Koecher principle for quaternionic discrete series

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Abstract

This article is the write-up of what the author talked about at the RIMS workshop on automorphic forms held in 2025, January. The Koecher principle usually means that holomorphic automorphic forms of several variables (other than elliptic modular forms) automatically satisfy the holomorphy around cusps of modular varieties on which they are defined. The aim of this article is to report author's result of the principle for automorphic forms generating quaternionic discrete representations, which are real analytic but non-holomorphic. The fundamental method is a theory of a Fourier-Jacobi expansion for such automorphic forms, which was presented at the previous RIMS workshop.

1 Background of this study

The usual defining conditions of automorphic forms include the moderate growth condition, which means that automorphic forms are at most of polynomial order. Due to Harish-Chandra [7], this condition leads to the well known fact that the spaces of automorphic forms (given an automorphy and an analytic condition etc.) are finite dimensional.

If we reformulate this condition for holomorphic automorphic forms on Hermitian symmetric domains, it says that they satisfy the holomorphy around cusps for modular varieties, which are arithmetic quotients of the Hermitian symmetric domains. A well known reference is due to M. Koecher [8] published in 1954, which takes up the case of holomorphic Siegel modular forms. For this we remark that F. Götzky [4] (published in 1928) already proved the Koecher principle for some Hilbert modular forms before [8]. At last, Baily and Borel [1] resolved the Koecher principle for general holomorphic modular forms by applying Serre's extension theorem for the Satake-Baily-Borel compactification of arithmetic quotients of Hermitian symmetric domains. We can find the Koecher principle for general holomorphic automorphic forms also in Piatetski-Shapiro [12].

We now formulate a general conjecture by Miatello and Wallach [9] as follows:

Conjecture. 1.1. *Automorphic forms on a semisimple real Lie group G of rank ≥ 2 (K-finite, $Z(\mathrm{Lie}(G))$-eigen) with respect to an irreducible arithemetic group would automatically satisfy the moderate growth condition.*

Miatello and Wallach proved this conjecture for Hilbert Maass forms on $O(1,n)$ ($n \geq 2$) over totally real number field other than the field of rational numbers.

By [10] we have examples of non-holomorphic automorphic forms satisfying the Koecher principle by forms on the indefinite symplectic group $Sp(1,q)$ generating quaternionic discrete series (cf. [6]) when $q > 1$. We remark that $Sp(1,q)$ is a group of rank one and that [10] provides examples of the non-holomorphic Koecher principle outside the conjecture by Miattelo and Wallach. In [11] further examples of non-holomorphic Koecher principle are provided by automorphic forms generating quaternionic discrete series (cf. [6]) in the setting of A. Pollack [13], which are often called quaternionic modular forms. The new examples of the Koecher principle just mentioned were presented in the RIMS workshop and this article is based on the presentation and the paper [11].

2 The statement of the result

A fundamental paper of quaternionic discrete series representations is due to Gross-Wallach [6]. We know that, in [6], there is the list of simple groups admitting quaternionic discrete series representations as follows:

- Classical groups
  - i) $SU(d,2)$ (type A)
  - ii) $SO(d,4)$ (type B or D)
  - iii) $Sp(1,d)$ (type C),
- Exceptional groups
  - i) G_2 (real rank 2)
  - ii) F_4 (real rank 4)
  - iii) $E_6,\ E_7,\ E_8$ (real rank 4).

For this article the targets of the groups are $SO(d,4)$ and all the exceptional groups above.

Hereafter, by $\mathcal{G}$ we denote a simple group of adjoint type over $\mathbb{Q}$ whose real groups $\mathcal{G}(\mathbb{R})$ correspond to the list above except for $SU(d,2)$ and $Sp(1,d)$.

Theorem. 2.1 (Koecher principle)**.** *Let $\mathcal{G}$ be a simple adjoint group as above, whose real group has quaternionic discrete series. We exclude the cases of special orthogonal groups $SO(4,N)$ ($PSO(4,N)$ for even N) and the exceptional group of type G_2. Automorphic forms on $\mathcal{G}(\mathbb{A})$ generating quaternionic discrete series representations at the archimedean place are automatically of moderate growth.*

For this theorem we cite [11, Theorem 3.9 (2)].

3 Ingredients of the proof

The algebraic group $\mathcal{G}$ has the maximal parabolic subgroup called the Heisenberg parabolic subgroup $\mathcal{H}_J \ltimes \mathcal{N}$ with the Levi part $\mathcal{H}_J$ and the unipotent radical $\mathcal{N}$, where $\mathcal{N}$ is a two-step nilpotent group and in fact, is called a Heisenberg group with the one-dimensional center. Here the notation J means the finite dimensional $\mathbb{Q}$-vector space equipped with a cubic norm structure as in [13, Sections 1,2 and 2]. The real vector space $J_{\mathbb{R}} := J \otimes_{\mathbb{Q}} \mathbb{R}$ is a formally real Jordan algebra and we have the Hermitian symmetric domain $D_J \subset J_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$.

We put $G :=$ the connected component of the identity for $\mathcal{G}(\mathbb{R})$, $N := \mathcal{N}(\mathbb{R})$ and $H_J^0 :=$ the connected component of the identity for $\mathcal{H}_J(\mathbb{R})$ in order to firstly state explicit formulas for generalized Whittaker functions attached to quaternionic discrete series representations. For this we note that H_J^0 is a Lie group corresponding to D_J.

Let K be a maximal compact subgroup of G. According to [6, Proposition 4.1], K is isomorphic to $M \times SU(2)/\langle(\epsilon,-1)\rangle$, where M is explained by [6, Table 2.6] and ϵ is the unique element of order two in the center of M. Let π_n be a quaternionic discrete series representation with minimal K type $(\tau_n, \mathbb{V}_n)$ given by the trivial extension of $2n$-th symmetric tensor representation of $SU(2)$, for which we assume $n > \dim J + 1$. As in [13, Section 7.3] the representation space $\mathbb{V}_n$ has a basis given by $\left\{ \frac{x^{n+v}}{(n+v)!}\frac{y^{n-v}}{(n-v)!} \;\middle|\; -n \le v \le n \right\}$.

3.1 Generalized Whittaker functions for quaternionic discrete series

For a character χ of N we define a generalized Whittaker function for π_n with K-type τ_n by an element in the image of the following map

$$I:\ \mathrm{Hom}_{(\mathfrak{g},K)}(\pi_n, C^\infty\text{-}\mathrm{Ind}_N^G(\chi)) \ni \Phi \mapsto \Phi\circ i \in \mathrm{Hom}_K(\tau_n, C^\infty\text{-}\mathrm{Ind}_N^G(\chi))$$

with the embedding $\iota:\tau_n\to\pi_n$. In view of Yamashita's characterization of $\mathrm{Im}(I)$ by means of the Dirac-Schmid operator (cf. [15, Proposition 2.1]) Pollack [13] obtains an explicit formula for generalized Whittaker functions of moderate growth, where we remark that the map I is verified to be a bijection for π_n in the proof of [11, Theorem 2.15] while I is proved to be an injection in general by [15, Proposition 2.1]. To review his formula we use the notation p_χ of the cubic polynomial on D_J attached to χ, as in [13, Section 1.2, p.1214]. In the following theorem we also include an explicit formula for generalized Whittaker functions not of moderate growth (see [11, Theorem 2.15] and the remark below) as well as Pollack's formula.

Theorem. 3.1. *Let π_n be a quaternionic discrete series representation of G. For a non-trivial character χ we have*

$$\dim\mathrm{Hom}_{(\mathfrak{g},K)}(\pi_n, C^\infty\text{-}\mathrm{Ind}_N^G(\chi)) = \begin{cases}2 & (p_\chi \text{ has no zero point in } D_J),\\ 1 & (p_\chi\text{:otheriwse}),\end{cases}$$
$$\dim\mathrm{Hom}_{(\mathfrak{g},K)}(\pi_n, C^\infty_{\mathrm{mod}}\text{-}\mathrm{Ind}_N^G(\chi)) = \begin{cases}1 & (p_\chi \text{ has no zero point in } D_J),\\ 0 & (p_\chi\text{:otheriwse}),\end{cases}$$

where $C^\infty_{\mathrm{mod}}\text{-}\mathrm{Ind}_N^G(\chi)$ denotes the space of moderate growth sections in $C^\infty\text{-}\mathrm{Ind}_N^G(\chi)$.

The generalized Whittaker functions are given by linear combinations of

$$W_\chi(g) := \sum_{-n\le v\le n} W_v^\chi(g)\frac{x^{n+v}y^{n-v}}{(n+v)!(n-v)!} \quad \text{and} \quad W'_\chi(g) := \sum_{-n\le v\le n} W'^\chi_v(g)\frac{x^{n+v}y^{n-v}}{(n+v)!(n-v)!}$$

for $g\in H_J^0$ with

$$W_v^\chi(g) = \left(\frac{|j(g,\sqrt{-1}1_J)p_\chi(g\cdot\sqrt{-1}1_J)|}{j(g,\sqrt{-1}1_J)p_\chi(g\cdot\sqrt{-1}1_J)}\right)^v \nu(g)^{n+1}K_v(|j(g,\sqrt{-1}1_J)p_\chi(g\cdot\sqrt{-1}1_J)|),$$
$$W'^\chi_v(g) = \left(-\frac{|j(g,\sqrt{-1}1_J)p_\chi(g\cdot\sqrt{-1}1_J)|}{j(g,\sqrt{-1}1_J)p_\chi(g\cdot\sqrt{-1}1_J)}\right)^v \nu(g)^{n+1}I_v(|j(g,\sqrt{-1}1_J)p_\chi(g\cdot\sqrt{-1}1_J)|),$$

where K_v (respectively I_v) denotes the K-Bessel function (respectively I-Bessel function) parametrized by v. Here $j(g,Z)$ and $\nu(g)>0$ with $(g,Z)\in H_J^0\times D_J$ denote the factor of automorphy (cf. [13, Proposition 2.3.1]) and the similitude factor as in [13, Section 2.2] respectively. When p_χ has a zero point on D_J, $\mathrm{Hom}_{(\mathfrak{g},K)}(\pi_n, C^\infty\text{-}\mathrm{Ind}_N^G(\chi))\neq 0$ is the $\mathbb{C}$-span of W'_χ, which is not of moderate growth.

When $\dim\mathrm{Hom}_{(\mathfrak{g},K)}(\pi_n, C^\infty_{\mathrm{mod}}\text{-}\mathrm{Ind}_N^G(\chi))=1$, the generalized Whittaker function is explicitly given by $W_\chi(g)$ up to scalars.

Remark. 3.2. *In [11, Theorem 2.15] it was stated that $\dim\mathrm{Hom}_{(\mathfrak{g},K)}(\pi_n, C^\infty\text{-}\mathrm{Ind}_N^G(\chi))=0$ when p_χ has a zero point on D_J. However, afterwards, the author has noticed that the analytic behavior of I_v around 0 implies that any zero point of p_χ does not give rise to the singularity for W'_χ while it does for W_χ as was pointed out in the proof of [13, Proposition 8.2.4]. Thus it turns out that W'_χ is smooth and of non-moderate growth.*

3.2 Fourier-Jacobi expansion

An automorphic form F on $\mathcal{G}(\mathbb{A})$ admits a Fourier expansion along the center of $\mathcal{N}(\mathbb{A})$. It is written as

$$F(g) = \sum_{\xi \in \mathbb{Q}} F_\xi(g),$$

with the Fourier transformation of F by the additive character of the center of $\mathcal{N}(\mathbb{A})$ with the parameter $\xi \in \mathbb{Q}$.

When F is an automorphic form of moderate growth generating quaternionic discrete series, a detailed description of F_0 is given by Pollack [13, Corollary 1.2.3, Proposition 11.1.1] in terms of the moderate growth generalized Whittaker functions for π_n. In this article we point out that F_ξ can be also described by the moderate growth generalized Whittaker functions. We need to note that $\mathcal{N}$ modulo center is given by the affine space $\mathcal{W}_J$ over $\mathbb{Q}$ with

$$\mathcal{W}_J(\mathbb{Q}) = \mathbb{Q} \oplus J \oplus J^\vee \oplus \mathbb{Q},$$

where $J^\vee$ denotes the dual space of J with respect to the trace pairing $(*, *)$ (cf. [13, Section 2.1]). The $\mathbb{Q}$-vector space $\mathcal{W}_J(\mathbb{Q})$ comes equipped with the symplectic form defined by

$$\langle(\alpha, \beta, \gamma, \delta), (\alpha', \beta', \gamma', \delta')\rangle := \alpha\delta' - (\beta, \gamma') + (\gamma, \beta') - \alpha'\delta$$

for $(\alpha, \beta, \gamma, \delta)$, $(\alpha', \beta', \gamma', \delta') \in \mathcal{W}_J(\mathbb{Q})$ with α, α', δ, $\delta' \in \mathbb{Q}$ and (β, γ) $(\beta', \gamma') \in J \oplus J^\vee$. We use the coordinate $n(\nu, t) \in \mathcal{N}(\mathbb{A})$ with $\nu \in \mathcal{W}_J(\mathbb{A})$ and $t \in \mathbb{A}$ and write $\nu = (\alpha, \beta, \gamma, \delta) \in \mathcal{W}_J(\mathbb{A})$.

For the description of F_ξ with $\xi \neq 0$ we furthermore need to review the element w_α of the Weyl group for $\mathcal{G}$. This comes from the Weyl reflexion of the root system of G_2 corresponding to the long root α in the set $\{\alpha, \beta\}$ of simple roots, where α (respectively β) denotes the long root (respectively short root). According to [14, Section 2.2], the exceptional group G_2 can be embedded into $\mathcal{G}$ and w_α can be regarded as an element of the Weyl group of $\mathcal{G}$. We can check this also by recalling that there is the $\mathbb{Z}/3$-grading of the Lie algebra $\mathfrak{g}_J$ defined over some general ground field (cf. [13, Section 4.2]).

Let F be an automorphic form on $\mathcal{G}(\mathbb{A})$. For a character χ of $\mathcal{N}(\mathbb{Q})\backslash\mathcal{N}(\mathbb{A})$, F_χ denotes the Fourier transformation of F by χ. We note that χ is parametrized via the symplectic form $\langle *, * \rangle$ by $(\alpha, \beta, \gamma, \delta) \in \mathbb{Q} \oplus J_{\mathbb{Q}} \oplus J_{\mathbb{Q}}^\vee \oplus \mathbb{Q} = \mathcal{W}_J(\mathbb{Q})$.

Theorem. 3.3. *Let F_ξ be the ξ-term of the Fourier-Jacobi expansion of an automorphic form F generating quaternionic discrete series for $\xi \in \mathbb{Q} \setminus \{0\}$. We do not assume the moderate growth condition of F.*

i) We have

$$F_\xi(g) = \sum_{\nu \in \mathbb{Q}} \sum_{\chi'_\xi} F_{\chi_\xi}(w_\alpha n((\nu, 0, 0, 0), 0)g),$$

where χ_ξ runs over characters of $\mathcal{N}(\mathbb{Q})\backslash\mathcal{N}(\mathbb{A})$ parametrized by $(\xi, \beta, \gamma, \delta) \in \mathcal{W}_J(\mathbb{Q})$ such that the associated cubic polynomial p_χ has no zero point on D_J.

ii) The archimedean part of F_{χ_ξ} is a constant multiple of the generalized Whittaker functions for π_n explicitly obtained by Pollack (cf. Theorem 3.1), which is of moderate growth.

For this theorem we cite [11, Theorem 3.9 (1)]. For the second assertion we remark that if F_{χ_ξ} is contributed by generalized Whittaker functions of non-moderate growth, some infinite sum of such Whittaker functions translated by a discrete group contributes to the Fourier series of F but it contradicts to the convergence of the Fourier series.

4 Outline of the proof and concluding remarks

4.1 Outline of the proof

Let F be an automorphic form on $\mathcal{G}(\mathbb{A})$ generating a quaternionic discrete series, for which we do not assume the moderate growth property in advance. The fundamental idea to show the Koecher principle of F is to reduce the moderate growth property of F to that of the constant term F_{00} of F, which contributes to F_0.

i) We begin with the following estimate

$$||F(g)||_{\tau_n} \geq ||F_{00}(g)||_{\tau_n}.$$

In [13, Proposition 11.1.1] F_{00} is written as a sum of a holomorphic modular form, its translation by some Weyl reflection and a constant function. When $\mathcal{G}$ is the exceptional group G_2 or a special orthogonal group $SO(4,N)$ (or $PSO(4,N)$), such holomorphic modular forms are elliptic modular forms or holomorphic forms on the direct product of the complex upper half plane and a symmetric domain of type IV. It is known that such holomorphic forms do not satisfy the Koecher principle. This is the reason why we exclude the two cases.

ii) The main difficulty of the proof is to show the moderate growth property of $F_0 - F_{00}$ and $F - \sum_{\xi\in\mathbb{Q}\setminus\{0\}} F_\xi$. As we have seen, these are sums of the moderate growth Whittaker functions W_χ with the explicit formula as in Theorem 3.1. Noting that automorphic forms are fixed by some open compact subgroup of the finite adeles of $\mathcal{G}$, we see that the expansions of $F_0 - F_{00}$ and $F - \sum_{\xi\in\mathbb{Q}\setminus\{0\}} F_\xi$ with respect to non-trivial characters over $\mathcal{W}_J(\mathbb{Q})$ is reduced to a sum over a lattice in $\mathcal{W}_J(\mathbb{Q})$. To carry out the estimate of these two we use the following formula (cf. [2, Section 1.6 (6.6)])

$$|K_v(y)| < \exp(-\frac{y}{2})K_v(2) \quad \text{if } y > 4. \tag{1}$$

Using this, we have

$$\begin{aligned}||F_0(g) - F_{00}(g)||_{\tau_n} \ll_{F,\tau_n,g_0} &\nu(g)^{n+1}\sum_{v_\chi} \exp(-\frac{1}{2}|\langle v_\chi, g\cdot r_0(\sqrt{-1}1_J)\rangle|) \\ \leq &\nu(g)^{n+1}\sum_{v_\chi}\exp(-\frac{1}{2}\frac{|\nu(g)|}{||g_1||_\infty}||v_\chi||),\end{aligned}$$

where $||v_\chi||$ denotes some norm of the vector $v_\chi \in \mathcal{W}_J(\mathbb{Q})$ parametrizing the character χ. This holds when $\frac{\nu(g)}{||g_1||_\infty||}v_\chi > 4$, The remaining summation running over χ such that $\frac{\nu(g)}{||g_1||_\infty||}v_\chi \leq 4$ is finite and estimated to be of moderate growth. The estimate of $F - \sum_{\xi\in\mathbb{Q}\setminus\{0\}} F_\xi$ can be given in a similar manner though we have to take the coordinate change by $w_\alpha n((\nu,0,0,0),0)$ into consideration.

4.2 Concluding remark

i) Though we exclude the cases of the exceptional group G_2 and the special orthogonal group $SO(4,N)$ (or $PSO(4,N)$) for the Koecher principle, it should be remarked that Theoren 2.1 does not imply that the Koecher principle is denied for the two cases. If we try to prove the principle for the two cases, we have to verify that elliptic modular forms without holomorphy around cusps do not contribute to the constant term F_{00} of an automorphic form F generating quaternionic discrete series.

ii) As for the case of $SU(d,2)$ we refer to [5], in which prove the "anti-Köcher principle" for the case of $SU(2,2)$. This does not mean the automatic validity of the moderate growth condition. The Koecher principle of automorphic forms on $SU(d,2)$ generating quaternionic discrete series is still open.

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