NON-VANISHING OF EISENSTEIN SERIES MOD p

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1. INTRODUCTION

This is a report on automorphic aspects of the author's joint work in progress with Raúl Alonso and Ashay Burungale. We propose a new approach to proving the non-vanishing modulo p of a specific p-adic family of Eisenstein series on the unitary group U(2, 1). This result was first proven by Hsieh in [Hsi14] and was a key step in his proof of one divisibility of the Iwasawa main conjecture for CM fields.

1.1. Background.
We first briefly explain the motivation for our work. A significantly more detailed account can be found in the introduction of [Hsi14].

Let $\mathcal{F}$ be a totally real field of degree d over $\mathbb{Q}$, and let $\mathcal{K}/\mathcal{F}$ be a CM extension. Let p be a rational prime which is unramified in $\mathcal{F}$ and such that every place of $\mathcal{F}$ above p splits completely in $\mathcal{K}$. Let ψ be a finite order character of Gal_K. For a fixed CM type Σ of $\mathcal{K}$, there is a p-adic L-function $\mathcal{L}_\psi$ interpolating algebraic parts of the L-values $L(0, \psi\chi)$ as χ runs over Hecke characters of p-power conductors satisfying certain conditions on its infinity type. This was constructed by Katz [Kat78] in the case $\psi = 1$ and later in general by Hida–Tilouine [HT93, Theorem II]. We refer the reader to the above papers for the exact interpolation relations.

The Iwasawa main conjecture predicts that $\mathcal{L}_\psi$ is equal to another function F_ψ constructed from Selmer groups. It has many applications to the arithmetic of Hecke characters, in particular the arithmetic of CM elliptic curves. One fruitful method to make progress towards this conjecture is the method of *Eisenstein congruence*, originating in Ribet's converse to Herbrand's theorem [Rib76]. The method requires a p-adic family of Eisenstein series $\mathcal{E}_\psi$ whose constant term is the p-adic L-function $\mathcal{L}_\psi$. This is based on Langland's formula for the constant term of Eisenstein series, and a crucial part is the choice of the local section at p to ensure that the resulting (automorphic) Eisenstein series interpolate in a p-adic family. A further technical computation is often needed to compare the interpolation factors. This is the main content of sections 6 and 7 of [Hsi14], which we will summarize later in this report.

Once $\mathcal{E}_\psi$ is constructed, the method requires a careful study of the points where its constant term $\mathcal{L}_\psi$ is zero modulo p. To obtain non-trivial results, it is crucial that $\mathcal{E}_\psi$ itself is non-trivial modulo p. Hsieh's approach is to compute the Fourier–Jacobi expansion of $\mathcal{E}_\psi$ along the boundary of a U(2, 1)-Shimura variety. A suitable combination of the coefficients is a product of two Hecke L-values, and Hida's non-vanishing result [Hid04] is used to conclude this part of the proof.

1.2. Our approach.
Instead of "evaluating" $\mathcal{E}_\psi$ at the boundary of the Shimura variety, we consider its values at CM points on the curve. This avoids the intricate geometry at the boundary and can be more easily generalized to higher rank groups.

More precisely, we consider the period integral

$$\int_{[\mathrm{U}(2)]} E_\psi(h)\varphi^+(h)\,dh,$$

where φ^+ is an auxiliary cusp form on U(2), and E_ψ denotes a specialization of the family $\mathcal{E}_\psi$. Geometrically, this is just a sum of the function $E_\psi \varphi^+$ over a finite set of points. On the other hand, by comparing with the Gan–Gross–Prasad formula, its square should factor as the product of two standard L-functions for φ^+, twisted by characters coming from the Eisenstein series.[1] We further choose φ^+ to be a CM form. Then the period integral squared factors as the product of four Hecke L-values. Doing this carefully shows that two of the values are central, and the other two are non-central. For the non-central values, we can use the Katz p-adic L-function to compare them with central values. At this point, we appeal to the forthcoming

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[1]The GGP formula serves only as motivation. The integral actually unfolds into an Euler product in our case.

work of Burungale–He–Tian–Ye, which establishes non-vanishing modulo p for *all but finitely many* auxiliary characters in this self-dual setting.

Among the various technical hypotheses Hsieh needed to prove his main theorem, we hope our method can eliminate the following two:

- $p \nmid h_{\mathcal{K}}^{-}$. This was needed in the Fourier inversion formula.
- ψ is congruent to a character which is unramified at p. This was used to simplify the computation of the Fourier–Jacobi coefficients.

However, we have to impose the following additional hypothesis

- ψ is residually anti-cyclotomic. This is needed to p-adically compare the aforementioned non-central L-values with central L-values.

Note that we cannot apply Hida's result: due to a gap in the proof, the main theorem gives the non-vanishing of Hecke L-values only for a *Zariski dense* set of auxiliary characters [Hid25]. A proof of Hida's original result would allow us to eliminate this hypothesis.

2. Eisenstein series

This section summarizes Hsieh's construction of the (ordinary) p-adic family of Eisenstein series and the computation of its constant term. A large part of the set-up generalizes to higher rank groups, but we will avoid this for notational simplicity.

2.1. **Notations.**

2.1.1. *Fields.* Recall that $\mathcal{K}/\mathcal{F}$ is a CM extension. Fix a complex CM type Σ. Further fix a totally imaginary element $\delta \in \mathcal{K}$ such that $\sqrt{-1}\sigma(\delta) > 0$ for all $\sigma \in \Sigma$.

2.1.2. *Hermitian spaces.* We introduce the following Hermitian spaces

- $W = \mathcal{K}\mathbf{w}$ is a skew-Hermitian line (i.e. 1-dimensional space). Moreover, let $\mathsf{a} := \langle \mathbf{w}, \mathbf{w} \rangle$, then $\delta^{-1}\mathsf{a}$ is a totally positive element of $\mathcal{F}$.
- $V = \mathcal{K}\mathbf{y} \oplus W \oplus \mathcal{K}\mathbf{x}$ is a 3-dimensional skew-Hermitian space with the Hermitian form

$$
\begin{pmatrix} & & -1 \\ & \mathsf{a} & \\ 1 & & \end{pmatrix}
$$

  with respect to the basis $(\mathbf{y}, \mathbf{w}, \mathbf{x})$. In particular, its signature is $(2, 1)$ at all archimedean places.
- $V^{+} = \mathcal{K}\mathbf{v}^{+} \oplus W$, $V^{-} = \mathcal{K}\mathbf{v}^{-}$ are definite Hermitian subspaces of V, where

$$
\mathbf{v}^{\pm} = \pm\mathbf{y} + \frac{\delta}{2}\mathbf{x}
$$

  With respect to the decomposition $V = V^{+} \oplus V^{-}$, the skew-Hermitian form is given by the diagonal matrix $\mathrm{diag}(\delta, \mathsf{a}, -\delta)$.
- $\mathbb{V} = V^{+} \oplus (-V^{+})$ is the double of V^{+}. It also has the decomposition $\mathbb{V} = V \oplus (-W)$.

2.1.3. *Unitary groups.* For the (skew-)Hermitian spaces introduced in the previous subsection, denote their unitary groups as follows

$$
H = \mathrm{U}(W), \quad G = \mathrm{U}(V), \quad G^{\pm} = \mathrm{U}(V^{\pm}), \quad \mathbb{G} = \mathrm{U}(\mathbb{V}).
$$

They are algebraic groups over $\mathcal{F}$.

The embeddings of Hermitian spaces give rise to the following three natural embeddings

$$
\nu : H \hookrightarrow G_{+}, \quad \kappa : G^{+} \times G^{-} \hookrightarrow G, \quad \iota : G \times H \hookrightarrow \mathbb{G},
$$

where for ι, we also view H as the unitary group for the second copy $-W$. We will commonly write $(g, h)_{\kappa}$ instead of $\kappa(g, h)$, and the same applies to ι.

2.1.4. *Parabolic subgroups.* Let $Q \subseteq \mathbb{G}$ denote the Siegel parabolic subgroup stabilizing the flag $0 \subseteq (V^+)^\Delta \subseteq \mathbb{V}$, where as usual, $(V^+)^\Delta = \{(v, v) \mid v \in V^+\} \subseteq \mathbb{V}$ is the diagonal copy of the subgroup. There is a bijection of sets

$$Q(\mathcal{F})\backslash\mathbb{G}(\mathcal{F}) \simeq (P(\mathcal{F})\backslash G(\mathcal{F})) \times H(\mathcal{F}).$$

The Levi subgroup of Q can be identified with $\mathrm{GL}((V^+)^\Delta) \simeq \mathrm{Res}_{\mathcal{K}/\mathcal{F}} \mathrm{GL}_2$. Given a (local or adèlic) character χ of $\mathcal{K}^\times$, let $I(\chi, s)$ be the set of functions ϕ such that

$$\phi(qg) = (\chi^{-1} |\cdot|_{\mathcal{K}}^{-s})(\det A_q) \cdot \phi(g), \text{ for all } q \in Q, g \in \mathbb{G}$$

where $A_q \in \mathrm{GL}((V^+)^\Delta)$ is the Levi component of q.

Let $P \subseteq G$ denote the Klingen parabolic subgroup stabilizing the flag $0 \subseteq X \subseteq V$. We have the following isomorphisms on their $\mathcal{F}$-points

$$P(\mathcal{F})\backslash G(\mathcal{F}) \simeq \big(H(\mathcal{F}) \times G^-(\mathcal{F})\big) \backslash \big(G^+(\mathcal{F}) \times G^-(\mathcal{F})\big) \simeq H(\mathcal{F})\backslash G^+(\mathcal{F}).$$

The Levi subgroup of P can be identified with $\mathrm{Res}_{\mathcal{K}/\mathcal{F}} \mathbf{G}_m \times \mathrm{U}(W)$. For $p \in P$, write its image in the first component as t_p. For χ as before, let $I(\chi, \mathbf{1}, s)$ be the set of functions ϕ such that

$$\phi(pg) = (\chi^{-1} |\cdot|_{\mathcal{K}}^{-s})(t_p) \cdot \phi(g), \ p \in P, g \in G.$$

In general (higher rank), the trivial character $\mathbf{1}$ should be replaced by a (smooth or automorphic) representation of $\mathrm{U}(W)$.

2.2. Choice of sections.
Our goal is to construct a p-adic family of Klingen Eisenstein series for the group G. For the purpose of p-adic interpolation, it is easy to work Siegel Eisenstein series on the group $\mathrm{U}(\mathbb{V})$, where we have q-expansions. Klingen Eisenstein series on the group $\mathrm{U}(V)$ are obtained by the pullback construction, cf. [Shi97].

2.2.1. *Pullback construction.* In the local setting, the pullback is given by the integral

$$I(\chi, s) \to I(\chi, \mathbf{1}, s), \quad \tilde{f}_s \mapsto \int_{H(\mathcal{F}_v)} \tilde{f}_s((g, h)_\iota)\chi(h) \, dh$$

This converges for $\mathrm{Re}(s)$ sufficiently large. The same expression is used to define pullbacks of adèlic functions and Eisenstein series, with obvious modifications.

2.2.2. *Siegel sections.* From now on, χ is an adèlic Hecke character of $\mathcal{K}$. It corresponds to the character $\psi\chi$ from the introduction. Let S be a finite set of bad places containing all places above p and ∞, ramified places for χ, and ramified places for the Hermitian spaces. Away from S, the Siegel section is the unramified section, normalized to equal to 1 at the identity. There are three more cases:

- Archimedean places: in [Hsi14], the standard weight-k section is used. For our purpose, it is useful to allow the weight to vary, so we apply certain standard differential operators, cf. [EL21].
- Ramified places away from p: we use the "volume section" introduced in [EHLS20, §4.2.2]. This is the section supported on the big cell QwN whose restriction to N is the indicator function of a small lattice (how small depends on certain fixed auxiliary data on $\mathrm{U}(V^+)$ to be described later). With this choice, the ramified local integrals from the doubling method become trivial to evaluate.
- Places above p: the choice is described in [Hsi14, §5.3.4]. Its exact form does not matter for our calculations. What we need to know are the following:
  (1) The resulting Eisenstein series form a p-adic family. This was proven by Hsieh by computing the q-expansion, cf. [Hsi14, Theorem 5.8].
  (2) The ordinary projection of its pullback to G. This was computed in [Hsi14, §6.3]. In particular, the original sections need to be chosen so that the result is non-zero.
  It would be interesting to find sections satisfying these two properties in higher rank settings.

Denote this Siegel section by $\tilde{f}_s$.

2.2.3. *Klingen sections.* Let $f_s^{\mathtt{pb}} \in I(\chi, \mathbf{1}, s)$ be the pullback of the Siegel section $\tilde{f}_s$ defined above. For the intended application to Iwasawa theory, it is necessary to consider its ordinary projection $f_s := e_{\mathrm{ord}} f_s^{\mathtt{pb}}$. This only changes the local section at places above p. From a purely automorphic point of view, the resulting section is an element in the image of the canonical lift of the Jacquet module, cf. [Cas95, §4.1].

As we will see shortly, our computations do not require the knowledge of f_s away from p: the local computations reduce to doubling local integrals for $\tilde{f}_s$.

3. Automorphic period

The section f_s above gives rise to a Klingen Eisenstein series $E(g, f_s)$. Let π^+ be a cuspidal automorphic representation of $G^+ = \mathrm{U}(V^+)$, and let $\varphi^+ \in \pi^+$. We now compute the G^+-period pairing between $E(g, f_s)|_{G^+}$ and φ^+

$$\mathtt{P}(f_s, \varphi^+) := \int_{[G^+]} E((g, 1)_\kappa, f_s)\varphi^+(g) dg$$

For the intended non-vanishing application, the role of π^+ and φ^+ is entirely auxiliary, and we have significant freedom in their choices.

3.1. **Factorization.** Using the identification $P(\mathcal{F})\backslash G(\mathcal{F}) \simeq H(\mathcal{F})\backslash G^+(\mathcal{F})$, we can unfold the integral

$$\mathtt{P} = \int_{[G^+]} \sum_{\gamma \in P(\mathcal{F})\backslash G(\mathcal{F})} f_s(\gamma(g, 1)_\kappa)\varphi^+(g)\, dg$$

$$= \int_{[G^+]} \sum_{\gamma \in H(\mathcal{F})\backslash G^+(\mathcal{F})} f_s(\kappa(\gamma g, 1))\varphi^+(g)\, dg$$

$$= \int_{H(\mathcal{F})\backslash G^+(\mathbb{A}_\mathcal{F})} f_s(\kappa(g, 1))\varphi^+(g)\, dg.$$

Now suppose f_s is obtained by pulling back a Siegel section $\tilde{f}_s$. Note that this holds by definition away from p. At a place v above p, the ordinary projection intervenes, but the statement still holds abstractly. Using the definition of the pullback construction, we get

$$\mathtt{P} = \int_{H(\mathcal{F})\backslash G^+(\mathbb{A}_\mathcal{F})} \int_{H(\mathbb{A}_\mathcal{F})} \tilde{f}_s(\iota((g, 1)_\kappa, h))\chi(h)\varphi^+(g)\, dh\, dg$$

$$= \int_{H(\mathcal{F})\backslash G^+(\mathbb{A}_\mathcal{F})} \int_{[H]} \sum_{\alpha \in H(\mathcal{F})} \tilde{f}_s(\iota((g, 1)_\kappa, \alpha h))\chi(h)\varphi^+(g)\, dh\, dg$$

$$= \int_{G^+(\mathbb{A}_\mathcal{F})} \int_{[H]} \tilde{f}_s(\iota((g, 1)_\kappa, h))\chi(h)\varphi^+(g)\, dh\, dg$$

$$= \int_{G^+(\mathbb{A}_\mathcal{F})} \tilde{f}_s(g, 1) \int_{[H]} \varphi^+(\nu(h)g)\, dh\, dg$$

In the final step, $\tilde{f}_s(g, 1)$ means evaluating $\tilde{f}_s$ at the image of $(g, 1)$ under the doubling embedding $G^+ \times G^+ \hookrightarrow \mathbb{G}$. Note that this is not in the Siegel parabolic Q unless $g = 1$.

The inner integral is an instance of the period integral appearing in the Gan–Gross–Prasad conjecture for the groups (H, G^+). Denote it by $P(\pi^+(g)\varphi^+)$. By multiplicity one, we can write down an *abstract factorization* $P = \prod_v P_v$. Given this, the above expression for $\mathtt{P}$ can be factored as follows.

$$\mathtt{P} = P(\varphi^+) \int_{G^+(\mathbb{A}_\mathcal{F})} \tilde{f}_s(g, 1) \frac{P(\pi^+(g)\varphi^+)}{P(\varphi^+)}\, dg$$

$$= P(\varphi^+) \prod_v \int_{G^+(\mathcal{F}_v)} \tilde{f}_{s,v}(g_v, 1) \frac{P_v(\pi_v^+(g_v)\varphi_v^+)}{P_v(\varphi_v^+)}\, dg_v$$

Since P_v is an element in a one-dimensional Hom-space, it may be replaced with any other non-zero element in the same Hom-space without changing the right hand side. We will do this using the canonical local integral from the GGP conjecture.

Let $\tilde{\pi}_v^+$ be the contragredient representations of π_v^+. For any $\tilde{\varphi}_v^+ \in \tilde{\pi}_v^+$, we may define

$$\alpha_{\tilde{\varphi}_v^+} \in \mathrm{Hom}_{H(\mathcal{F}_v)}(\pi_v^+, \mathbb{C}), \quad \alpha_{\tilde{\varphi}_v^+}(\varphi_v^+) = \int_{H(\mathcal{F}_v)} \langle \pi^+(\nu(h))\varphi_v^+, \tilde{\varphi}_v^+ \rangle \, dh$$

Using this linear form, we have

$$\int_{G^+(\mathcal{F}_v)} \tilde{f}_{s,v}(g_v, 1) \alpha_{\tilde{\varphi}_v^+}(\pi_v^+(g_v)\varphi_v^+) \, dg_v$$

$$= \int_{G^+(\mathcal{F}_v)} \int_{H(\mathcal{F}_v)} \tilde{f}_{s,v}(g_v, 1) \langle \pi_v^+(\nu(h)g_v)\varphi_v^+, \tilde{\varphi}_v^+ \rangle \, dh \, dg_v$$

$$= \int_{H(\mathcal{F}_v)} \int_{G^+(\mathcal{F}_v)} \tilde{f}_{s,v}(g_v, 1) \langle \pi_v^+(g_v)\varphi_v^+, \tilde{\pi}_v^+(\nu(h)^{-1})\tilde{\varphi}_v^+ \rangle \, dg_v \, dh$$

Substituting this into the expression for P, we get a factorization

$$\text{(1)} \qquad \mathtt{P}(f_s, \varphi^+) = P(\varphi^+) \prod_v \frac{I(\tilde{f}_{s,v}, \varphi_v^+, \tilde{\varphi}_v^+)}{\alpha_{\tilde{\varphi}_v^+}(\varphi_v^+)}$$

where the local factors are

$$I(\tilde{f}_{s,v}, \varphi_v^+, \tilde{\varphi}_v^+) = \int_{H(\mathcal{F}_v)} \int_{G^+(\mathcal{F}_v)} \tilde{f}_{s,v}(g_v, 1) \langle \pi_v^+(g_v)\varphi_v^+, \tilde{\pi}_v^+(\nu(h)^{-1})\tilde{\varphi}_v^+ \rangle \, dg_v \, dh$$

This factorization is valid for any choice of the auxiliary vector $\tilde{\varphi}_v^+$ for which the denominator is non-zero. Moreover, each term in the product independent of the normalization of the matrix coefficients on H and G^+. Away from p, we will compute these local terms using known results from the theory of doubling integrals.

The integral $I(\tilde{f}_{s,v}, \varphi_v^+, \tilde{\varphi}_v^+)$ can be written in a way which only depends on the original section f_s.

$$\int_{H(\mathcal{F}_v)} \int_{G^+(\mathcal{F}_v)} \tilde{f}_{s,v}(g, 1) \langle \pi_v^+(g)\varphi_v^+, \tilde{\pi}_v^+(\nu(h)^{-1})\tilde{\varphi}_v^+ \rangle \, dg \, dh$$

$$= \int_{H(\mathcal{F}_v)} \int_{G^+(\mathcal{F}_v)} \tilde{f}_{s,v}(\nu(h)^{-1}g, 1) \langle \pi_v^+(g)\varphi_v^+, \tilde{\pi}_v^+(\nu(h)^{-1})\tilde{\varphi}_v^+ \rangle \, dg \, dh$$

$$= \int_{G^+(\mathcal{F}_v)} \left(\int_{H(\mathcal{F}_v)} \tilde{f}_{s,v}(g, \nu(h))\chi(h) \, dh \right) \langle \pi^+(g)\varphi^+, \tilde{\varphi}^+ \rangle \, dg$$

By unwinding the various choices of embeddings, the integral in the parenthesis is the integral defining the local pullback section, so the entire expression simplifies to

$$I(\tilde{f}_{s,v}, \varphi_v \otimes \varphi_v^+, \tilde{\Phi}_v) = \int_{G^+(\mathcal{F}_v)} f_{s,v}(\kappa(g)) \langle \pi^+(g)\varphi_v^+, \tilde{\varphi}_v^+ \rangle \, dg$$

This form will be useful for the p-adic local computations, where we know the pullback section but not the Siegel section.

For the first term, Waldspurger's formula expresses its square as an L-function multiplied by local integrals, and we use the explicit version of the formula given in [CST14]. It remains to consider the other terms.

3.2. **Local computation.** There are four cases.

- Unramified places: there is a dependency on h in the inner integral in $I(\tilde{f}_{s,v}, \varphi_v^+, \tilde{\varphi}_v^+)$. However, the integral is an unramified linear form, so the standard unramified computation in the doubling method [GPSR87, Li92] gives

$$\int_{G^+(\mathcal{F}_v)} \tilde{f}_{s,v}(g_v, 1) \langle \pi_v^+(g_v)v, w \rangle \, dg_v = \frac{L(s - \frac{1}{2}, \pi_v^+, \chi_v)}{L(s, \chi_v|_{\mathcal{F}_v}) L(s - 1, \chi_v|_{\mathcal{F}_v} \eta_{\mathcal{K}_v/\mathcal{F}_v})} \cdot \langle v, w \rangle$$

After factoring out the ratio of local L-factors, the outer integral over $H(\mathcal{F}_v)$ is exactly equal to the denominator, so we are done. In particular, this gives one of the main terms $L\left(s - \frac{1}{2}, \pi^+, \chi\right)$ in the final result.

– Ramified places away from p: the same idea as above applies. More precisely, it is explained in [EHLS20, §4.2.2] that the inner integral is a volume multiplied by the matrix coefficient, provided that the support of $\tilde{f}_s$ is "small compared to the ramification of π_v^+". In fact, the p-adic valuation of this volume term does not change when shrinking the support, since the relevant group in question is pro-ℓ for $\ell \neq p$.

– Archimedean places: we directly apply the result of [EL21, Thm. 5.3.1]. This uses the low-rank fact that $H \simeq \mathrm{U}(1)$ is abelian, so its action on $\tilde{\varphi}_\infty^+$ is easy to describe.

– Places above p: we use the second version of the local integral, involving only f_s without referencing Siegel sections. Recall that all places of $\mathcal{F}$ above p splits in $\mathcal{K}$, so this is a GL_2-integral. We use an explicit description of the ordinary section to compute this integral.

3.3. **Relation to L-values.** By combining the local computations above and Waldspurger's formula, we get a relation of the form

$$|\mathrm{P}|^2 = (*) \cdot L\left(\frac{1}{2}, \pi^+\right) \cdot \left|L\left(-\frac{1}{2}, \pi^+, \chi\right)\right|^2,$$

where $(*)$ is completely explicit and contains certain Petersson inner products, which will act as transcendental periods for the L-values.

It is still a difficult problem to understand mod p non-vanishing of GL_2 L-functions in general. For our work, we choose π^+ to be a CM form $\theta(\chi')$. In this case, each of the two L-functions factor as a product of two Hecke L-functions. We then need to find a Hecke character χ' such that all four Hecke L-values are non-zero mod p.

For the two central L-values arising from the first term, all but finitely many χ' satisfy this condition, by the forthcoming work of Burungale–He–Tian–Ye. The other two terms are non-central, so we need to relate them to central L-values (more precisely, symplectic self-dual characters) using p-adic arguments. While this can be done by a careful comparison with Katz's p-adic L-function, we opt for the more direct approach of extending Hsieh's p-adic family to vary the weight. This is the reason we inserted differential operators in our archimedean sections.

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