Local Intertwining Relations

Hiraku Atobe
Department of Mathematics, Kyoto University

Abstract

The local intertwining relations are the second main local theorem in Arthur's theory of endoscopic classification. They are identities that give precise information about the action of normalized intertwining operators on parabolically induced representations. In this report, we explain the tempered generic case, the untwisted case for general linear groups, and the co-tempered case for quasi-split classical groups. These are required as the seed cases in the inductive proof of the endoscopic classification for quasi-split classical groups due to Arthur [Ar] and Mok [Mok]. This is based on a joint work [AGIKMS] with Atsushi Ichino (Kyoto), Wee Teck Gan (Singapore), Alberto Minguez (Vienna), Tasho Kaletha (Bonn, Michigan), and Sug Woo Shin (UC Berkeley).

1 Background

Class Field Theory (1920–1940), developed by Takagi and reformulated by Artin, was one of the greatest results in Number Theory in the early 20th century. It has may applications, for example:

- Decomposition law of the prime ideals for abelian extensions of number fields.
- Reciprocity law of the power residues.
- Čebotarev density theorem (theorem on arithmetic progressions).
- Kronecker's Jugendtraum (proven by Takagi).
- Grunwald–Wang theorem.

One of the most amazing statement in the late 20th century is *Shimura–Taniyama conjecture* (1955–1960's), which asserts that all elliptic curves over $\mathbb{Q}$ are modular. Before the conjecture was proven, many examples and many applications were founded. One of the most famous applications would be as follows.

Theorem 1.1 (Frey, Serre, Ribet (1984–1986))**.** *There would exist a counterexample for Shimura–Taniyama conjecture* if *Fermat's Last Theorem were to be false.*

After this theorem, Wiles together with Taylor (1995) established Shimura–Taniyama conjecture (for the semistable case), and has concluded that Fermat's Last Theorem is also true.

Now we are in the 21st century so that let us go to the next stage. One of the ultimate goal in Number Theory nowadays is the *Langlands conjecture*. It is very roughly stated as follows.

Conjecture 1.2 (Langlands conjecture (1967, 1970)). *For a connected (split) reductive group G over a number field F, there would be a correspondence between*

- *automorphic representations of G; and*
- *"Galois representations" valued in $\widehat{G}$,*

where $\widehat{G}$ is the Langlands dual group of G.

For example, if $G = \mathrm{GL}_N$ (resp. $G = \mathrm{Sp}_{2n}$), then $\widehat{G} = \mathrm{GL}_N(\mathbb{C})$ (resp. $\widehat{G} = \mathrm{SO}_{2n+1}(\mathbb{C})$). The Langlands conjecture for GL_1 (resp. GL_2) is Class Field Theory (resp. contains Shimura–Taniyama conjecture). In general, this conjecture is beyond our current technology.

Since "Galois representations" valued in $\mathrm{SO}_{2n+1}(\mathbb{C})$ are regarded as ones valued in $\mathrm{GL}_{2n+1}(\mathbb{C})$, assuming the Langlands conjecture for Sp_{2n} and GL_{2n+1}, one might expect that (discrete) automorphic representations of Sp_{2n} could be classified by (discrete) automorphic representations of GL_{2n+1}. Arthur studied this observation well, and obtained a more precise statement as a conjecture in 1980's.

Arthur continued to study this conjecture, and as a crowning achievement of the theory of endoscopy, he has proved:

Theorem 1.3 (Arthur (2013) [Ar], Mok (2015) [Mok]). *Arthur's conjecture and the local Langlands conjecture hold for quasi-split classical groups.*

Similar to Class Field Theory and the Shimura–Taniyama conjecture, this result has many applications. For example:

- new instances of Langlands conjecture by Scholze [Sc] and Kret–Shin [KS1, KS2];
- p-adic Gross–Zagier and Beilinson–Bloch–Kato conjectures by Y. Liu et al. [DL, LL, LTXZZ];
- Gan–Gross–Prasad conjecture by Waldspurger [W] and Beuzart-Plessis et al. [BP1, BP2, BP3, BPLZZ, BPCZ, BPC];
- an extension of Shimura correspondence by Gan–Ichino [GI] and W.-W. Li [L];
- Harder's conjecture by Chenevier–Lannes [CL] and Ibukiyama–Katsurada et al [ACIKY1, ACIKY2].

Arthur's result is really magnificent, but in his book, there are three issues.

1. Although the unweighted version (resp. weighted version for split groups) of Fundamental Lemma established by Ngô, Waldspurger and others (see [N] and [LMW]) (resp. Chaudouard–Laumon [CL1, CL2]), Arthur's argument also needs the weighted Fundamental Lemma (WFL) for quasi-split groups, which is still open.
2. Stabilization of the twisted trace formula and local trace formula were required, but they were established by Mœglin–Waldspurger [MW1, MW2, MW3] assuming twisted WFL.

3. Arthur's 4 papers [A24, A25, A26, A27] in preparation. Among of them, [A24] is included in [MW1], but the others are open.

The purpose of our paper [AGIKMS] is to give proofs of the theorems in [A25, A26, A27], which concern the *Local Intertwining Relations*. In conclusion, Arthur's endoscopic classification for quasi-split classical groups and their several applications will become unconditional as soon as the twisted weighted fundamental lemma is fully verified.

2 Local Intertwining Relation (LIR)

Let F be a p-adic field (for simplicity), W_F the Weil group of F, $\psi_F\colon F \to \mathbb{C}^\times$ a fixed non-trivial additive character. Consider $G = \mathrm{GL}_n(F)$ or $G = \mathrm{Sp}_{2n}(F)$ (for simplicity). Fix a standard parabolic subgroup $P = MN$ of G. Assume Arthur's results for M as an induction hypothesis. Let $\psi_M\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{M}$ be a local A-parameter for M, and let Π_{ψ_M} be the A-packet of ψ_M, which is a multi-set of $\mathrm{Irr}_{\mathrm{unit}}(M)$. Note that ψ_M gives an A-parameter ψ of G by the embedding $\widehat{M} \hookrightarrow \widehat{G}$. Then we define Π_ψ by the multi-set consisting of irreducible components π of $I_P(\pi_M) = \mathrm{Ind}_P^G(\pi_M)$ for $\pi_M \in \Pi_{\psi_M}$. For a Weyl element $w \in W^G$, one can choose its representative $\widetilde{w} \in G$. Then we can define a representation $w\pi_M$ of $\widetilde{w}M\widetilde{w}^{-1}$ by $(w\pi_M)(m) = \pi_M(\widetilde{w}^{-1}m\widetilde{w})$.

If $\widetilde{w}M\widetilde{w}^{-1} = M$ and $w\pi_M \cong \pi_M$, one can define the normalized intertwining operator

$$R_P(w, \widetilde{\pi}_M, \psi_M)\colon I_P(\pi_M) \to I_P(\pi_M)$$

by the meromorphic continuation of

$$f \mapsto \gamma_A(0, \psi_M, \mathrm{Ad}_{\mathfrak{n}}^\vee, \psi_F) \int_{(\widetilde{w}N\widetilde{w}^{-1}\cap N)\backslash N} \mathcal{A}_w(f(\widetilde{w}^{-1}ug))du,$$

where $\mathcal{A}_w\colon w\pi_M \xrightarrow{\sim} \pi_M$ is a normalized isomorphism, and

$$\gamma_A(s, \psi_M, \rho, \psi_F) = \varepsilon(s, \rho \circ \phi_{\psi_M}, \psi_F) \frac{L(1+s, \rho \circ \phi_{\psi_M})}{L(s, \rho \circ \phi_{\psi_M})}$$

is Arthur's gamma factor.

Our main theorem is stated as follows.

Theorem 2.1. *Suppose that $w\pi_M \cong \pi_M$. Let $\pi \subset I_P(\pi_M)$ be an irreducible summand.*

1. *([A27], [AGIKMS, Theorem 1.8.1])*
   If π is tempered and generic, then $R_P(w, \widetilde{\pi}_M, \psi_M)|_\pi = \mathrm{id}_\pi$.

2. *([A26], [AGIKMS, Theorem 1.9.1])*
   If $G = \mathrm{GL}_n(F)$, then $R_P(w, \widetilde{\pi}_M, \psi_M)|_\pi = \mathrm{id}_\pi$.

3. *([A25], [AGIKMS, Theorem 1.10.5 (2)])*
   If $G = \mathrm{Sp}_{2n}(F)$ and if ψ_M is co-tempered, i.e., $\psi_M|_{\{1_{W_F}\}\times \mathrm{SL}_2(\mathbb{C}) \times \{\mathbf{1}_2\}} = \mathbf{1}$, then

   $$R_P(w, \widetilde{\pi}_M, \psi_M)|_\pi = \langle s_u, \pi \rangle_\psi \cdot \mathrm{id}_\pi \quad \text{(LIR)}$$

   for $w = w_u$ corresponding to $s_u \in \mathcal{S}_\psi$.

Note that to show (LIR) in general by a global argument, Arthur used the co-tempered case as a seed case.

For the tempered and generic case, almost the same statement was known by Shahidi [Sh]. What we have to do is to compare Arthur's normalizing factors with Shahidi's local coefficients, as Arthur suggested in [Ar, Section 2.5].

For the non-tempered cases, Arthur gave some hints, but we gave a different approach.

3 LIR for non-tempered representations of $\mathrm{GL}_n(F)$

In this section, we explain our idea for the second part of Theorem 2.1 in the non-tempered case.

For $s_1, \dots, s_n \in \mathbb{C}$, write

$$I(s_1, \dots, s_n) = \mathrm{Ind}_B^{\mathrm{GL}_n(F)}(|\cdot|^{s_1} \boxtimes \cdots \boxtimes |\cdot|^{s_n})$$

for the principal series representation. It is called a *standard module* if $\mathrm{Re}(s_1) \geq \cdots \geq \mathrm{Re}(s_n)$, in which case, it has the unique irreducible quotient called the *Langlands quotient*. For example,

$$I\left(s + \frac{n-1}{2}, s + \frac{n-3}{2}, \dots, s - \frac{n-1}{2}\right) \twoheadrightarrow |\mathrm{det}_{\mathrm{GL}_n(F)}|^s.$$

Let S_d be the unique irreducible algebraic representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension d. In this report, we only explain a particular example as follows.

Suppose that $G = \mathrm{GL}_6(F) \supset P = MN \supset M = \mathrm{GL}_3(F) \times \mathrm{GL}_3(F)$, and $\psi_M = S_3 \oplus S_3$ so that $\pi_M = \mathbf{1}_{\mathrm{GL}_3(F)} \boxtimes \mathbf{1}_{\mathrm{GL}_3(F)}$. Then $\pi = I_P(\pi_M) = \mathbf{1}_{\mathrm{GL}_3(F)} \times \mathbf{1}_{\mathrm{GL}_3(F)}$ is irreducible.

The standard module of π_M is $I(1,0,-1) \boxtimes I(1,0,-1)$. We can realize π as the unique subrepresentation of $I(-1,0,1,-1,0,1)$, which is the image of the normalized intertwining operator

$$R(w_1)\colon I(1,0,-1,1,0,-1) \to I(-1,0,1,-1,0,1)$$

with $w_1 = ((1,3),(4,6)) \in \mathfrak{S}_6 = W^G$. On the other hand, the standard module of π is $I(1,1,0,0,-1,-1)$. There is a normalized intertwining operator

$$R(w_2)\colon I(1,1,0,0,-1,-1) \to I(1,0,-1,1,0,-1)$$

with $w_2 = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 2 & 5 & 3 & 6 \end{smallmatrix}\right) \in \mathfrak{S}_6 = W^G$. The following lemma is a consequence of the multiplicity one result for the Langlands quotient.

Lemma 3.1 ([AGIKMS, Lemma 3.3.1])**.** *The composition $R(w_1) \circ R(w_2)$ is nonzero. Hence it realizes the Langlands quotient map*

$$R(w_1) \circ R(w_2)\colon I(1,1,0,0,-1,-1) \twoheadrightarrow \mathbf{1}_{\mathrm{GL}_3(F)} \times \mathbf{1}_{\mathrm{GL}_3(F)}.$$

Now the main idea to establish the local intertwining relation (LIR) in this case is to show the following theorem.

Theorem 3.2 ([AGIKMS, Theorem 3.4.2])**.** *The "main diagram"*

$$\begin{array}{ccc} I(1,1,0,0,-1,-1) & \xrightarrow{R((1,2),(3,4),(5,6))} & I(1,1,0,0,-1,-1) \\ {\scriptstyle R((2,4,5,3))}\downarrow & & \downarrow{\scriptstyle R((2,4,5,3))} \\ I(1,0,-1,1,0,-1) & & I(1,0,-1,1,0,-1) \\ {\scriptstyle R((1,3),(4,6))}\downarrow & & \downarrow{\scriptstyle R((1,3),(4,6))} \\ \mathbf{1}_{\mathrm{GL}_3(F)} \times \mathbf{1}_{\mathrm{GL}_3(F)} & \xrightarrow{R\left(\left(\begin{smallmatrix} 0 & \mathbf{1}_3 \\ \mathbf{1}_3 & 0 \end{smallmatrix}\right)\right)} & \mathbf{1}_{\mathrm{GL}_3(F)} \times \mathbf{1}_{\mathrm{GL}_3(F)} \end{array}$$

is commutative.

This is not so immediate from the previous results. Since the above horizontal arrow is the identity by the LIR for the tempered (and generic) case ([A27]), so is the bottom one, which is the LIR for the non-tempered representation $\pi_M = \mathbf{1}_{\mathrm{GL}_3(F)} \boxtimes \mathbf{1}_{\mathrm{GL}_3(F)}$.

The general case ([A26]) is proven similarly. A twisted analogue was supposed to be proven in [A26] too, and our method still works for the twisted case.

4 LIR for co-tempered parameters for $\mathrm{Sp}_{2n}(F)$

Now we explain the LIR for $\mathrm{Sp}_{2n}(F)$ in the co-tempered case. First of all, it is reduced to the maximal parabolic case. So we consider the following situation.

Suppose that $G = \mathrm{Sp}_{2n}(F) \supset P = MN \supset M = \mathrm{GL}_k(F) \times \mathrm{Sp}_{2n_0}(F)$. Let $\phi_M = \phi_{\mathrm{GL}} \oplus \phi_0$ be a tempered L-parameter for M, $\psi_M = \widehat{\phi}_M = \widehat{\phi}_{\mathrm{GL}} \oplus \widehat{\phi}_0$ the dual A-parameter defined by

$$\psi_M(w, g_1, g_2) = \phi_M(w, g_2, g_1),$$

and let ψ (resp. ϕ) be the A-parameter for G given by ψ_M (resp. ϕ_M). We may further assume that ϕ_{GL} is irreducible and self-dual. Then $I_P(\pi_M)$ is a direct sum of at most two irreducible representations for any $\pi_M \in \Pi_{\psi_M}$.

The A-packet Π_ψ is the multi-set of irreducible summands of $I_P(\pi_M)$ for $\pi_M \in \Pi_{\psi_M}$. By *Aubert duality*, we can define the map

$$\Pi_\psi \ni \pi \mapsto \langle \cdot, \pi \rangle_\psi \in \widehat{\mathcal{S}_\psi}$$

from $\Pi_\phi \to \widehat{\mathcal{S}_\phi}$. Let $w_u \in W^G$ be the unique non-trivial Weyl element preserving M, and let $s_u \in \mathcal{S}_\psi$ be the corresponding element. The goal is to show the *local intertwining relation*

$$R_P(w_u, \widetilde{\pi}_M, \psi_M)|_\pi = \langle s_u, \pi \rangle_\psi \cdot \mathrm{id}_\pi \tag{LIR}$$

for any irreducible summand $\pi \subset I_P(\pi_M)$ for $\pi_M \in \Pi_{\psi_M}$.

We tried to extend our method for $\mathrm{GL}_n(F)$ to classical groups, but found an irreducible representation π of $\mathrm{Sp}_{2n}(F)$ to which our method cannot be applied. According to Tadić's idea (told by Mínguez), our method may be applied to only one irreducible summand $\pi \subset I_P(\pi_M)$ for each $\pi_M \in \Pi_{\psi_M}$. Since the length of $I_P(\pi_M)$ is at most 2, our method has "half possibility" to prove (LIR).

On the other hand, since Π_ψ is constructed by Aubert duality from Π_ϕ, (LIR) for ψ would follow from (LIR) for ϕ *if* we were able to show that Aubert duality commutes with $R_P(w_u, \widetilde{\pi}_M, \psi_M)$. Arthur suggested proving this commutativity by the Hecke algebra method, but we could not do it. However, the commutativity *up to a scalar* has been known for 10 years. This implies the following key lemma.

Lemma 4.1. *For $\pi_M \in \Pi_{\psi_M}$, if $I_P(\pi_M) = \pi_1 \oplus \pi_2$, then (LIR) holds for π_1 if and only if (LIR) for π_2.*

This lemma says that "half" is enough! Together with the "half possibility", we can attack (LIR) now.

For $\pi_M \in \Pi_{\psi_M}$, the *main diagram* is as follows.

$$\begin{array}{ccc} \mathcal{I}_\pi^G & \xrightarrow{R(w_2^{-1}w_1^{-1}w_uw_1w_2)} & \mathcal{I}_\pi^G \\ {\scriptstyle R(w_2)}\downarrow & & \downarrow{\scriptstyle R(w_2)} \\ I_P(\mathcal{I}^M_{\pi_M}) & & I_P(\mathcal{I}^M_{\pi_M}) \\ {\scriptstyle R(w_1)}\downarrow & & \downarrow{\scriptstyle R(w_1)} \\ I_P(\pi_M) & \xrightarrow{c^{-1}R_P(w_u,\widetilde{\pi}_M,\psi_M)} & I_P(\pi_M), \end{array}$$

where $\mathcal{I}^M_{\pi_M}$ and $\mathcal{I}^G_\pi$ are the standard modules of π_M and $\pi \subset I_P(\pi_M)$, respectively. We choose $\pi \subset I_P(\pi_M)$ so that $\mathcal{I}^G_\pi$ is easy to describe, which exists (uniquely in many cases) by Tadić. See [AGIKMS, Lemma 6.4.2]. In this diagram, $R(w_1)$, $R(w_2)$ and $R(w_2^{-1}w_1^{-1}w_uw_1w_2)$ are intertwining operators normalized using the *L-parameters* such that

$$\mathcal{I}^G_\pi \xrightarrow{R(w_2)} I_P(\mathcal{I}^M_{\pi_M}) \xrightarrow{R(w_1)} I_P(\pi_M)$$

realizes the Langlands quotient map $\mathcal{I}^G_\pi \twoheadrightarrow \pi$. However, since $R_P(w_u, \widetilde{\pi}_M, \psi_M)$ is normalized by the *A-parameter*, we need a constant $c \in \mathbb{C}^\times$ to make the main diagram commutative. It is given by

$$c = \left.\frac{\gamma_A(s, \psi_{\mathrm{GL}} \otimes \psi_0, \psi_F)}{\gamma_A(s, \psi_{\mathrm{GL}} \otimes \phi_{\pi_0}, \psi_F)}\right|_{s=0},$$

where $\psi_M = \psi_{\mathrm{GL}} \oplus \psi_0$, $\pi_M = \pi_{\mathrm{GL}} \boxtimes \pi_0$, and ϕ_{π_0} is the L-parameter of π_0.

By the commutativity of the main diagram ([AGIKMS, Theorem 6.5.1]) together with (LIR) for tempered case, we obtain:

Theorem 4.2 ([AGIKMS, Corollary 6.5.2])**.** *Suppose that $\psi = \psi_{\mathrm{GL}} \oplus \psi_0$ where $\psi_{\mathrm{GL}} = \rho_{\mathrm{GL}} \boxtimes \mathbf{1} \boxtimes S_d$ with d even (for simplicity). For $\pi_M = \pi_{\mathrm{GL}} \boxtimes \pi_0 \in \Pi_{\psi_M}$, (LIR) holds for a specific $\pi \subset I_P(\pi_M)$ if and only if*

$$\left.\frac{\gamma_A(s, \psi_{\mathrm{GL}} \otimes \psi_0, \psi_F)}{\gamma_A(s, \psi_{\mathrm{GL}} \otimes \phi_{\pi_0}, \psi_F)}\right|_{s=0} = \langle s_u, \pi \rangle_\psi$$

holds, where ϕ_{π_0} is the L-parameter of π_0.

Note that if d is even, then $R(w_2^{-1}w_1^{-1}w_uw_1w_2) = \mathrm{id}$. When d is odd, we need a slight modification.

Therefore, the final task is to show the equation

$$\left.\frac{\gamma_A(s, \psi_{\mathrm{GL}} \otimes \psi_0, \psi_F)}{\gamma_A(s, \psi_{\mathrm{GL}} \otimes \phi_{\pi_0}, \psi_F)}\right|_{s=0} = \langle s_u, \pi \rangle_\psi .$$

In general, it is very difficult to list ϕ_{π_0} for $\pi_0 \in \Pi_{\psi_0}$. Instead to show the equation directly, we compute the left-hand side inductively. To compute ϕ_{π_0} and $\langle s_u, \pi \rangle_\psi$, we use our Jacquet module computations in [At]. In conclusion, we achieve (LIR) for the co-tempered case ([A25]).

References

[Ar] J. Arthur, *The endoscopic classification of representations. Orthogonal and symplectic groups. American Mathematical Society Colloquium Publications,* **61**. American Mathematical Society, Providence, RI, 2013. xviii+590 pp.

[A24] J. Arthur, *Endoscopy and singular invariant distributions.* In preparation.

[A25] J. Arthur, *Duality, Endoscopy and Hecke operators.* In preparation.

[A26] J. Arthur, *A nontempered intertwining relation for GL(N).* In preparation.

[A27] J. Arthur, *Transfer factors and Whittaker models.* In preparation.

[At] H. Atobe, *Jacquet modules and local Langlands correspondence. Invent. Math.* **219** (2020), no. 3, 831–871.

[ACIKY1] H. Atobe, M. Chida, T. Ibukiyama, H. Katsurada and T. Yamauchi, *Harder's conjecture I. J. Math. Soc. Japan* **75** (2023), no. 4, 1339–1408.

[ACIKY2] H. Atobe, M. Chida, T. Ibukiyama, H. Katsurada and T. Yamauchi, *Harder's conjecture II.* Preprint, arXiv:2306.07582v2.

[AGIKMS] H. Atobe, W. T. Gan, A. Ichino, T. Kaletha, A. Mínguez and S. W. Shin, *Local Intertwining Relations and Co-tempered A-packets of Classical Groups*, Preprint, arXiv:2410.13504v1.

[BP1] R. Beuzart-Plessis, *Endoscopie et conjecture locale raffinée de Gan–Gross–Prasad pour les groupes unitaires. Compos. Math.* **151** (2015), no. 7, 1309–1371.

[BP2] R. Beuzart-Plessis, *A local trace formula for the Gan–Gross–Prasad conjecture for unitary groups: the Archimedean case. Astérisque* No. 418 (2020), ix+305 pp.

[BP3] R. Beuzart-Plessis, *Plancherel formula for* $\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(E)$ *and applications to the Ichino-Ikeda and formal degree conjectures for unitary groups. Invent. Math.* **225** (2021), no. 1, 159–297.

[BPC] R. Beuzart-Plessis and P.-H. Chaudouard, *The global Gan–Gross–Prasad conjecture for unitary groups. II. From Eisenstein series to Bessel periods.* Preprint, arXiv:2302.12331v1.

[BPCZ] R. Beuzart-Plessis, P.-H. Chaudouard and M. Zydor, *The global Gan–Gross–Prasad conjecture for unitary groups: the endoscopic case. Publ. Math. Inst. Hautes Études Sci.* **135** (2022), 183–336.

[BPLZZ] R. Beuzart-Plessis, Y. Liu, W. Zhang and X. Zhu, *Isolation of cuspidal spectrum, with application to the Gan–Gross–Prasad conjecture. Ann. of Math. (2)* **194** (2021), no. 2, 519–584.

[CL1] P.-H. Chaudouard and G. Laumon, *Le lemme fondamental pondéré. I. Constructions géométriques. Compos. Math.* **146** (2010), no. 6, 1416–1506.

[CL2] P.-H. Chaudouard and G. Laumon, *Le lemme fondamental pondéré. II. Énoncés cohomologiques. Ann. of Math. (2)* **176** (2012), no. 3, 1647–1781.

[CL] G. Chenevier and J. Lannes, *Automorphic forms and even unimodular lattices. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics,* **69**. Springer, Cham, 2019. xxi+417 pp.

[DL] D. Disegni and Y. Liu, *A p-adic arithmetic inner product formula. Invent. Math.* **236** (2024), no. 1, 219–371.

[GI] W. T. Gan and A. Ichino, *The Shimura–Waldspurger correspondence for* Mp_{2n}. *Ann. of Math. (2)* **188** (2018), no. 3, 965–1016.

[KS1] A. Kret and S. W. Shin, *Galois representations for general symplectic groups. J. Eur. Math. Soc. (JEMS)* **25** (2023), no. 1, 75–152.

[KS2] A. Kret and S. W. Shin, *Galois representations for even general special orthogonal groups. J. Inst. Math. Jussieu* **23** (2024), no. 5, 1959–2050.

[LMW] B. Lemaire, C. Mœglin and J.-L. Waldspurger, *Le lemme fondamental pour l'endoscopie tordue: réduction aux éléments unités. Ann. Sci. Éc. Norm. Supér. (4)* **51** (2018), no. 2, 281–369.

[LL] C. Li and Y. Liu, *Chow groups and L-derivatives of automorphic motives for unitary groups. Ann. of Math. (2)* **194** (2021), no. 3, 817–901.

[LTXZZ] Y. Liu, Y. Tian, L. Xiao, W. Zhang, and X. Zhu, *On the Beilinson–Bloch–Kato conjecture for Rankin–Selberg motives. Invent. Math.* **228** (2022), no. 1, 107–375.

[L] W.-W. Li, *Arthur packets for metaplectic groups*. Preprint, arXiv:2410.13606v2.

[MW1] C. Mœglin and J.-L. Waldspurger, *Stabilisation de la formule des traces tordue. Vol. 1.* Progress in Mathematics, **316**. *Birkhäuser/Springer, Cham,* 2016. xviii+587 pp.

[MW2] C. Mœglin and J.-L. Waldspurger, *Stabilisation de la formule des traces tordue. Vol. 2.* Progress in Mathematics, **317**. *Birkhäuser/Springer, Cham,* 2016. pp. v–xxviii and 589–1315.

[MW3] C. Mœglin and J.-L. Waldspurger, *La formule des traces locale tordue. Mem. Amer. Math. Soc.* **251** (2018), no. 1198, v+183 pp.

[Mok] C. P. Mok, *Endoscopic classification of representations of quasi-split unitary groups. Mem. Amer. Math. Soc.* **235** (2015), no. 1108, vi+248 pp.

[N] B. C. Ngô, *Le lemme fondamental pour les algèbres de Lie. Publ. Math. Inst. Hautes Études Sci.* No. 111 (2010), 1–169.

[Sc] P. Scholze, *On torsion in the cohomology of locally symmetric varieties. Ann. of Math. (2)* **182** (2015), no. 3, 945–1066.

[Sh] F. Shahidi, *A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups. Ann. of Math. (2)* **132** (1990), no. 2, 273–330.

[W] J.-L. Waldspurger, *La conjecture locale de Gross–Prasad pour les représentations tempérées des groupes spéciaux orthogonaux*. Sur les conjectures de Gross et Prasad. II. *Astérisque* No. 347 (2012), 103–165.

Department of Mathematics
Kyoto University
Kitashirakawa-Oiwake-cho, Sakyo-ku, Kyoto, 606-8502
JAPAN
E-mail address: atobe@math.kyoto-u.ac.jp