. (4) **26** (1993), no. 2, 189–259. MR 1209708

[HT94] ———, *On the anticyclotomic main conjecture for CM fields.*, Inventiones mathematicae **117** (1994), no. 1, 89–148.

[Kat78] Nicholas M. Katz, *p-Adic L-functions for CM fields*, Inventiones Mathematicae **49** (1978), no. 3–4, 199–297.

[Lee24] Yu-Sheng Lee, *Hida family of theta lift from U(1) to definite U(2)*, 2024, preprint at https://arxiv.org/abs/2406.12351.

[Lee25] ———, *Anticyclotomic Euler Systems for CM Fields*, 2025, preprint.

[Och05] Tadashi Ochiai, *Euler system for Galois deformations*, Annales de l'institut Fourier **55** (2005), no. 1, 113–146.

[Pan22] Lue Pan, *The Fontaine-Mazur conjecture in the residually reducible case*, Journal of the American Mathematical Society **35** (2022), no. 4, 1031–1169.

[Pas13] Vytautas Paskūnas, *The image of Colmez's Montreal functor*, Publications Mathématiques de l'IHÉS **118** (2013), 1–191 (en). MR 3150248

[Rib76] Kenneth A. Ribet, *A modular construction of unramified p-extensions of $Q(\mu_p)$*, Inventiones Mathematicae **34** (1976), no. 3, 151–162.

[Roh82] David E. Rohrlich, *Root Numbers of Hecke L-Functions of CM Fields*, American Journal of Mathematics **104** (1982), no. 3, 517.

[Rub00] Karl Rubin, *Euler systems*, Annals of Mathematics Studies, vol. 147, Princeton University Press, Princeton, NJ, 2000, Hermann Weyl Lectures. The Institute for Advanced Study. MR 1749177

[SV16] Peter Schneider and Otmar Venjakob, *Coates–Wiles Homomorphisms and Iwasawa Cohomology for Lubin–Tate Extensions*, p. 401–468, Springer International Publishing, 2016.

[Urb20] Eric Urban, *Euler system and Eisenstein Congruences*, 2020, preprint at https://www.math.columbia.edu/~urban/EURP.html.

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA
Email address: yushglee@umich.edu