ADMISSIBLE $\mathfrak{sl}(2)$ AND $\mathcal{N} = 2$, AND THEIR FUSION PRODUCT

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1. Introduction

This talk is based on joint work with Hiromu Nakano, Florencia Orosz Hunziker and Ana Ros Camacho. All results discussed here can be found in [1].

Within the body of literature on conformal field theory and vertex operator algebras the class of rational theories (satisfying a number of technical niceness conditions including C_2-cofiniteness and the category of admissible modules being semisimple) stand out as being exceptionally intensively studied and well understood. One particularly compelling aspect of these rational theories is the abundance of rich mathematical structures they exhibit. For example, the fact that their categories of modules are modular tensor categories and that the categorical action of the modular group via Hopf links and twists matches the modular transformations of characters [2]. Apart from their intrinsic mathematical beauty these modularity results also have practical implications: they allow the efficient computation of tensor products (also called fusion) of vertex operator algebra modules in terms of the modular transformation formulae of characters. In practice, this provides an enormous reduction in the computational effort required to understand fusion products.

There is ample evidence to suggest that the modularity properties enjoyed by rational theories generalise to large classes of non-rational ones. The purpose of this paper is to prove that these conjectured modularity properties do indeed hold for so called admissible $\mathfrak{sl}(2)$ and $\mathcal{N} = 2$ superconformal theories. In order to better understand these conjectures, we begin with a historical overview. This paper makes use of a coset realisation of the $\mathcal{N} = 2$ superconformal algebra in terms of $\mathfrak{sl}(2)$ and a fermionic ghost (or *bc*) system [3, 4, 5, 6]. Because of this the representation theories of $\mathcal{N} = 2$ and $\mathfrak{sl}(2)$ are closely intertwined and any statement about the representation theory for one algebra has a corresponding version for the other.

A first hint of modularity beyond rationality was discovered by Kac and Wakimoto [7, 8, 9] when they computed modular transformation formulae for characters of simple highest weight modules over affine Lie algebras at admissible levels and weights. Since the non-negative integral levels (which are the only ones giving rise to rational theories) are a proper subset of all admissible levels, it is somewhat surprising for such formulae to exist at all. Shortly afterwards Koh and Sorba [10] plugged these modular transformation properties for $\mathfrak{sl}(2)$ into the Verlinde formula, which, rather startlingly, predicted negative multiplicities for some summands appearing in certain fusion products. This in turn led to some concern within the academic community that these non-integral admissible (also called fractional admissible) theories may suffer from some intrinsic "sickness" [11].

New light was later shed on this riddle by Ridout after a careful analysis of $\mathfrak{sl}(2)$ at level $k = -\frac{1}{2}$ [12] led him to note that the characters of highest weight modules at this level were only convergent in certain domains and that the modular transformation formulae of Kac and Wakimoto hold only for the analytic continuations of characters rather then their series expansions. This is because these modular transformations do not preserve domains of convergence. He also noted that in the category of weight modules the analytic continuation of characters of certain highest weight

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modules was the negative of the analytic continuation of characters of certain other weight modules. This gave a first hint as to why signs were appearing in the Verlinde formula of Koh and Sorba.

Creutzig and Ridout then studied the category of weight modules over affine $\mathfrak{gl}(1|1)$ and the modular properties of characters [13]. While category $\mathcal{O}$ is semisimple and finite, the category of weight modules is neither (this is also true for affine $\mathfrak{sl}(2)$ at non-integral admissible levels and weights). However, the span of characters (taken as specific series expansions rather than their analytic continuations) of a distinguished class of weight modules (called *standard modules*) carries an action of the modular group. Further, evaluating the Verlinde formula (in a generalised version conjectured to hold for infinite categories of modules) using this action predicts non-negative fusion multiplicities. This new generalised Verlinde formula sheds light on the riddle of negative fusion multiplicities of Koh and Sorba: there are linear relations between the analytic continuations of characters of simple weight modules. So a negative multiplicity in the Verlinde formula for category $\mathcal{O}$ can be interpreted as a positive multiplicity of a different weight module outside of category $\mathcal{O}$. This work was then generalised to $\mathfrak{sl}(2)$ at all admissible levels [14, 15] and together with other authors to many other families of algebras [16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26].

Two key conjectures or hopes that crystallised from Creutzig and Ridout's work in [13, 14, 15] were that the category of weight modules is rigid (this implies that the fusion product is exact, a necessary condition for anything akin to a Verlinde formula to be well defined) and that the fusion product decomposition formulae predicted by the generalised Verlinde formulae of Creutzig and Ridout are true (at levels $k = -\frac{1}{2}, -\frac{4}{3}$ the fusion product formulae were explicitly checked using the Nahm-Gaberdiel-Kausch algorithm).

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2. The main result

Let $u, v \in \mathbb{Z}_{>0}$, $\gcd(u, v) = 1$, $u \geq 2$. Denote the simple quotient of the universal affine $\mathfrak{sl}(2)$ vertex operator algebra at level $k = \frac{u}{v} - 2$ by $\mathsf{A}_1(u, v)$ and the simple quotient of the universal $\mathcal{N} = 2$ superconformal vertex operator superalgebra a central charge $c = 1 - \frac{6v}{u}$ by $\mathsf{N}(u, v)$. In addition to the conformal grading by Virasoro L_0-generator eigenvalues, both of these algebras admit an additional grading called $\mathfrak{sl}(2)$-weight and J-weight, respectively. These gradings are given by the eigenvalues of field modes respectively denoted h_0 and J_0.

Definition 2.1. A module over $\mathsf{A}_1(u, v)$ or $\mathsf{N}(u, v)$ called *weight* if all of the following hold.

(1) The modules is finitely generated.
(2) The module is smooth.
(3) All weight spaces (simultaneous homogeneous spaces for generalised L_0-eigenvalues and the additional grading) are finite dimensional and the module is a direct sum of its weight spaces.

Theorem 2.2. *Let* $u, v \in \mathbb{Z}_{\geq 2}$, $\gcd(u, v) = 1$. *Then the categories of weight modules over* $\mathsf{A}_1(u, v)$ *and* $\mathsf{N}(u, v)$ *are rigid. Additionally the projective fusion product decomposition formulae conjectured by Creutzig and Ridout in* [14, 15] *hold.*

3. Sketch of proof

The algebras $\mathsf{A}_1(u, v)$ and $\mathsf{N}(u, v)$ are interrelated by a coset construction

$$\mathsf{N}(u, v) \cong \mathrm{Com}(\mathsf{H}, \mathsf{A}_1(u, v) \otimes \mathsf{BC}), \tag{3.1}$$

where BC is a fermionic ghost system and H is the Heisenberg vertex operator algebra. As a consequence the categories of weight modules over $\mathsf{A}_1(u, v)$ and $\mathsf{N}(u, v)$ are block-wise equivalent as linear categories and their tensor products are also closely related (morally there is a monoidal induction functor from $mmNuv$ modules to $\mathsf{A}_1(u,v)$ modules.

The proof of rigidity requires first understanding and proving the conjectured some of the conjectured fusion product decomposition formulae of Creutzig and Ridout. These have equivalent $\mathsf{N}(u,v)$ and $\mathsf{A}_1(u,v)$ versions (that is, the decomposition formulae hold for one algebra if and only if they also hold for the other). On the $\mathsf{N}(u,v)$-side one can make use of the Zhu algebra formalism to compute upper bounds on multiplicities of direct summands appearing in fusion product decompositions, while on the $\mathsf{A}_1(u,v)$-side one can use free field realisations to compute lower bounds. These two bounds match, hence proving the decomposition formulae.

The rigidity proof then requires the construction of certain intertwining operators to construct evaluation and coevaluation morphisms. The construction of these intertwining operators uses novel techniques for twisting free field intertwining operators coming from the free field realisation of $\mathsf{A}_1(u,v)$ by screening currents. Verifying that these evaluation and coevaluation morphisms satisfy the necessary zigzag relations then reduces to verifying that certain correlation functions (constructed from the previously mentioned intertwining operators) are non-zero.

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