On the Fourier coefficients of Siegel Eisenstein series of an odd prime level

Keiichi Gunji
Department of Mathematics
Chiba Institute of Technology

1 Introduction

In this article, we study the Fourier coefficients of Siegel Eisenstein series of level p. Although it is one of the most fundamental topics in the theory of Siegel modular forms, we have not yet solve this problem completely. The author tried to write down an explicit formula of the Fourier coefficients for a long time, to get a big progress recently. The precise arguments are written in the preprint ([9]), thus we explain mainly the basic ideas in the present article.

2 Fourier expansion of Siegel Eisenstein series

First we would like to recall the basic facts and previous results about the Fourier coefficients of Siegel Eisenstein series.

2.1 Full modular case

We start to explain the full modular case. Let $Sp(n,\mathbb{Z})$ be the simplex modular group of rank n, matrix-size $2n$. For $\gamma \in Sp(n,\mathbb{Z})$, we write $\gamma = \begin{pmatrix} A_\gamma & B_\gamma \\ C_\gamma & D_\gamma \end{pmatrix}$. Let $P_n(\mathbb{Z}) = \{\gamma \in Sp(n,\mathbb{Z}) \mid C_\gamma = 0\}$. For $Z \in \mathbb{H}_n$ of Siegel upper half space, the Siegel Eisenstein series of weight $k \in 2\mathbb{Z}$ is defined by

$$E_k^n(Z) = \sum_{\gamma \in P_n(\mathbb{Z})\backslash Sp(n,\mathbb{Z})} \det(C_\gamma Z + D_\gamma)^{-k},$$

that converges when $k > n+1$. Let $\mathrm{Sym}^n(\mathbb{Z})^*$ be the set of half integral symmetric matrices of size n, consisting of symmetric matrices whose diagonal entries belong to $\mathbb{Z}$ and off-diagonal entries belong to $\frac{1}{2}\mathbb{Z}$. We write the Fourier expansion of $E_k^n(Z)$ as

$$E_k^n(Z) = \sum_{\substack{N \in \mathrm{Sym}^n(\mathbb{Z})^* \\ N \geq 0}} A(N,k)\mathbf{e}(NZ),$$

here we write $\mathbf{e}(NZ) = \exp\bigl(2\pi\sqrt{-1}\,\mathrm{Tr}(NZ)\bigr)$.

Let us investigate the Fourier coefficient $A(N,k)$. First we remark that it suffices to consider the case for positive definite N; if $\operatorname{rank} N = r < n$, then $A(N,k)$ comes from the Fourier coefficient of the Siegel Eisenstein series $E_k^r(Z)$ of degree r. Now we assume that $\det N \neq 0$. Then $A(N,k)$ has an Euler product expression

$$A(N,k) = \Xi(N,k) \prod_{q:\mathrm{prime}} S_{n,q}(N,s)$$

(cf. [15, p. 306]). The infinite part $\Xi(N,k)$ is calculated by Siegel ([22, (111)]) (see also [20, (4.34K),(4.35K)]). One has

$$\Xi(N,k) = \frac{2^{-n(n-1)/2}(-2\pi\sqrt{-1})^{nk}(\det N)^{k-(n+1)/2}}{\Gamma_n(k)},$$

with $\Gamma_n(s) = \pi^{n(n-1)/4}\prod_{i=0}^{n-1}\Gamma(s-i/2)$. The term $S_{n,q}(N,s)$ is called the *Siegel series*, defined as follows. Let

$$\mathcal{M}_n(q) = \left\{ (C,D) \in M_{n,2n}(\mathbb{Z}_q) \;\middle|\; \begin{array}{l} (C,D) \text{ is symmetric and co-prime,} \\ \det C = q^r \text{ for some } r \geq 0 \end{array} \right\},$$

here we say (C,D) is symmetric if $C\,{}^tD \in \mathrm{Sym}^n(\mathbb{Q}_p)$ and we say (C,D) is co-prime if $CX + DY = \mathbf{1}_n$ for some $X, Y \in M_n(\mathbb{Z}_q)$. Any $R \in \mathrm{Sym}^n(\mathbb{Q}_q)$ is written by $R = C^{-1}D$ with $(C,D) \in \mathcal{M}_n(q)$, unique up to the left action by $SL_n(\mathbb{Z}_q)$. For $R = C^{-1}D \in \mathrm{Sym}^n(\mathbb{Q}_q)$ we set $\delta_q(R) = \det C$. Then for $s \in \mathbb{C}$ we define

$$S_{n,q}(N,s) = \sum_{R \in \mathrm{Sym}^n(\mathbb{Q}_q/Z_q)} \delta_q(R)^{-s}\mathbf{e}(NR),$$

that converges if $\mathrm{Re}(s) \gg 0$. It is known that the Siegel series is a polynomial in q^{-s}, thus we may also write it as $S_{n,q}(N, q^{-s})$. Let

$$\gamma_{n,q}(N,q^{-s}) = \begin{cases} (1-q^{-s})(1-\chi_N(q)q^{n/2-s})^{-1}\prod_{i=1}^{n/2}(1-q^{2i-2s}) & n \text{ is even,} \\ (1-q^{-s})\prod_{i=1}^{(n-1)/2}(1-q^{2i-2s}) & n \text{ is odd.} \end{cases}$$

Here we denote by χ_N the quadratic character associated with the quadratic extension $\mathbb{Q}(\sqrt{(-1)^{n/2}(\det 2N)})/\mathbb{Q}$. Then we have

$$S_{n,q}(N,q^{-s}) = \gamma_{n,q}(N,q^{-s})F_{n,q}(N,q^{-s})$$

with a certain polynomial $F_{n,q}(N,T)$, such that $F_{n,q} = 1$ for almost all q. This fact is proved independently by Kitaoka ([14]), Feit ([3]) and Shimura ([20]). Thus our aim is to find an explicit formula of $F_{n,q}(N,q^{-s})$.

For the case of $n = 2$, it is given by Kaufhold ([11]). After a long time (more than 30 years), an explicit formula for $n = 3$ was found by Katsurada ([12]), and a few years later he also gave an explicit formula of $F_{n,q}(N,q^{-s})$ for any degree n ([13]). Thus our problem was settled for full modular case.

2.2 The case with level and characters

Next we consider the case of Siegel Eisenstein series with level and characters. For $l \in \mathbb{Z}_{>0}$, we put

$$\Gamma_0^n(l) = \{\gamma \in Sp(n,\mathbb{Z}) \mid C_\gamma \equiv 0 \bmod l\}.$$

Let ψ be a Dirichlet character modulo l. Then the space of holomorphic Siegel modular forms $M_k(\Gamma_0^n(l),\psi)$ of weight k with character ψ, is defined as follows. For a holomorphic function $f\colon \mathbb{H}_n \to \mathbb{C}$ and $\gamma \in Sp(n,\mathbb{Z})$ we put

$$f|_k\gamma(Z) = \det(C_\gamma Z + D_\gamma)^{-k} f\big((A_\gamma Z + B_\gamma)(C_\gamma Z + D_\gamma)^{-1}\big).$$

Then we define

$$M_k(\Gamma_0^n(l),\psi) = \{f\colon \mathbb{H}_n \xrightarrow{\text{hol}} \mathbb{C} \mid f|_k\gamma(Z) = \psi(\det D_\gamma)f(Z),\ \gamma \in \Gamma_0^n(l)\}.$$

If $n = 1$, we also require the holomorphic condition at each cusp. If ψ is the trivial character modulo l, we simply denote it by $M_k(\Gamma_0^n(l))$.

Let k be a positive integer such that $\psi(-1) = (-1)^k$. We define the Siegel Eisenstein series of weight k, level l with character ψ by

$$E_{k,l,\psi}^n(Z) = \sum_{\gamma \in P_n(\mathbb{Z})\backslash \Gamma_0^n(l)} \overline{\psi}(\det D_\gamma)\det(C_\gamma Z + D_\gamma)^{-k},$$

that converges when $k > n+1$, then it is contained in $M_k(\Gamma_0^n(l),\psi)$. However when the level $l > 1$, the space of Siegel Eisenstein series $\mathcal{E}_k(\Gamma_0^n(l),\psi)$ in $M_k(\Gamma_0^n(l),\psi)$ has dimension greater than 1. In detail, we can define the Siegel Eisenstein series corresponding to each 0-dimensional cusp of $\Gamma_0^n(l)\backslash\mathbb{H}_n$. Take a representative set $\Gamma_0^n(l)\backslash Sp(n,\mathbb{Z})/P_n(\mathbb{Z}) = \{M_\lambda\}$. If $k > n+1$ is even and ψ is the trivial character for example, then we can construct the Siegel Eisenstein series $E_{k,l}^n(M_\lambda; Z)$ attached to cusp M_λ such that

$$C_0\big(E_{k,l}^n(M_\lambda;Z)|_k M_\nu\big) = \begin{cases} 1 & \nu = \lambda \\ 0 & \text{otherwise,} \end{cases}$$

here $C_0(f)$ denote the constant term of the Fourier expansion of f. In particular we know that $\dim \mathcal{E}_n(\Gamma_0^n(l)) = \sharp\{M_\nu\}$. In the case that ψ is a non-trivial Dirichlet character, Siegel Eisenstein series attached to cusp M_λ may vanish, thus we can only say $\dim \mathcal{E}_n(\Gamma_0^n(l),\psi) \le \sharp\{M_\lambda\}$.

For simplicity, we explain the case of odd prime level. Let p be a fixed odd prime number. Then we can take a set of representative $\Gamma_0^n(p)\backslash Sp(n,\mathbb{Z})/P_n(\mathbb{Z})$ by $\{w_\nu^{-1}\}$ $(0 \le \nu \le n)$ with

$$w_{n,\nu} = w_\nu = \left(\begin{array}{cc|cc} 0_\nu & & -\mathbf{1}_\nu & \\ & \mathbf{1}_{n-\nu} & & 0_{n-\nu} \\ \hline \mathbf{1}_\nu & & 0_\nu & \\ & 0_{n-\nu} & & \mathbf{1}_{n-\nu} \end{array}\right).$$

We assume that the Dirichlet character ψ is χ_0 or χ_p, here χ_0 stands for the trivial character modulo p and χ_p stands for the quadratic character modulo p. Then for $k > n+1$ such that $\psi(-1) = (-1)^k$, the Siegel Eisenstein series attached to w_ν is defined by

$$E^n_{k,\psi}(w_\nu; Z) = \sum_{\gamma \in P_n(\mathbb{Z}) \backslash P_n(\mathbb{Z}) w_\nu \Gamma^n_0(p)} \psi_{w_\nu}(\gamma) \det(C_\gamma Z + D_\gamma)^{-k},$$

here the function ψ_{w_ν} on $P_n(\mathbb{Z})\omega_\nu \Gamma^n_0(p)$ is given by

$$\psi_{w_\nu}(\eta w_\nu \kappa) = \psi(\det D_\eta)\psi(\det D_\kappa), \quad \eta \in P_n(\mathbb{Z}), \kappa \in \Gamma^n_0(p),$$

that is well-defined since $\psi^2 = 1$. We note that if ψ is neither trivial nor quadratic, then $E^n_{k,\psi}(w_\nu; Z)$ is defined only for $\nu = 0$ or n.

The Fourier coefficient $A^{w_\nu}_\psi(N, k)$ of $E^n_{k,\psi}(w_\nu; Z)$ at $0 < N \in \mathrm{Sym}^n(\mathbb{Z})^*$ is given by

$$A^{w_\nu}_\psi(N, k) = \Xi(N, k) S^{w_\nu}_n(\psi, N, k) \prod_{q \neq p} S_{n,q}(N, \psi(q)q^{-k}).$$

We call the Euler p-factor $S^{w_\nu}_n(\psi, N, s)$ as *ramified Siegel series*. It is defined as follows. For $0 \le \nu \le n$ we put

$$\mathcal{M}^\nu_n(p) = \{(C, D) \in \mathcal{M}_n(p) \mid \mathrm{rank}(C \bmod p) = \nu\},$$
$$\mathrm{Sym}^n(\mathbb{Q}_p)^{(\nu)} = \{R = C^{-1}D \in \mathrm{Sym}^n(\mathbb{Q}_p) \mid (C, D) \in \mathcal{M}^\nu_n(p)\}.$$

Take $R = C^{-1}D \in \mathrm{Sym}^n(\mathbb{Q}_p)^{(\nu)}$, then there exists $\gamma = \left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix} \right) \in P_n(\mathbb{Z}_p) w_\nu K_p$, with $K_p = \{\kappa \in Sp(n, \mathbb{Z}_p) \mid C_\kappa \equiv 0 \bmod p\}$. We set $\widetilde{\psi}_{w_\nu}(R) = \psi_{w_\nu}(\gamma)$. The ramified Siegel series is defined by

$$S^{w_\nu}_n(\psi, N, s) = \sum_{\substack{R \in \mathrm{Sym}^n(\mathbb{Q}_p)^{(\nu)} \\ \mathrm{mod}\ \mathrm{Sym}^n(\mathbb{Z}_p)}} \widetilde{\psi}_{w_\nu}(R) \delta_p(R)^{-s} \mathbf{e}(NR).$$

Thus our final goal is to give an explicit formula of $S^{w_\nu}_n(\psi, N, s)$ for all ν.

Remark By definition we easily see that $S^{w_n}_n(\psi, N, s) = 1$ for all N.

Remark One can easily extend the result of odd prime level case to the case of any odd square-free level l, since we have computed every local p-factor with $p \mid l$.

2.3 Known results

The first concrete result about the Fourier coefficients of Siegel Eisenstein series with level, was due to Mizuno ([17]) in the case of $n = 2$; he found the explicit formula of the Fourier coefficients of $E^2_{k,l,\psi}(Z)$ for odd square-free level l with primitive character ψ. For the proof, he showed that the Maass lift of the Eisenstein series of Jacobi form coincides with our $E^2_{k,l,\psi}(Z)$, then he got the results by utilizing the Fourier coefficients of the Eisenstein series of Jacobi form.

After that the author calculate $S_2^{w_0}(\psi, N, s)$ directly ([4]) for an odd prime number p and primitive character ψ. By rewriting $T \in \mathrm{Sym}^2(\mathbb{Q}_p)^{(0)}$ as $T = C^{-1}D$ with $(C, D) \in \mathcal{M}_2^0(p)$, the calculation of ramified Siegel series reduces to the computation of exponential sums. By using the same method, Takemori computed the Fourier coefficients of $E^2_{k,l,\psi}(Z)$ for any level l and primitive character ψ ([23]). We note that his results contains the non square-free level and also even level.

About the Siegel Eisenstein series attached to other cusps, Dickson succeeded to give the Fourier coefficients of $E^2_{k,l}(M_\lambda; Z)$ (the case of trivial character and $n = 2$) for square free level l with *every* cusp M_λ ([2]). He studied the action of the Hecke operator to the space of Siegel Eisenstein series, especially the action to the Fourier coefficients, and using the formula for $E^2_k(Z)$ (full modular Siegel Eisenstein series), to get the results. The author also calculated $S_2^{w_1}(\chi_p, N, s)$ and $S_2^{w_1}(\chi_0, N, s)$ by a direct computation ([6]).

Next we explain the results for higher degree case due to Takemori ([24]). Let l be square-free and ψ a primitive Dirichlet character modulo l. For the decomposition $\psi = \prod_{l|p} \psi_p$, assume that $\psi_p^2 \not\equiv 1$ for any $p \mid l$. Under this condition, Takemori gave an explicit formula of the Fourier coefficients of $E^n_{k,l,\psi}(Z)$ for any degree n. For the proof, first he constructed the Siegel Eisenstein series such that the ramified Siegel series becomes quite simple by using the functional equation of Siegel series, next he showed that this Eisenstein series coincides our $E^n_{k,l,\psi}(Z)$, if ψ satisfies the above condition.

As the remaining case that ψ is the quadratic character χ_p, the author calculated $S_3^{w_0}(\chi_p, N, s)$ directly by the same methods as degree 2 case ([7]). For the exponential sum, we use the explicit formula due to Saito ([18]) of the generalized Gauss sum

$$\sum_{T \in \mathrm{Sym}^n(\mathbb{Z}/p)} \psi(\det T)\mathbf{e}(TN).$$

By using it, in principle, we can compute ramified Siegel series directly for any degree n, however it seems actually to be impossible to compute if $n \geq 4$.

Another point of view, the author ([5]) and Watanabe ([25]) independently gave an explicit formula of ramified Siegel series for general degree. Both results deeply depend on the paper by Sato and Hironaka ([19]), that treated local densities. We call it here the Sato-Hironaka type formula. Although ramified Siegel series is expressed by finite sum, this formula is too complicated to handle at least by hands. In fact, to get the same result for degree 3 case as [7], using the Sato-Hironaka type formula seems as tedious as the direct computation method.

3 Katsurada's recursion formula

As we already mentioned, Katsurada and Ikeda-Katsurada ([10]) gave a recursion formula for ordinary Siegel series to get an explicit formula. We recall there argument. There are two key ingredients; one is inductive relation and the other is functional equation of Siegel series. The inductive relation is as follows. For simplicity we assume that q is an odd prime. Then we may assume that N is diagonal for the computation of $S_{n,q}(N,s)$. Let $N = \mathrm{diag}(\alpha_1 q^{u_1}, \dots, \alpha_n q^{u_n})$ with $\alpha_i \in \mathbb{Z}_q^\times$, $u_1 \le \cdots \le u_n$. We put $N' = \mathrm{diag}(\alpha_0 q^{u_1}, \dots, \alpha_{n-1} q^{u_{n-1}}, \alpha_n q^{u_n+2})$, and $N_0 = \mathrm{diag}(\alpha_1 p^{u_1}, \dots \alpha_{n-1} p^{u_{n-1}}) \in \mathrm{Sym}^{n-1}(\mathbb{Z}_q)$. Then the following relation holds.

Proposition 3.1 ([13, Theorem 2.6], [10, Theorem 5.2]) *We have the following formula.*

$$S_{n,q}(N',s) - q^{n+1-2s} S_{n,q}(N,s) = (1-q^{-s})(1+q^{1-s}) S_{n-1,q}(N_0, s-1). \quad (3.1)$$

In order to explain the functional equations of Siegel series, we prepare some notations. For $N \in \mathrm{Sym}^n(\mathbb{Z})^*$ with $\det N \neq 0$, we set $D_N = (-4)^{[n/2]} \det N$ and $d_q(N) = \mathrm{ord}_q D_N$. For the Hasse invariant $h_q(N)$ of N, we define

$$\zeta_q(N) = \begin{cases} 1 & n \text{ is even} \\ h_q(N)\,(\det N, (-1)^{\frac{n-1}{2}} \det N)_q\,(-1,-1)_q^{\frac{n^2-1}{8}} & n \text{ is odd,} \end{cases}$$

here $(\ ,\ )_q$ stands for the Hilbert symbol. We set

$$e_q(N) = \begin{cases} d_q(N) - 1 & n \text{ is even and } d_q(N) \text{ is odd,} \\ d_q(N) & \text{otherwise.} \end{cases}$$

Proposition 3.2 ([13, Theorem 3.2]) *Under the above notations, the functional equation of Siegel series is formulated as follows.*

$$F_{n,q}(N, n+1-s) = \zeta_q(N) p^{(s-\frac{n+1}{2})e_q(N)} F_{n,q}(N,s).$$

Now change the variable $s \mapsto n+1-s$ in (3.1) and apply the functional equation. Since $e_q(N') = e_q(N) + 2$ we have the formula of the form

$$S_{n,q}(N',s) - S_{n,q}(N,s) = H_n(N,s) S_{n-1,q}(N_0,s). \quad (3.2)$$

Together with (3.1), we can get the desired recursion formula of the form

$$S_{n,q}(N,s) = C(N,s) S_{n-1,q}(N,s-1) + D(N,s) S_{n-1,q}(N,s).$$

Please see [13, Theorem 4.1] or [10, Theorem 1.1] for details.

4 Recursion formulas of the ramified Siegel series

The main result of this article is to give a recursion formula of the ramified Siegel series. We use the same arguments as the full modular case, thus we use inductive relations and functional equations. For simplicity we only explain the case of the quadratic character. Please see [9] in detail including the case of the trivial character.

4.1 $U(p)$-operators in the space of Siegel Eisenstein series

In order to find the functional equation of ramified Siegel series, we recall the theory of the functional equations of Siegel Eisenstein series of level p, that is investigated by the author. Since the space of Siegel Eisenstein series is $(n+1)$-dimensional, we have to take suitable basis so that the functional equations becomes simple. For that we consider the $U(p)$-operator and take the eigenfunctions as the basis.

We set the Siegel Eisenstein series with parameter $s \in \mathbb{C}$. From now on we only treat the case $\psi = \chi_p$. Take $k \in \mathbb{Z}_{\geq 0}$ such that $\chi_p(-1) = (-1)^k$. We define

$$\begin{aligned} &E^n_{k,\chi_p}(w_\nu; Z, s) \\ &= \det(Y)^{(s-k)/2} \sum_{\gamma} \chi_{p,w_\nu}(\gamma) \det(C_\gamma Z + D_\gamma)^{-k} |\det(C_\gamma Z + D_\gamma)|^{-2s+k}, \end{aligned}$$

here γ runs $P_n(\mathbb{Z})\backslash P_n(\mathbb{Z}) w_\nu \Gamma_0^n(p)$. Then $E^n_{k,\chi_p}(w_\nu; Z) = E^n_{k,\chi_p}(w_\nu; Z, k)$. Let

$$\mathcal{E}_{k,s}(\Gamma_0^n(p), \chi_p) = \langle E^n_{k,\chi_p}(w_\nu; Z, s) \mid 0 \leq \nu \leq n \rangle_{\mathbb{C}}$$

be the space of Siegel Eisenstein series. The $U(p)$-operator acts on it as follows. If the Fourier expansion of $E(Z,s) \in \mathcal{E}_{k,s}(\Gamma_0^n(p), \chi_p)$ is given by

$$E(Z,s) = \sum_{T \in \mathrm{Sym}^n(\mathbb{Z})^*} b(T, Y, s) \mathbf{e}(TX),$$

then

$$U(p)E(Z,s) = \sum_{T \in \mathrm{Sym}^n(\mathbb{Z})^*} b(pT, p^{-1}Y, s) \mathbf{e}(TX).$$

The $U(p)$-action is studied in [8] explicitly. Let

$$m_{\chi_p}(s)_{ij} = \begin{cases} \chi_p(-1)^{(j-i)/2} p^{si-j(j+1)/2+(j-i)^2/4} & \\ \qquad \times \dfrac{\prod_{r=i+1}^{j}(p^r-1)}{\prod_{r=1}^{(j-i)/2}(p^{2r}-1)} & \text{if } i \leq j \text{ and } j-i \text{ is even}, \\ 0 & \text{otherwise}. \end{cases}$$

Then

$$U(p)E^n_{k,\chi_p}(w_i;Z,s) = p^{n(k-s)/2} \sum_{\substack{j=i \\ j\equiv i \bmod 2}}^{n} m_{\chi_p}(s)_{ij}\, E^n_{k,\chi_p}(w_j;Z,s).$$

In particular the eigenvalues of $U(p)$ are easily seen. For $0 \le \nu \le n$, we put $l(s,\nu) = n(s-k)/2 + s\nu - \nu(\nu+1)/2$

Lemma 4.1 *The eigenvalues of $U(p)$ on $\mathcal{E}_{k,s}(\Gamma^n_0(p),\psi)$ are given by $p^{l(\nu,s)}$ with $0 \le \nu \le n$.*

We note that this lemma also holds for $\psi = \chi_0$.

Since the representative matrix of the $U(p)$-operator is triangular, we can construct the eigenvectors easily as follows from a theory of linear algebra. Let

$$\begin{array}{ll} \text{if } i > j & b_{\chi_p}(s)_{ij} = 0 \\ \text{if } i = j & b_{\chi_p}(s)_{ii} = 1 \\ \text{if } i < j & b_{\chi_p}(s)_{ij} = \dfrac{p^{-js+j(j+1)/2}}{p^{(j-i)((j+i+1)/2-s)}-1} \displaystyle\sum_{k=i}^{j-1} m_{\chi_p}(s)_{kj} b_{\chi_p}(s)_{ik}. \end{array}$$

Then

$$E^{n,(\nu)}_{k,\chi_p}(Z,s) = E^n_{k,\chi_p}(w_\nu;Z,s) + \sum_{\substack{r=\nu+2 \\ r\equiv\nu \bmod 2}}^{n} b_{\chi_p}(s)_{\nu r}\, E^n_{k,\chi_p}(w_r;Z,s)$$

is the eigenfunction of $U(p)$ with eigenvalue $p^{l(s,\nu)}$. We also need the inverse relation. Let $c_{\chi_p}(s)_{ij}$ $(0 \le i,j \le n)$ be the (i,j)-component of $\big(b_{\chi_p}(s)_{ij}\big)^{-1}$, then

$$E^n_{k,\chi_p}(w_\nu;Z,s) = E^{n,(\nu)}_{k,\chi_p}(Z,s) + \sum_{\substack{r=\nu+2 \\ r\equiv\nu \bmod 2}}^{n} c_{\chi_p}(s)_{\nu r} E^{n,(r)}_{k,\chi_p}(Z,s).$$

The first result of this article is to give explicit forms of $b_{\chi_p}(s)$ and $c_{\chi_p}(s)$.

Proposition 4.2 *Let $i \le j$ and $i \equiv j$* mod 2. *We have the following explicit formulas.*

$$b_{\chi_p}(s)_{ij} = \frac{\big(-\chi_p(-1)\big)^{(j-i)/2} p^{(j-i)(\frac{1}{2}-s)} \prod_{k=i+1}^{j}(p^k-1)}{\prod_{k=1}^{(j-i)/2}(1-p^{2i+1+2k-2s})(p^{2k}-1)},$$

$$c_{\chi_p}(s)_{ij} = \frac{\chi_p(-1)^{(j-i)/2} p^{(j-i)^2/4-(j-i)s} \prod_{k=i+1}^{j}(p^k-1)}{\prod_{k=1}^{(j-i)/2}\big(1-p^{i+j-1+2k-2s}\big)(p^{2k}-1)}.$$

For the proof, first we compute $b_{\chi_p}(s)_{ij}$ and $c_{\chi_p}(s)_{ij}$ for small i, j by using calculation software (Maple), to guess the general term. Then we check that it satisfies our induction formula in fact. Then our calculation reduces to the following formula.

Lemma 4.3 *Let X and Y be indeterminates.*

(1) $$\sum_{t=1}^{r} \prod_{k=1}^{t} \frac{(Y - X^{k-1})(X^{r+1-k} - 1)}{X^k - 1} = Y^r - 1.$$

(2) $$\sum_{t=1}^{r} \prod_{k=1}^{t} \frac{(X^{k-1}Y - 1)(X^{r+1-k} - 1)}{(X^{r-k} - Y)(X^k - 1)} = -1.$$

One can prove (2) easily, however (1) is not so straightforward. It is pointed out by Watanabe, (1) follows from q-analogue of Chu-Vandermonde formula, that is famous in the area of q-series.

Now we define

$$S_n^{(\nu)}(\chi_p, N, s) = \sum_{\substack{r=\nu \\ r \equiv \nu \bmod 2}}^{n} b_{\chi_p}(s)_{\nu r} \, S_n^{w_r}(\chi_p, N, s), \tag{4.1}$$

that is the Euler p-factor of the Fourier coefficients of $E_{k,\chi_p}^{n,(\nu)}(Z, s)$. Then

$$S_n^{w_\nu}(\chi_p, N, s) = \sum_{\substack{r=\nu \\ r \equiv \nu \bmod 2}}^{n} c_{\chi_p}(s)_{\nu r} \, S_n^{(r)}(\chi_p, N, s) \tag{4.2}$$

holds. Hence our goal is to get the formula of $S_n^{(\nu)}(\chi_p, N, s)$ for each ν.

4.2 Functional equations of ramified Siegel series

We shall give functional equations of ramified Siegel series in this subsection. For that first we write down the functional equations of Siegel Eisenstein series explicitly. Let $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ be the completed Riemann zeta function, that satisfies $\xi(1-s) = \xi(s)$. We set $\delta_p = 0$ or 1 according as $p \equiv 1$ or $p \equiv 3 \bmod 4$ respectively. Then the completed Dirichlet L-function for χ_p is defined by $\xi(\chi_p, s) = (p/\pi)^{(s+\delta_p)/2}\Gamma((s+\delta_p)/2)L(\chi_p, s)$, that also satisfies $\xi(\chi_p, 1-s) = \xi(\chi_p, s)$.

For $0 \le \nu \le n$, we set

$$\alpha_\nu(\chi_p, s) = (\varepsilon_p p^{-1/2})^\nu \prod_{i=1}^{[\nu/2]} \frac{1 - p^{2i-2s}}{1 - p^{2\nu+1-2i-2s}},$$

here $\varepsilon_p = 1$ or $\sqrt{-1}$ according as $p \equiv 1$ or $p \equiv 3 \bmod 4$. Now we define

$$\mathbb{E}^{n,(\nu)}_{k,\chi_p}(Z,s) = \alpha_\nu(\chi_p,s)\frac{\Gamma_n\left(\dfrac{s+k}{2}\right)}{\Gamma_n\left(\dfrac{s+\delta_p}{2}\right)}\xi(\chi_p,s)\prod_{j=1}^{[n/2]}\xi(2s-2j)E^{n,(\nu)}_{k,\chi_p}(Z,s).$$

Theorem 4.4 ([8, Theorem 4.8]) *The functional equation*

$$\mathbb{E}^{n,(\nu)}_{k,\chi_p}(Z,s) = \mathbb{E}^{n,(n-\nu)}_{k,\chi_p}(Z,n+1-s)$$

holds.

Our functional equation of ramified Siegel series follows directly from that. We use the same calculation as [1], but our argument is simpler than that. One of the most important idea of [1] is how to deduce a local functional equation from the global one, i.e. when we only know the functional equation of the infinite product of factors, how to pick up the information of single factor from that. However in our case, we have already known the functional equation of ordinary Siegel series (Proposition 3.2), thus the information of ramified Siegel series follows automatically.

We need some notations adding to the notation in Section 3. For $N \in \mathrm{Sym}^n(\mathbb{Z})^*$ with $\det N \neq 0$, we define D'_N so that $D_N = p^{d_p(N)}D'_N$. We put

$$\eta_p(N) = \begin{cases} 1 & n \text{ is even,} \\ \chi_p(D'_N)\zeta_p(N) & n \text{ is odd.} \end{cases}$$

We define

$$a_N = \begin{cases} 0 & n \text{ is even and } d_p(N) \text{ is even} \\ 1 & \text{otherwise.} \end{cases}$$

The functional equation of ordinary Siegel series is formulated to its polynomial part $F_n(N,s)$. Similarly, in order to write down the functional equation, we have to remove suitable Euler factors of zeta functions from ramified Siegel series. Roughly speaking, we should define the zeta part $\beta_n^{(\nu)}(\chi_p,N,s)$ so that $\alpha_\nu(\chi_p,s)\beta_n^{(\nu)}(\psi,N,s) = \gamma_{n,p}(N,p^{-s})$ holds, here $\gamma_{n,p}(N,p^{-s})$ is defined in Subsection 2.1. We need a little modification from that, since the Dirichlet character $\chi_N \cdot \chi_p$ appears but it would be non-primitive. We denote χ_N^* the primitive Dirichlet character arising from $\chi_N \cdot \chi_p$. Then we define

$$\beta_n^{(\nu)}(\chi_p,N,s) = \frac{\prod_{j=1}^{[\nu/2]}(1-p^{2\nu+1-2j-2s})\prod_{j=[\nu/2]+1}^{[n/2]}(1-p^{2j-2s})}{(\varepsilon_p p^{-1/2})^\nu(1-\chi_N^*(p)p^{n/2-s})},$$

here we understand $\chi_N^*(p) = 0$ when n is odd.

We write $S_n^{(\nu)}(\chi_p,N,s) = \beta_n^{(\nu)}(\chi_p,s)F_n^{(\nu)}(\chi_p,N,s)$ and call $F_n^{(\nu)}(\chi_p,N,s)$ the *principal part* of ramified Siegel series.

Theorem 4.5 *The principal part of ramified Siegel series satisfies the following functional equation.*

$$F_n^{(n-\nu)}(\chi_p, N, n+1-s) = \eta_p(N) p^{(s-\frac{n+1}{2})(d_p(N)+a_N)} F_n^{(\nu)}(\chi_p, N, s).$$

We know $S_n^{w_n}(\chi_p, N, s) = 1$, as we have remarked at the end of Subsection 2.2. Using the functional equation one can get an explicit formula of $S_n^{w_0}(\chi_p, N, s)$ as follows.

Lemma 4.6 *We have*

$$S_n^{(0)}(\chi_p, N, s) = \eta_p(N)\, \varepsilon_p^n\, p^{(\frac{n+1}{2}-s)(d_p(N)+a_N)-\frac{n}{2}} \times \frac{(1-\chi_N^*(p)p^{s-\frac{n}{2}-1}) \prod_{j=1}^{[n/2]}(1-p^{2j-2s})}{(1-\chi_N^*(p)p^{\frac{n}{2}-s}) \prod_{j=1}^{[n/2]}(1-p^{2s-1-2j})},$$

here we understand $\chi_N^*(p) = 0$ if n is odd.

4.3 Inductive relations

In this subsection, we study inductive relation, that is another key ingredient. Let $0 \le u_1 \le \cdots \le u_n$ be the sequence of integers. Take $\alpha_i \in \mathbb{Z}_p^\times$ $(1 \le i \le n)$ and put $N_0 = \mathrm{diag}(\alpha_1 p^{u_1}, \ldots, \alpha_{n-1} p^{u_{n-1}}) \in \mathrm{Sym}^{n-1}(\mathbb{Z})^*$. We set $N = N_0 \perp \langle \alpha_n p^{u_n} \rangle$, $N' = N_0 \perp \langle \alpha_n p^{u_n+2} \rangle$. The following inductive relation is due to Watanabe.

Theorem 4.7 ([26, Theorem 6.1]) *For any $0 \le \nu \le n$ we have*

$$S_n^{w_\nu}(\chi_p, N', s) - S_n^{w_\nu}(\chi_p, N, s) = H_n(\chi_p, N, s) S_{n-1}^{w_\nu}(\chi_p, N_0, s),$$

here $H_n(\chi_p, N, s)$ does not depend on ν. We understand $S_{n-1}^{w_n}(\chi_p, N, s) = 0$.

Remark In [26] Watanabe gave an explicit form of $H_n(\chi_p, N, s)$ by dividing 4 cases, n is even or odd and $d_p(N)$ is even or odd. We shall give another description of $H_n(\chi_p, N, s)$ in the proposition below.

For the proof of this theorem, Watanabe used his Sato-Hironaka type formula of the explicit expression of ramified Siegel series, introduced in Subsection 2.3. Though Sato-Hironaka type formula is too complicated, it is quite useful for our argument.

The important point of the above theorem is that $H_n(\chi_p, N, s)$ is independent of ν. Therefore linear combinations of $S_n^{w_\nu}(\chi_p, N, s)$ also satisfies the same relation and we have the following formula.

$$S_n^{(\nu)}(\chi_p, N', s) - S_n^{(\nu)}(\chi_p, N, s) = H_n(\chi_p, N, s) S_{n-1}^{(\nu)}(\chi_p, N, s). \quad (4.3)$$

Since we know the explicit formula of $S_n^{(0)}(\chi_p, N, s)$ in Lemma 4.6, by substituting it one can get the following description of $H_n(\chi_p, N, s)$.

Lemma 4.8 (1) *If n is even then*

$$H_n(\chi_p, N, s) = \varepsilon_p \eta_p(N_0) p^{\frac{1}{2}d_p(N_0)+(\frac{n+1}{2}-s)(u_n+a_N+1)} \times \frac{(1-p^{n-2s})(1-\chi_N^*(p)p^{s-\frac{n}{2}-1})}{1-\chi_N^*(p)p^{\frac{n}{2}-s}}.$$

(2) *If n is odd then*

$$H_n(\chi_p, N, s) = -\varepsilon_p \eta_p(N) p^{\frac{1}{2}d_p(N)+(\frac{n}{2}-s)(u_n+a_{N_0}+1)} \times \frac{(1-p^{n+1-2s})(1-\chi_{N_0}^*(p)p^{s-\frac{n-1}{2}})}{1-\chi_{N_0}^*(p)p^{\frac{n+1}{2}-s}}.$$

4.4 Main result

Using inductive relations and functional equations, we can get the recursion formula as follows. Change $s \mapsto n+1-s$ and $\nu \mapsto n-\nu$ in (4.3) then we have

$$S_n^{(n-\nu)}(\chi_p, N', n+1-s) - S_n^{(n-\nu)}(\chi_p, N, n+1-s) = H_n(\chi_p, N, n+1-s) S_{n-1}^{(n-\nu)}(\chi_p, N, n+1-s).$$

Apply the functional equation in Theorem 4.5, we have

$$\eta_p(N) p^{(s-\frac{n+1}{2})(d_p(N)+a_N+2)} \Big(S_n^{(\nu)}(\chi_p, N', s) - p^{n+1-2s} S_n^{(\nu)}(\chi_p, N, s) \Big) = \frac{\beta_n^{(\nu)}(N,s)\,\beta_{n-1}^{(n-\nu)}(N_0, n+1-s)}{\beta_n^{(n-\nu)}(N, n+1-s)\,\beta_{n-1}^{(\nu-1)}(N_0, s-1)} H_n(\chi_p, N, n+1-s) \times \eta_p(N_0) p^{(s-1-\frac{n}{2})(d_p(N_0)+a_N)} S_{n-1}^{(\nu-1)}(\chi_p, N_0, s-1).$$

By a direct computation using Lemma 4.8, we can simplify the above as

$$S_n^{(\nu)}(\chi_p, N', s) - p^{n+1-2s} S_n^{(\nu)}(\chi_p, N, s) = (1-p^{\nu+1-2s}) S_{n-1}^{(\nu-1)}(\chi_p, N_0, s-1). \tag{4.4}$$

Gathering (4.3) and (4.4) we have the following main theorem.

Theorem 4.9 (Recursion formula of ramified Siegel series) *Let $\alpha_i \in \mathbb{Z}_p^\times$ and $0 \le u_1 \le \cdots \le u_n \in \mathbb{Z}$. For $N = \mathrm{diag}(\alpha_1 p^{u_1}, \dots, \alpha_n p^{u_n})$, we set $N_0 = \mathrm{diag}(\alpha_1 p^{u_1}, \dots, \alpha_{n-1} p^{u_{n-1}})$. Then we have*

$$S_n^{(\nu)}(\chi_p, N, s) = \frac{1-p^{\nu+1-2s}}{1-p^{n+1-2s}} S_{n-1}^{(\nu-1)}(\chi_p, N_0, s-1) - \frac{H_n(\chi_p, N, s)}{1-p^{n+1-2s}} S_{n-1}^{(\nu)}(\chi_p, N_0, s).$$

Remark For the ordinary Siegel series, as we explained in Section 3, Katsurada first found the inductive relation (3.1), that is the relation between $S_n(N', s) - p^{n+1-2s}S_n(N, s)$ and $S_{n-1}(N_0, s-1)$. Then applying the functional equation, he got the relation (3.2) between $S_n(N', s) - S_n(N, s)$ and $S_n(N_0, s)$. However for the ramified Siegel series, Watanabe's first relation is an analogue of (3.2). We apply the functional equation to the relation of $U(p)$-characteristic series, we get the relation analogue to (3.1). This type of relation is only for $U(p)$-characteristic series, i.e. it seems that there are no simple inductive relation between $S_n^{w_\nu}(\chi_p, N', s) - p^{n+1-2s}S_n^{w_\nu}(\chi_p, N, s)$ and $S^{w_{\nu'}}(\chi_p, N_0, s-1)$.

By usign Theorem 4.9, (4.2) and Proposition 4.2, we can write down the Fourier coefficients of $E^n_{k,\chi_p}(w_\nu; Z)$ explicitly. In particular almost all previous results in Subsection 2.3 will be covered with our argument (see the Remark at the end of Subsction 2.2).

For example, let us consider the degree 2 case. Put $N = \mathrm{diag}(p^m\alpha, p^{m+t}\beta)$ with $\alpha, \beta \in \mathbb{Z}_p^\times$. By (4.2) we have

$$S_2^{w_0}(\chi_p, N, s) = S_2^{(0)}(\chi_p, N, s) + \frac{\chi_p(-1)p^{1-2s}(p-1)}{1-p^{3-2s}},$$

and $S_2^{(0)}(\chi_p, N, s)$ can be computed by Lemma 4.6. We have

$$S_2^{(0)}(\chi_p, N, s) = -\frac{\chi_p(-1)p^{2-2s}p^{(3/2-s)(2m+t+a_N)}(1-p^{2-2s})(1-\chi_N^*(p)p^{s-2})}{(1-p^{3-2s})(1-\chi_N^*(p)p^{1-s})},$$

that coincides with the result of [4] or [23].

Also for the $S_2^{w_1}(\chi_p, N, s) = S_2^{(1)}(\chi_p, N, s)$, Theorem 4.9 says

$$S_2^{(1)}(\chi_p, N, s) = \frac{1-p^{2-2s}}{1-p^{3-2s}}S_1^{(0)}(\chi_p, N_0, s-1) - \frac{H_2(\chi_p, N, s)}{1-p^{3-2s}}.$$

together with $S_1^{(0)}(\chi_p, N_0, s-1) = \chi_p(\alpha)\varepsilon_p p^{(2-s)m+3/2-s}$ and

$$H_2(\chi_p, N, s) = \varepsilon_p\chi_p(\alpha)p^{(2-s)m+(3/2-s)(t+a_N+1)}\frac{(1-p^{2-2s})(1-\chi_N^*(p)p^{s-2})}{1-\chi_N^*(p)p^{1-s}},$$

thus we get the same result as [6].

References

[1] S. Böcherer and W. Kohnen "On the functional equation of singular series" Abh. Math. Sem. Univ. Hamburg **70** (2000), 281-286.

[2] M. Dickson, "Fourier coefficients of degree two Siegel-Eisenstein series with trivial character at squarefree level", Ramanujan J. **37** (2015), no. 3, 541–562.

[3] P. Feit *Poles and residules of Eisenstein series for sumplectic and unitary groups*, Memoirs. Am. Math. Soc. **61**, No. 346 (1986)

[4] K. Gunji, On the Siegel Eisenstein series of degree two for low weights, J. Math. Soc. Japan **67** (2015), no. 3, 1043–1067.

[5] K. Gunji "On the Fourier coefficients of the Siegel Eisenstein series of odd level and the genus theta series", J. Number Theory **240** (2022), 105–138.

[6] K. Gunji, "Fourier coefficients of the Siegel Eisenstein series of degree 2 with odd prime level, corresponding to the middle cusp", Abh. Math. Semin. Univ. Hambg. **92** (2022), 69–83.

[7] K. Gunji, "On the Fourier coefficients of Siegel Eisenstein series of degree 3 of an odd prime level with the quadratic character", J. math. soc. Japan **76** (2024), No. 1, 173–216.

[8] K. Gunji, "On the functional equations of Siegel Eisenstein series of an odd prime level p", Int. J. Number Theory **21** (2025), no. 7, 1513–1546.

[9] K. Gunji, "Recursion formulas for the Fourier coefficients of Siegel Eisenstein series of an odd prime level", preprint, arXiv:2504.17245.

[10] T. Ikeda and H. Katsurada, "An explicit formula for the Siegel series of a quadratic form over a non-archimedean local field", J. Reine Angew. Math. **783**, (2022), 1–47.

[11] G. Kaufhold "Dirichletsche Reihe mit Funktionalgleichung in der Theorie der Modulfunktion 2. Grades", Math. Ann. **137** (1959), 454–476.

[12] H. Katsurada, "An explicit formula for the Fourier coefficients of Siegel-Eisenstein series of degree 3", Nagoya Math. J. **146** (1997), 199–223.

[13] H. Katsurada, "An explicit formula for Siegel series", Amer. J. Math. **121** (1999), 415–452.

[14] Y. Kitaoka, "Dirichlet seres in the theory of Siegel modular forms", Nagoya Math. J. **95** (1984), 73–84.

[15] H. Maass *Siegel's modular forms and Dirichlet series*, Lecture Notes in Mathematics, Vol. 216. Springer-Verlag, Berlin-New York, 1971

[16] S. Mizumoto "Eisenstein series for Siegel modular groups", Math. Ann. **297** (1993) 581–625.

[17] Y. Mizuno, "An explicit arithmetic formula for the Fourier coefficients of Siegel-Eisenstein series of degree two and square-free odd levels", Math. Z. **263** (2009), no. 4, 837–860.

[18] H. Saito, "A generalization of Gauss sums and its application to Siegel modular forms and L-functions associated with the vector space of quadratic forms" J. reine. ang. Math. **416** (1991), 91–142.

[19] F. Sato and Y. Hironaka, Local densities of representations of quadratic forms over p-adic integers (the non dyadic case). J. Number Theory **83** (2000), 106–136.

[20] G. Shimura, "Confluent hypergeometric functions on tube domains", Math. Ann. **260** (1982), No. 3, 269–302.

[21] G. Shimura, "Euler products and Fourier coefficients of automorphic forms on symplectic groups", Invent. Math. **116** (1994), 531–576.

[22] C.L. Siegel, Über die analytische Theorie der quadratischen Formen I, Ann. Math. **36** (1935), 527-606.

[23] S. Takemori, "p-adic Siegel Eisenstein series of degree two", J. Number Theory **132** (2012), 1203–1264.

[24] S. Takemori, "Siegel Eisenstein series of degree n and Λ-adic Eisenstein series", J. Number Theory **149** (2015), 105–138.

[25] M, Watanabe, "On the calculation of the ramified Siegel series", preprint, arXiv:2302.04429

[26] M. Watanabe, "On the ramified Siegel series", preprint, Kyoto University Research Information Repository.

Department of Mathematics, Chiba Institute of Technology,
2-1-1 Shibazono, Narashino, Chiba, 275-0023, Japan
E-mail: `keiichi.gunji@it-chiba.ac.jp`