SOLUTIONS OF CERTAIN HOLONOMIC SYSTEM OF RANK 8 ARISING FROM DIFFERENTIAL OPERATORS ON SIEGEL MODULAR FORMS

TOMOYOSHI IBUKIYAMA
OSAKA UNIVERSITY

For a variable Z in the Siegel upper half space H_n, we fix sizes n_i of diagonal blocks of Z with $n = n_1 + \cdots + n_r$ and consider the restriction of Z to $(Z_{11}, Z_{22}, \ldots, Z_{rr})$ where Z_{ii} are the i-th $n_i \times n_i$ diagonal block of Z. We consider actions of $Sp(n, \mathbb{R})$ and $Sp(n_1, \mathbb{R}) \times \cdots \times Sp(n_r, \mathbb{R})$ associated with fixed automorphy factors. We consider linear differential operators with constant coefficients operating on holomorphic functions on H_n so that operation is equivariant with the actions associated with automorphy factors under the restriction to the diagonal blocks. Our problem is to describe such operators. We had a long history on a theory on such operators and this is related to various arithmetic problems which will be explained shortly later. Our aim of this note is much more special. We consider a holonomic system of three variables and of rank 8 whose polynomial solutions are associated with differential operators of the above type on Siegel modular forms of degree three preserving automorphy under the restriction to diagonal components. We will describe polynomial solutions of the system (unique up to constant for each system for generic parameters) and also a basis of its eight dimensional solutions including seven non-polynomial solutions. This is a joint work with Don Zagier and it is a part of a various theory independently done since 1990. The above holonomic system appeared first time in [3] as a Pfaffian of rank 8 (in Japanese). English explanation has been also in [5]. A general characterization of the differential operators under the restriction to any diagonal blocks was given in [3] and [4]. Universal generating functions of such differential operators are in [7]. A completely different intrinsic characterization of the differential operators by the action on the scalar valued automorphy factors with applications to pullback formula of Eisenstein series has been given in [8]. Reference [6] treated another kind of holonomic systems of rank 2^m coming from the case of two diagonal blocks of general size m.

In case of the diagonal restriction (not diagonal blocks), a detailed explanation on polynomials which give the above described differential operators on Siegel modular forms has been given in [9]. The details of the theory including holonomic systems in the present note will be written in [10]. This is still in preparation but mostly written and we hope to publish it soon as a book.

1. Introduction and problem settings

We start from a prototype of the theory.

(1) The classical Gegenbauer polynomials $C_\nu^{d/2-1}(t)$ are defined by the following generating series:

$$\frac{1}{(1-2tx+x^2)^{(d-2)/2}} = \sum_{\nu=0}^{\infty} C_\nu^{(d-2)/2}(t)x^\nu.$$

When $d=3$, these are called Legendre polynomials.

(2) The associated Gegenbauer differential equation is defined by

$$(1-t^2)\frac{d^2y}{dt^2} - (d-1)t\frac{dy}{dt} + \nu(\nu+d-2)y = 0.$$

For integral $\nu \geq 0$, a basis of this equation is spanned by $C_\nu^{(d-2)/2}(t)$ and a Gegenbauer function.

To explain an application of Gegenbauer polynomials to differential operators, we review Siegel modular forms. Let H_n be the Siegel upper half space defined by

$$H_n = \{Z \in M_n(\mathbb{C}); Z = {}^tZ, Im(Z) > 0\}.$$

The symplectic group of rank n defined by

$$Sp(n,\mathbb{R}) = \{g \in GL_{2n}(\mathbb{R}); {}^t gJ_ng = J_n\}, \quad J_n = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix},$$

acts on H_n as usual by

$$gZ = (AZ+B)(CZ+D)^{-1} \text{ for } g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R}).$$

Let $\mathrm{Hol}(H_n\mathbb{C})$ be the space of holomorphic functions of H_n. On $F \in \mathrm{Hol}(H_n,\mathbb{C})$, the group $Sp(n,\mathbb{R})$ acts by

$$F|_k[g] = \det(CZ+D)^{-k}F(gZ).$$

Let $\Gamma \subset Sp(n,\mathbb{R})$ be a discrete subgroup of covolume finite. We say $F \in \mathrm{Hol}(H_n,\mathbb{C})$ is a Siegel modular form of weight k with respect to Γ if $F|k[\gamma] = F$ for all $\gamma \in \Gamma$ (plus holomorphy at cusps if $n=1$).

Now we explain the simplest restriction of domains. We have an embedding

$$SL_2(\mathbb{R}) \times \cdots \times SL_2(\mathbb{R}) \hookrightarrow Sp(n,\mathbb{R}).$$

by setting

$$\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \ldots, \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}\right) \to \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C, D are diagonal matrices $A = \mathrm{diag}(a_i)$, $B = \mathrm{diag}(b_i)$, $C = \mathrm{diag}(c_i)$, $D = \mathrm{diag}(d_i)$. For $F(Z) \in Hol(H_n, \mathbb{C})$, we write

$$(Res(F))(z_{11}, \dots, z_{nn}) := F \begin{pmatrix} z_{11} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & z_{nn} \end{pmatrix}.$$

Then for $g = (g_1, \dots, g_n) \in SL_2(\mathbb{R}) \times \cdots \times SL_2(\mathbb{R}) \subset Sp(n, \mathbb{R})$, we have

$$Res(F|_k[g]) = \prod_{i=1}^{n} (c_i z_{ii} + d_i)^{-k} Res(F)(g_1 z_{11}, \dots, g_n z_{nn})$$

Now we would like to consider a differential operators such that the above relation holds also for $\mathbb{D}F$ (details will be explained later). First we consider the case $n = 2$ for simplicity. For a 2×2 symmetric matrix $T = \begin{pmatrix} t_{11} & t_{12} \\ t_{12} & t_{22} \end{pmatrix}$, we define $P_\nu(T)$ by

$$\frac{1}{(1 - 2t_{12}x + t_{11}t_{22}x^2)^{(d-2)/2}} = \sum_{\nu=0}^{\infty} P_\nu(T) x^\nu.$$

For $Z = (z_{ij}) \in H_2$, put

$$\partial_Z = \begin{pmatrix} \dfrac{\partial}{\partial z_{11}} & \dfrac{1}{2}\dfrac{\partial}{\partial z_{12}} \\ \dfrac{1}{2}\dfrac{\partial}{\partial z_{12}} & \dfrac{\partial}{\partial z_{22}} \end{pmatrix}.$$

Put $\mathbb{D} = P_\nu(\partial_Z)$.

Proposition 1.1. *For $g = (g_1, g_2) \in SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \subset Sp(2, \mathbb{R})$, $F \in \mathrm{Hom}(H_2, \mathbb{C})$, and $k = d/2$, we have*

$$Res(\mathbb{D}(F|_k[g]) = (c_1 z_{11} + d_1)^{-k-\nu}(c_2 z_{22} + d_2)^{-k-\nu} Res(\mathbb{D}F)(g_1 z_{11}, g_2 z_{22}).$$

This is essentially due to Eichler and Zagier [2]. (They treated Jacobi forms instead of F.)

Now we shortly explain what happens when we consider the same problem for

$$Sp(n_1, \mathbb{R}) \times \cdots \times Sp(n_r, \mathbb{R}) \subset Sp(n, \mathbb{R})$$

and the corresponding diagonal block restriction.
(1) For any irreducible representations (ρ_{n_i}, V_i) of $GL_{n_i}(\mathbb{C})$, we define automorphy factor of $Sp(n_i, \mathbb{R})$ by $J_i(g, Z_{ii}) = \rho_i(CZ + D)$ and put $V = V_1 \otimes \cdots \otimes V_r$. We consider V-valued differential operators on $\mathrm{Hol}(H_n, \mathbb{C})$ which commute with the automorphy factor $\det(CZ + D)^k$ and $\det(C_{11}Z_{11} + D_{11})^k J_1(g_1, Z_{11}) \otimes \cdots \otimes \det(C_{rr}Z_{rr} + D_{rr})^k J_r(g_r, Z_{rr})$ under the restriction to the diagonal blocks. We can give a complete description of this kind of vector valued differential operators for general n, representations ρ_i, and arbitrary diagonal block restrictions ([7],

[8]). This has an application to doubling method(=pullback formula of Eisenstein series), explicit calculation of critical values, Harder conjecture on congruneces, and so on ([1], [8]).
(2) In particular, we can give an explicit generating series of polynomials $P(T)$ so that $P(\partial_Z)$ gives the differential operator associated with diagonal restrictions, where we put

$$\partial_Z = \left(\frac{1+\delta_{ij}}{2} \frac{\partial}{\partial z_{ij}} \right).$$

([9], [7].)
(3) We can give a system of differential equations whose polynomial solutions give our differential operators.
(4) In some cases, this system is holonomic, in particular, when $n = r = 3$, $n_1 = n_2 = n_3 = 1$. This case is a subject of this note.

2. The case of the diagonal restriction

Now in order to make the Gegenbauer polynomials associate with our differential operators, we explain a modification of the usual definition. The Gegenbauer polynomials are often explained as harmonic polynomials on the sphere $S^{d-1} = SO(d)/SO(d-1)$ invariant by $SO(d-1)$. But our interpretation of the Gegenbauer polynomial is as follows.
Let $f(x,y)$ be a polynomial in $2d$ variables $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$. We impose the following three conditions.
(1) $f(x,y)$ is homogeneous for each x and y.
(2) $f(x,y)$ is harmonic for each x and y, i.e.

$$\sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2} = \sum_{i=1}^{d} \frac{\partial^2 f}{\partial y_i^2} = 0.$$

(3) $f(xh, yh) = f(x,y)$ for all $h \in O(d)$.
By the fundamental theorem of invariants, (3) means that for $d \geq n$, there exists a polynomial $P(T)$ in t_{ij} with $T = \begin{pmatrix} t_{11} & t_{12} \\ t_{12} & t_{22} \end{pmatrix}$ such that

$$P\begin{pmatrix} (x,x) & (x,y) \\ (x,y) & (y,y) \end{pmatrix} = f(x,y). \qquad \text{(Here } (*,*) \text{ is the inner product.)}$$

If we restrict $f(x,y)$ to $(x,x) = (y,y) = 1$, then this is the usual Gegenbauer polynomial in one variable (x,y).

This formulation can be easily generalized to general setting. Let $f(x_1, \ldots, x_n)$ be a polynomial in $x_i \in \mathbb{R}^d$ homogeneous of degree a_i, and harmonic for each x_i, such that

$$f(x_1h, \ldots, x_nh) = f(x_1, \ldots, x_n) \qquad \text{for all } h \in O(d).$$

If $d \geq n$, then there exists $P(T)$ ($T = (t_{ij}) = {}^tT$) such that

$$f(x_1, \ldots, x_n) = P((x_i, x_j)).$$

Proposition 2.1. *For a holomorphic function* $F : H_n \to \mathbb{C}$, $g = (g_1, \ldots, g_n) \in SL_2(\mathbb{R})^n \subset Sp(n, \mathbb{R})$ *and a differential operator*

$$D = P(\partial_Z) \qquad \text{for } P(T) \text{ as above,}$$

we have

$$Res(\mathbb{D}(F|[g]) = Res(\mathbb{D}F)(g_1 z_{11}, \ldots, g_n z_{nn}) \prod_{i=1}^{n} (c_i z_{ii} + d_i)^{d/2+a_i}$$

In order to develop more general theory, it is indispensible to consider everything in T variable and not in x variable. We describe this coordinate change. For an $n \times n$ symmetric matrix $T = (t_{ij})$ of variables, we consider the polynomial ring $\mathbb{C}[T]$ in variables t_{ij} $(1 \le i \le j \le n)$. We consider a map $x = (x_1, \ldots, x_n) \in (\mathbb{R}^d)^n$ to $T = ((x_i, x_j)) = (t_{ij})$, where (x_i, x_j) are the usual inner product of x_i and x_j. Put $\partial_{ij} = (1+\delta_{ij})\dfrac{\partial}{\partial t_{ij}}$ and

$$D_{ij} = d\partial_{ij} + \sum_{k,l=1}^{n} t_{kl} \partial_{ik} \partial_{jl}. \tag{1}$$

Then we have

$$\sum_{\nu=1}^{d} \frac{\partial^2 P((x_i, x_j))}{\partial x_{i\nu} \partial x_{j\nu}} = (D_{ij} P)((x_i, x_j)).$$

We call a polynomial $P(T)$ such that $D_{ii} P = 0$ for all $i = 1, \ldots, n$ a spherical polynomial. In x-coordinate, this just means that $P((x_i, x_j))$ is harmonic for each x_i. Although d was originally the length of the vectors x_i, once we define D_{ij} by (1), we can forget x_i, and d can be any complex number.

In order to describe the dimension of P, we prepare notation. For any diagonal matrix $\mathbf{c} = diag(c_1, \ldots, c_n)$, we say $P(T)$ is of multi-degree $\mathbf{a} = (a_1, \ldots, a_n)$ if $P(\mathbf{c}T\mathbf{c}) = \prod_{i=1}^n c_i^{a_i} P(T)$.

For a fixed multi-degree $\mathbf{a} = (a_1 \ldots, a_n)$, the dimension of the space of spherical polynomials $P(T)$ is equal to the cardinality of the following set $\mathcal{N}_0(\mathbf{a})$:

$$\begin{aligned}\mathcal{N}_0(\mathbf{a}) = \Big\{ &\boldsymbol{\nu} = (\nu_{ij})_{1 \le i,j \le n} : \nu_{ij} = \nu_{ji} \in \mathbb{Z}_{\ge 0}, \\ &\nu_{ii} = 0 \ (i = 1, \ldots, n), \quad \sum_{j=1}^{n} \nu_{ij} = a_i \text{ for each } i \Big\}.\end{aligned}$$

We call an element $\boldsymbol{\nu} \in \mathcal{N}_0(\mathbf{a})$ a multi-index.

When d is *generic* (which is defined that d is a complex number which is not an integer less than n), there exists a nice basis $P_{\boldsymbol{\nu}}(T)$ of spherical polynomials such that

$$P_{\boldsymbol{\nu}}(\partial_Z) \det(Z)^s = \phi_{\boldsymbol{\nu}}(Z^{-1}) \det(Z)^s$$

where $\phi_{\boldsymbol{\nu}}$ is an explicitly written monomial in variable T, where we do not explain here (See [8]).

Now when we consider the Gegenbauer polynomial, we usually consider not a polynomial $f(x, y)$ in x, $y \in \mathbb{R}^d$, but take the restriction to $(x, x) = (y, y) = 1$ and regard $f(x, y)$ as a polynomial in one variable (x, y). This polynomial in one variable is usually called a radial part of f. We consider the same thing in general case.

Let $P(T)$ be a polynomial of a multi-degree $\mathbf{a}$, and we put $\delta(T)^{\mathbf{a}/2} := \prod_{i=1}^n t_{ii}^{a_i/2}$. Put $\tau_{ij} = \dfrac{t_{ij}}{\sqrt{t_{ii}t_{jj}}}$. Then we have $P(T) = \prod_{i=1}^n \delta(T)^{\mathbf{a}/2} Q(\tau_{ij})$ for some polynomial $Q(\tau)$ in τ_{ij} for $\tau = (\tau_{ij})$. Then there exists a linear differential operator $\mathbb{D}_{ij}(\mathbf{a})$ with respect to τ_{ij} such that

$$D_{ij}P(T) = \delta(T)^{(\mathbf{a}-\mathbf{e}_i-\mathbf{e}_j)/2}\mathbb{D}_{ij}(\mathbf{a})Q(\tau) \qquad (1 \leq i, j \leq n),$$

where $\mathbf{e}_i$ are unit vector whose i-th component is 1 and the other components are 0.

Example: If $n = 2$, then for generic d, there exists a non-zero spherical polynomial P if and only if $a_1 = a_2$. Put $t = \dfrac{t_{12}}{\sqrt{t_{11}t_{22}}}$. Then the differential equation $D_{11}P = D_{22}P = 0$ can be written as a constant times of the Gegenbauer differential equation

$$(1 - t^2)\frac{d^2Q}{dt^2} + (d - 1)t\frac{dQ}{dt} + \nu(\nu + d - 2)Q = 0,$$

where $\nu = a_1 = a_2$. (Two equations $D_{ii}P = 0$ for $i = 1$ and 2 gives the same equation up to constant.)

In case of $n = 3$, for a 3×3 symmetric matric $T = (t_{ij})$, we put

$$t_1 = \frac{t_{23}}{\sqrt{t_{22}t_{33}}}, \quad t_2 = \frac{t_{13}}{\sqrt{t_{11}t_{33}}}, \quad t_3 = \frac{t_{12}}{\sqrt{t_{11}t_{22}}}.$$

For each $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{C}^3$ and $d \in \mathbb{C}$, we denote by $\mathcal{S}(d, \mathbf{a})$ the radial part of the system of the differential equation $D_{11}P = D_{22}P = D_{33}P = 0$ in variables t_i. This can be written as the system of following three equations.

$$\begin{aligned} D_{kk}(\mathbf{a}) =& (1 - t_i^2)\frac{\partial^2 Q}{\partial t_i^2} + 2(t_k - t_i t_j)\frac{\partial^2 Q}{\partial t_i \partial t_j} + (1 - t_j^2)\frac{\partial^2 Q}{\partial t_j^2} \\ &+ (d - 1)\left(t_i\frac{\partial Q}{\partial t_i} + t_j\frac{\partial Q}{\partial t_j}\right) + a_k(a_k + d - 2)Q = 0, \end{aligned}$$

where $\{i, j, k\} = \{1, 2, 3\}$. This is the equation to treat in this note.

Theorem 2.2. *For any complex parameters d, a_1, a_2, a_3, this system $\mathcal{S}(d, \mathbf{a})$ is holonomic and equivalent to a Pfaffian system of rank eight.*

Originally, d and a_i are positive integers, but here we do not assume that at all.

Here the Pfaffian system can be written as follows. Let $Q(t_1,t_2,t_3)$ be a solution of the system $\mathcal{S}(d,\mathbf{a})$. We write $Q_i = \dfrac{\partial Q}{\partial t_i}$ $(i=1,2,3)$, $Q_{ij} = \dfrac{\partial^2 Q}{\partial t_i \partial t_j}$ $(1 \le i \le j \le 3)$, $Q_{123} = \dfrac{\partial^3 Q}{\partial t_1 \partial t_2 \partial t_3}$ and define a vector u by $u = {}^t(Q, Q_1, Q_2, Q_3, Q_{12}, Q_{23}, Q_{13}, Q_{123})$.

Theorem 2.3. *There exists an explicitly written* 1*-form* $\Omega = \Omega_1(t)dt_1 + \Omega_2(t)dt_2 + \Omega_3(t)dt_3$ *of variable* $t = (t_1,t_2,t_3)$*, where* $\Omega_i(t)$ *are* 8×8 *explicit matrices of rational functions of* t *that are holomorphic at* $(t_1,t_2,t_3) = 0$ *such that*

$$du = \Omega u.$$

This system is integralable, i.e. $d\Omega - \Omega \wedge \Omega = 0$.

Explicit shape of Ω is a bit too long to write here and omitted, but the integrability is explicitly checked by using a concrete formula for Ω.

As a conclusion, there exist 8 linearly independent solutions holomorphic arount $t = (0,0,0)$ which is determined by its Taylor coefficients at 1, t_1, t_2, t_3, t_1t_2, t_2t_3, t_1t_3, $t_1t_2t_3$.

(By the way, for $n \ge 4$, the radial part of the system $D_{11}P = \cdots = D_{nn}P = 0$ for a fixed multi-degree is not holonomic, and we do not know how to define a holonomic system. For other examples, see [6].)

Next we would like to describe solutions of the system $\mathcal{S}(\mathbf{a},d)$. First we consider polynomial solutions. Here we assume that a_i are non-negative integers. Assume that d is a complex number such that

$$d \notin 2\mathbb{Z}_{\le 0} \cap \bigcup_{a_i \ge 2} [4 - 2a_i, 2 - a_i].$$

Then the system $D_{11}P(T) = D_{22}P(T) = D_{33}P(T) = 0$ has a polynomial solution if ant only if there exist integers $\nu_i \ge 0$ $(i = 1,2,3)$ such that $a_i = \nu_j + \nu_k$ $(\{i,j,k\} = \{1,2,3\})$. This is a condition such that $|\mathcal{N}_0(\mathbf{a})| = 1$. So in this case, for each such multidegree $\mathbf{a}$, polynomial solution of multidegree $\mathbf{a}$ is unique up to constant.
Naturally, for such solution $P(T)$, a polynomial $Q(\tau)$ defined by

$$P(T) = \prod_{i=1}^{n} t_{ii}^{a_i/2} Q(\tau)$$

is a solution of $\mathcal{S}(d,\mathbf{a})$, again a polynomial in τ_{ij}.

For example, when $\mathbf{a} = (0,0,0)$, then 1 is the unique polynomial solution up to constant for generic d.

We give a generating function of polynomials solutions $P(T)$ for $\mathbf{a}$ such that $\mathcal{N}_0(\mathbf{a}) \ne \emptyset$. Put $X = \begin{pmatrix} 0 & x_3 & x_2 \\ x_3 & 0 & x_1 \\ x_2 & x_1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{12} & t_{22} & t_{23} \\ t_{13} & t_{23} & t_{33} \end{pmatrix}$

and define polynomials $\sigma_i(T,X)$ in t_{ij} and x_i by the following relation.

$$\det(\lambda 1_3 - TX) = \lambda^3 - \sigma_1(T,X)\lambda^2 + \sigma_2(T,X)\lambda - \sigma_3(T,X).$$

For the multi-index $\boldsymbol{\nu} = (\nu_i)$ such that $a_i = \nu_j + \nu_k$, we define polynomials $P_{\boldsymbol{\nu}}(T)$ as coefficients of the following formal power series in x_1, x_2, x_3.

$$G = \frac{1}{\sqrt{\Delta_0^2 - 8\sigma_3}} \left(\frac{\Delta_0 + \sqrt{\Delta_0^2 - 8\sigma_3}}{2} \right)^{-(d-4)/2} = \sum_{\boldsymbol{\nu}} P_{\boldsymbol{\nu}}(T) x_1^{\nu_1} x_2^{\nu_2} x_3^{\nu_3},$$

where $\Delta_0 = (1 - \sigma_1/2)^2 - \sigma_2$. Then the radial part of $P_{\boldsymbol{\nu}}(T)$ is the solution of the system $\mathcal{S}(d, \mathbf{a})$ for $\mathbf{a}$ such that $\mathcal{N}_0(\mathbf{a}) = \{\boldsymbol{\nu}\}$.

3. Non-polynomial solutions

Next we return to general solutions. Let d be any complex number as before and assume that $\mathbf{a} = (a_1, a_2, a_3)$ are non-negative integers and $a_1 = \nu_2 + \nu_3$, $a_2 = \nu_3 + \nu_1$, $a_3 = \nu_1 + \nu_2$ for some non-negative integers ν_i. Under these conditions, we see that the space of solutions of $\mathcal{S}(d, \mathbf{a})$ is reduced to solutions of $\mathcal{S}(d, \mathbf{0})$ where $\mathbf{0} = (0,0,0)$. That is, we have an explicit linear isomorphism from $\mathcal{S}(d, \mathbf{0})$ onto $\mathcal{S}(d, \mathbf{a})$. This is actually given as the iteration of operators from $\mathcal{S}(d, \mathbf{a})$ to $\mathcal{S}(d, \mathbf{a} + \mathbf{e}_i + \mathbf{e}_j)$, where $\mathbf{e}_i$ are the unit vectors. We explain this now. First we fix *generic* d. Denote by $\mathcal{F}(\mathbf{a})$ the 8 dimensional vector space of solutions of the system $\mathcal{S}(d, \mathbf{a})$. We define $\mathbb{D}_{ij}(\mathbf{a})$ as a radial part of $\mathbb{D}_{ij}$ for a fixed $\mathbf{a}$ as before. Put $\mathbf{1} = (1,1,1)$. Then we have

$$\mathbb{D}_{ij}((2-d)\mathbf{1} - \mathbf{a})\mathcal{F}(\mathbf{a}) \subset \mathcal{F}(\mathbf{a} + e_i + e_j).$$

This inclusion is proved by the following relation.

$$\mathbb{D}_{\ell\ell}(\mathbf{a} + \mathbf{e}_i + \mathbf{e}_j)\mathbb{D}_{ij}((2-d)\mathbf{1} - \mathbf{a}) = \mathbb{D}_{ij}((2-d)\mathbf{1} - \mathbf{a} - 2e_\ell)\mathbb{D}_{\ell\ell}(\mathbf{a}).$$

These relations are easily proved by rewriting the trivial relations
$D_{ij}D_{kl} = D_{kl}D_{ij}$ of operators in T by the radial part. We put

$$\mathbb{R}_{ij}(\mathbf{a}) = \mathbb{D}_{ij}((2-d)\mathbf{1} - \mathbf{a})$$

and call these *raising operators*. Under the assumption that d is *generic*, we can prove the surjectivity of the raising operators as follows.

By using the Pfaffian system, we identify $\mathcal{F}(\mathbf{a})$ as 8 dimensional vectors of coefficients of 1, t_1, t_2, t_3, t_1t_2, t_2t_3, t_1t_3, $t_1t_2t_3$. Then we can write down an explicit representation matrix of the raising operator $\mathbb{R}_{ij}(\mathbf{a}) := \mathbb{D}_{ij}((2-d)\mathbf{1} - \mathbf{a})$ on these vectors and its determinant:

$$\begin{aligned}\det \mathbb{R}_{ij}(\mathbf{a}) &= (2 + a_i + a_j - a_k)^2 (a_i + a_j - a_k + d - 2)^2 \\ &\quad \times (a_1 + a_2 + a_3 + d)^2 (a_1 + a_2 + a_3 + 2d - 4)^2 / 256.\end{aligned}$$

So if $a_i = \nu_j + \nu_k$ for $\nu_i \geq 0$ and d is generic (for example if $d > 2$), then the raising operators are all surjective.
Actually, as far as $\mathbf{a}$ corresponds with a multi-index $\boldsymbol{\nu}$, then for any

(non-generic) d, by a suitable base change, we can define a modified surjective (and injective) raising operator.

So from now on we consider solutions only for the case $\mathbf{a} = \mathbf{0}$.

The easiest solution in case $\mathbf{a} = \mathbf{0}$ is the constant 1. We explain next easy solutions for $\mathbf{a} = 0$ (called bottom solutions). We define a one variable function $f_s(t)$ by

$$f_s(t) = \int_0^t \frac{1}{(1-x^2)^{s+3/2}} dx = t + (2s+3)/6t^3 + \cdots$$

This is a classical solution of the Gegenbauer differential equation

$$(1-t^2)y'' - (2s+3)ty' = 0 \qquad \text{for } \nu = 0.$$

If we put $s = (d-4)/2$, then we see that $f_s(t_i)$ are solutions in $\mathcal{F}(\mathbf{0})$. So we get four simple linearly independent solutions 1, $f_s(t_1)$, $f_s(t_2)$, $f_s(t_3)$ whose coefficients at t_1t_2, t_2t_3, t_1t_3, $t_1t_2t_3$ all vanish.

Remark. In general, there is no one variable solution in $\mathcal{F}(\mathbf{a})$ for a multi-degree $\mathbf{a} \neq \mathbf{0}$.

Another special case is the case $d = 4$ (i.e. $s = 0$). In this case, we can give all eight linearly independent solutions explicitly. We put $\Delta = 1 - t_1^2 - t_2^2 - t_3^2 + 2t_1t_2t_3$ and

$$v_i = \frac{t_i - t_jt_k}{\sqrt{(1-t_j^2)(1-t_k^2)}} \qquad \text{for } \{i,j,k\} = \{1,2,3\}.$$

When $d = 4$ ($s = (d-4)/2 = 0$), a basis of $\mathcal{F}(\mathbf{0})$ will be given below.

(1) Trivial solution: 1,

(2) Three Bottom solutions: $f_0(t_i) = \dfrac{t_i}{\sqrt{1-t_i^2}} \quad (i = 1,2,3)$.

(3) Three Middle solutions:

$$\sin^{-1}(t_i) - f_0(t_j)f_{-1/2}(v_k) - f_0(t_k)f_{-1/2}(v_j) = t_i - 2t_jt_k + \cdots$$

for all $\{i,j,k\} = \{1,2,3\}$. This is invariant if we exchange j and k.

(4) One Top solution

$$\log(\Delta) + \sum_{i=1}^{3} \frac{t_i}{1-t_i^2} \sin^{-1}(t_i) = 2t_1t_2t_3 + \cdots$$

For general d or s, solutions are much more complicated, so we need more explanation. We need to describe solutions whose coefficients at t_1t_2, t_2t_3, t_1t_3, or $t_1t_2t_3$ do not vanish.

We can show that there exists a *middle* solution $G_{jk}(t)$ invariant by $(t_j, t_k) \to (t_k, t_j)$, and $(-t_j, -t_k)$ such that

$$G_{jk}(t) = t_i - 2(s+1)t_jt_k + \cdots .$$

We can prove that the Taylor coefficients $c_{ijk}(s)$ of $G_{ij}(t)$ around $t=0$ defined by

$$G_{ij}(t) = \sum_{i,j,k\geq 0} c_{ijk}(s) \frac{t_1^i t_2^j t_3^l}{i!j!k!},$$

are given by

$$c_{ijk}(s) = (-1)^i (i+j-2)!!(i+k-2)!!(2s+2)(2s+4)\cdots(2s+j+k).$$

if $j \equiv k \equiv i+1 \bmod 2$, and

$$c_{ijk}(s) = 0, \qquad \text{otherwise.}$$

Here we put $n!! = n\cdot(n-2)\cdots$ and $(-1)!! = 1$, as usual.
When $s \neq -1$, this is a new solution different from 1 and $f_s(t_i)$, but when $s=-1$, this degenerates to the bottom solution $f_{-1}(t_k)$. so to obtain a new solution, we must modify the solution by taking a derivative with respect to s.

Although Taylor coefficients are explicitly written as above, it is a different matter to write down a (kind of algebraic) closed formula. Now we would like to write down a formula for $G_{jk}(t)$ more concretely. As before, we put $\Delta = 1 - t_1^2 - t_2^2 - t_3^2 + 2t_1t_2t_3$ and put

$$u_i = \frac{t_jt_k - t_i}{\sqrt{\Delta}} \qquad \text{for } \{i,j,k\} = \{1,2,3\}.$$

We also put

$$h_s(t,u) = \frac{t}{(1-t^2)^{s+1/2}} \int_0^u \frac{(1+x^2)^{s+1/2}}{t^2+x^2} dx.$$

Theorem 3.1. *For $\{i,j,k\} = \{1,2,3\}$, the function*

$$G_{s,jk}(t_1,t_2,t_3) = h_s(t_j,u_k) + h_s(t_k,u_j) = t_i - 2(s+1)t_jt_k + \cdots$$

are middle solutions of the system $\mathcal{S}(\mathbf{0},d)$ for any $d = 2s+4$ except for $s=-1$. If $s=-1$, take $(G_{s,jk}(t) - \sin^{-1}(t_i))/(s+1)$ instead.

As for top solutions, the Taylor expansions are known (omitted since complicated), but general closed formula is not known. If we consider the case of integral $s = (d-4)/2 = m \geq 0$, then we can give a closed formula, and the formula is described by using the polynomial solution of non-zero multi-index as follows.

For $s = (d-4)/2$ and multi-index $\boldsymbol{\nu}$, denote by $P_{\boldsymbol{\nu},s}(T)$ the polynomial solution defined by the generating function (**??**) as the coefficient of $x_1^{\nu_1}x_2^{\nu_2}x_3^{\nu_3}$. Let $Q_{\boldsymbol{\nu},s}(t)$ be the inhomogeneous version of $P_{\boldsymbol{\nu},s}(T)$. For any integer $m \geq 0$, define

$$R_m(t) = \left.\frac{dQ_{(m,m,m),s}}{ds}\right|_{s=-m}.$$

Also define

$$g_s(t) = \sum_{n=0}^{\infty} \frac{4^n n!^2}{(2n+2)!} \binom{n+s+1}{n} t^{2n+2} = \int_0^t \int_0^x \frac{(1-y^2)^{s+1/2}}{(1-x^2)^{s+3/2}} dydx.$$

Theorem 3.2. *For any integer $m \geq 0$, the function*

$$F_m(t) = \log(\Delta) + (2m+2) \sum_{i=1}^{3} g_s(t_i) + 2^{-2m} \frac{R_m(t)}{\Delta^m}$$

is a top solution for $d = 2m + 4$.

When s is not a non-negative integer, we have several sporadic explicit examples of top solutions for various s, but $\log(\Delta)$ does not appear there, and we do not have general idea how it is like.

4. How to obtain the general middle solution.

In the above explanation, we gave the result on middle solutions without explanation. But it was very hard to get the theorem, so it would be interesting to explain how we could reach the solution. We will give a very rough sketch of our ideas. (For details, see [10]).

- First we consider the case integral $s = m \geq 0$. By examples of small m obtained by brute force, we can guess that the solution is

$$\begin{aligned} G_m = & f_m(t_2) f_{-1/2}(v_3) + f_m(t_3) f_{-1/2}(v_2) + d_m \sin^{-1}(t_1) \\ & + \frac{\sqrt{1-t_1^2}(H_m(t_1,t_2,t_3) + H_m(t_1,t_3,t_2))}{\Delta^m} \end{aligned}$$

  for some rational function $H_m(t)$ and a constant d_m, where $v_i = (t_i - t_j t_k)/\sqrt{(1-t_j^2)(1-t_k^2)}$ as before.
- We put $D_i(\boldsymbol{\nu}, s) = D_{ii}(\mathbf{a})$ for $d = 2s + 4$ and $a_i = \nu_j + \nu_k$. We first consider the condition $D_1(\mathbf{0}, s) G_m = 0$, neglecting other D_2 and D_3. Here note that

$$D_1(\mathbf{0}, s)(\sin^{-1}(t_1)) = 0,$$

$$D_1(\mathbf{0}, s)(f_m(t_j) f_{-1/2}(v_k)) = \frac{2s f_m(t_2)}{\Delta}.$$

Ansatz for integral m**.** By our demand, the denominator Δ^m of $\sqrt{1-t_1^2} H_m(t)/\Delta^m$ should become Δ under $D_1(\mathbf{0}, s)$. This seems very strong condition. So we consider how we can erase the denominator in a neat way. To see this mechanic, we consider Ansatz that we have

$$\frac{\sqrt{1-t_1^2} H_m(t_1,t_2,t_3)}{\Delta^m} = \sum_{j=0}^{m-1} \frac{h_j(t)}{\Delta^{m-j}}$$

for some suitable $h_j(t)$ and then try to give suitable $h_j(t)$ by recursion. Here we note that we can prove the following inversion formula to change the denominator of powers of Δ.

Lemma 4.1 (Inversion formula of Δ)**.**

$$\Delta^{-\lambda} D_i(\boldsymbol{\nu}, s)\Delta^{\lambda} Q(t) = D_i(\boldsymbol{\nu} - \lambda \mathbf{1}, s + 2\lambda)Q(t) + \frac{4\lambda(s+\lambda)(1-t_i^2)Q(t)}{\Delta}.$$

This is a powerful trick and by vanishing under $D_1(0, m)$, the following recursive relations are expected.

$$D_1(0,m)\left(\frac{h_j(t)}{\Delta^{m-j}}\right) = \frac{D_i((m-j)\mathbf{1}, 2j-m)h_j(t)}{\Delta^{m-j}} - 4(m-j)j\frac{(1-t_1^2)h_j(t)}{\Delta^{m-j+1}}.$$

Another trick is a separation of variables giving the operation of $D_1(\boldsymbol{\nu}, s)$ on special product of functions.

Lemma 4.2. *For a function $F(t) = f(t_2)g(v_3)$ for some functions f and g in one variable, where v_3 is as before, we have*

$$\begin{aligned} D_1(\boldsymbol{\nu}, s)F = &\Big((1-v_3^2)g''(v_3) - (2s+2)v_3 g'(v_3)\Big) f(t_2)/(1-t_2^2) \\ &+ g(v_3)\Big((1-t_2^2)f''(t_2) - (2s+3)t_2 f'(t_2) + a_1(a_1+2s+2)\Big). \end{aligned}$$

So if we put $h_0(t) = v_3\eta_0(t_2)$, then $h_{m-1}(t) = v_3\eta_{m-1}(t_2)$ for some η_{m-1}, which fits our condition since the final term should be

$$D_1(\mathbf{0}, m)(f_m(t_2)f_{-1/2}(v_3)) = -\frac{2m f_m(t_2)v_3}{\Delta}.$$

By using this, we guess what should be suitable $h(t)$ for us. If we put a "half of" $h_j(t)$ corresponding to $f_m(t_2)f_{-1/2}(v_3)$ by $(1-t_1^2)^{m-j}(1-t_2^2)^{m-j}v_3\psi_{m,j}(t_2)$, then we should expect the following recursion:

$$\begin{aligned} &(1-t_2^2)\psi''_{m,j}(t_2) - (2m+3)t_2\psi''_{m,j}(t_2) + \frac{2(j+1)(2-2j-1)}{1-t_2^2}\psi_{m,j}(t_2) \\ &= \begin{cases} 4(m-j-1)(j+1)\psi_{m,j+1}(t_2)/(1-t_2^2) & j \le m-2, \\ 2m f_m(t_2)/(1-t_2^2) & j = m-1. \end{cases} \end{aligned}$$

We must solve this equation. By the parity condition of the middle solution, the function $\psi_{m,j}(t_2)$ be an odd function with respect to t_2. Then a basis of solution of this second order equation is given by

$$\psi_{m,j}(t_2) = \frac{f_m(t_2)}{c_{m-j-1}}, \text{ or } const \cdot f_0(t_2)$$

where $c_n = \binom{n+1/2}{n}$. These are conditions for $D_1 G_m = 0$ and we have not yet considered the other conditions $D_2 G_m = D_3 G_m = 0$. Applying $D_2(G_m) = D_3(G_m) = 0$, we can determine remaining constants d_m.

By these considerations, for any integral $s = m \geq 0$, the middle solution G_m can be given as follows, defining u_i and v_j as before.

$$\begin{aligned}-G_m/d_m =& - c_m(f_m(t_2)f_{-1/2}(v_3) + f_m(t_3)f_{-1/2}(v_2)) + \sin^{-1}(t_1)\\ &+ \sum_{n=0}^{m-1}\left(\frac{c_m}{c_n}f_m(t_2) - f_n(t_2)\right)u_3(1+u_3^2)^{n+1/2}\\ &+ \sum_{n=0}^{m-1}\left(\frac{c_m}{c_n}f_m(t_3) - f_n(t_3)\right)u_2(1+u_2^2)^{n+1/2}.\end{aligned}$$

Now in order to treat the case of general s, we must interpolate integers $s = m$ to a variable s. We see this now. Put

$$\tilde{h}_m(t,u) = -c_m f_m(t)f_{-1/2}(v) + \sum_{c=0}^{m-1}\left(\frac{c_m}{c_n}f_m(t) - f_n(t)\right)u(1+u^2)^{n+1/2}.$$

for variables t, u, v. Here we assume $v(1-v^2)^{-n-1} = -u(1+u^2)^{n+1/2}$ and t and u are independent variables. We also put

$$F_s(u) = \sum_{n=0}^{\infty}\binom{s-1/2}{n}\frac{u^{2n+1}}{2n+1}.$$

Then we have following relations.

$$\begin{aligned}&2sF_s(u) - (2s-1)F_{s-1}(u) = u(1+u^2)^{s-1/2},\\ &F_m(u)/\binom{m-1/2}{m} = F_0(u) + \sum_{n=0}^{m-1}\frac{u(1+u^2)^{n+1/2}}{c_n},\\ &\tilde{h}_m(t,u) - \tilde{h}_{m-1}(t,u) = F_m(u)(1-t^2)^{-m-1/2}.\end{aligned}$$

By these, we can show that

$$\begin{aligned}\tilde{h}_m(t,u) =&(2m+1)f_m(t)F_m(u) - u\sum_{n=0}^{m-1} f_n(t)u(1+u^2)^{n+1/2}\\ =&\sum_{n=0}^{m}\frac{t}{(1-t^2)^{n+1/2}}F_n(u).\end{aligned}$$

Since we have

$$\frac{dF_s(u)}{du} = (1+u^2)^{s-1/2}$$

we have

$$\frac{d\tilde{h}_m(t,u)}{du} = \frac{t}{t^2+u^2}\left(\left(\frac{1+u^2}{1-t^2}\right)^{m+1/2} - \left(\frac{1-t^2}{1+u^2}\right)^{1/2}\right).$$

In this last formula, we can change m to s. This is a key for our interpolation. Now for variable s, we define

$$\tilde{h}_s(t,u) = \frac{t}{(1-t^2)^{s+1/2}} \int_0^u \frac{(1+u^2)^{s+1}-(1-t^2)^{s+1}}{(t^2+u^2)\sqrt{1+u^2}}du.$$
$$= h_s(t,u) - \int_0^u \frac{\sqrt{1-t^2}}{(t^2+u^2)\sqrt{1+u^2}}$$

The integral of the second term can be directly calculated and using the addition formula of tangent, we have

$$\tan^{-1}\left(\frac{u_3\sqrt{1-t_2^2}}{t_2\sqrt{1+u_3^2}}\right) + \tan^{-1}\left(\frac{u_2\sqrt{1-t_3^2}}{t_3\sqrt{1+u_2^2}}\right) = -\sin^{-1}(t_1).$$

This proves Theorem 3.1.

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Department of Mathematics, Graduate School of Mathematics, Osaka University, Machikaneyama 1-1, Toyonaka,Osaka, 560-0043 Japan
Email address: ibukiyam@math.sci.osaka-u.ac.jp