A QUANTIZATION OF MODULAR FORMS

XUANZHONG DAI

A vertex algebra is an algebraic system with infinitely many products parameterized by integers. The Rankin-Cohen bracket, a bilinear operation on modular forms parameterized by non-negative integers, has long been conjectured by Eholzer, Manin, and Zagier to have certain connections with vertex algebras [Z]. To address this problem, we employ the sheaf-theoretic construction of vertex algebras, known as the chiral de Rham complex (or its purely even subsheaf, the chiral differential operators), to provide a quantization of modular forms.

In this work, we provide an overview of the papers [D1, D2, DS]. For any congruence subgroup Γ, we study the vertex algebra of Γ-invariant sections of chiral de Rham complex on the upper half-plane that are holomorphic at the cusps. We describe the linear structure of the Γ-invariant vertex algebra by constructing a linear basis determined by modular forms. Furthermore, we demonstrate that the vertex operations are entirely governed by a slight modification of the Rankin-Cohen bracket. We then extend this construction to the meromorphic setting and establish that the Γ-invariant vertex algebra is always simple.

1. Sheaf Construction of Vertex algebras

1.1. **Vertex algebra.** A vertex (super)algebra is a superspace V, equipped with a vector $1 \in V_{\bar{0}}$, a parity preserving linear map (called the state-field correspondence) from V to $\operatorname{End} V[[z, z^{-1}]]$,

$$\begin{aligned} V &\longrightarrow \operatorname{End} V[[z, z^{-1}]] \\ u &\longmapsto Y(u, z) = \sum_{n \in \mathbb{Z}} u_{(n)} z^{-n-1} \end{aligned}$$

a linear map $T \in (\operatorname{End} V)_{\bar{0}}$, satisfying the following axioms

(1) (the truncation condition): For every two vectors $u, v \in V$,

$$u_{(n)} v = 0$$

for n sufficiently large.

(2) (vacuum): $T1 = 0, Y(1, z) = id$, $Y(u, z)1 \in V[[z]]$ and $Y(u, z)1|_{z=0} = u$.

(3) (translation covariance):

$$[T, Y(u, z)] = \partial_z Y(u, z),$$

(4) (locality): For every $u, v \in V$, $Y(u, z)$ and $Y(v, z)$ are mutually local, namely, there exists $N \in \mathbb{Z}_{>0}$, such that

$$[Y(u, z), Y(v, w)](z - w)^N = 0.$$

A vertex algebra is called a vertex operator algebra if there is a distinguished vector ω (called the Virasoro element), such that L_0 is semisimple and

$$L_{-1} = T,\ [L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m,-n}c$$

where $L_n = \omega_{(n+1)}$, and $c \in \mathbb{C}$ is a constant called the central charge. An element $v \in V$ is said to have conformal weight n if $L_0 v = nv$.

1.2. **$\beta\gamma$-bc system.** We present several examples of vertex operator algebras that serve as fundamental building blocks for the chiral de Rham complex. Let l be a positive integer. The $\beta\gamma$-system of rank l, denoted by $\mathcal{S}_l$ is defined to be the vacuum representation of Heisenberg Lie algebra with the basis $\beta_n^i, \gamma_n^i, i = 1, \cdots, l, n \in \mathbb{Z}$, the central element C, and the nontrivial commutation relations

$$[\beta_m^i, \gamma_n^j] = \delta_{i,j}\delta_{n+m,0}C,$$

which is generated by the vacuum vector 1 with the following relations

$$\beta_m^i 1 = 0, \quad \text{if } m \geq 0; \;\; \gamma_n^i 1 = 0, \quad \text{if } n \geq 1; \;\; C1 = 1.$$

As a vector space, $\mathcal{S}_l$ is the symmetric algebra generated by elements $\beta_m^i, \gamma_n^i, m < 0, n \leq 0, 1 \leq i \leq d$. Let $\beta^i = \beta_{-1}^i \cdot 1$, and $\gamma^i = \gamma_0^i \cdot 1$. The nontrivial operator product expansion of the generators are

$$\beta^i(z)\gamma^j(w) \sim \delta_{ij}(z-w)^{-1}.$$

As a vector space, $\mathcal{S}_l$ is the symmetric algebra generated by elements β_m^i and γ_n^i for $m < 0$, $n \leq 0$, and $1 \leq i \leq d$. We define $\beta^i = \beta_{-1}^i \cdot 1$ and $\gamma^i = \gamma_0^i \cdot 1$. The nontrivial operator product expansion (OPE) of the generators is given by

$$\beta^i(z)\gamma^j(w) \sim \delta_{ij}(z-w)^{-1}.$$

The bc-system of rank l, denoted by $\mathcal{E}_l$ is defined to be the vacuum representation of the Lie superalgebra with the basis consisting of odd elements $b_n^i, c_n^i, i = 1, \cdots, l, n \in \mathbb{Z}$, and the even center C, and the nontrivial commutation relations

$$[b_m^i, c_n^j] = \delta_{i,j}\delta_{m+n,0}C,$$

which is generated by the vacuum vector 1, and the relations

$$c_m^i 1 = 0, \quad \text{if } m \geq 1; \;\; b_n^i 1 = 0, \quad \text{if } n \geq 0; \;\; C1 = 1.$$

As a vector space, $\mathcal{E}_l$ is the exterior algebra generated by elements b_n^i and c_m^i for $m \leq 0$, $n < 0$, and $1 \leq i \leq l$. We define $b^i = b_{-1}^i \cdot 1$ and $c^i = c_0^i \cdot 1$. The nontrivial OPE of the generators is given by

$$b^i(z)c^j(w) \sim \delta_{ij}(z-w)^{-1}.$$

The $\beta\gamma$-bc system of rank l is the tensor product vertex algebra $\Omega_l := \mathcal{S}_l \otimes \mathcal{E}_l$, where the Virasoro element is given by $\omega = \sum_{i=1}^l :\beta^i\partial\gamma^i: - :b^i\partial c^i:$ with central charge 0. The Virasoro element together with an even element $J = \sum_{i=1}^l :c^i b^i:$ of conformal weight 1, two odd elements $Q = \sum_{i=1}^l :\beta^i c^i:$ and $G = \sum_{i=1}^l :b^i\partial\gamma^i:$ of conformal weight 1 and 2 respectively, makes Ω_l into a topological vertex algebra or $N = 2$ superconformal vertex algebra [MSV].

The zeroth Fourier coefficient of the field $J(z)$ is called the fermionic charge operator. It acts semisimply on Ω_l, with its eigenvalue referred to as the fermionic charge. We denote by $\Omega_l(m)$ the subspace spanned by elements with fermionic charge m. It is evident that the zeroth Fourier mode of $Q(z)$, denoted by $Q_{(0)}$, increases the fermionic charge by 1, and its square vanishes. Consequently, the space $\Omega_l = \bigoplus_{m\in\mathbb{Z}} \Omega_l(m)$, together with the chiral de Rham differential $d := -Q_{(0)}$, forms a complex, which is called the chiral de Rham complex on the affine space.

1.3. **Chiral de Rham complex.** The chiral de Rham complex, introduced in [MSV], is a sheaf of vertex algebras defined on any complex manifold or nonsingular algebraic variety. Let U be an open subset of an l-dimensional complex manifold (or smooth algebraic variety) with local coordinates $\gamma^1, \ldots, \gamma^l$. The space of holomorphic (or algebraic) functions on U, denoted by $\mathcal{O}(U)$, is a $\mathbb{C}[\gamma_0^1, \ldots, \gamma_0^l]$-module, where γ_0^i acts as multiplication by γ^i. The localization of Ω_l on U is defined as

$$\Omega^{\mathrm{ch}}(U) = \Omega_l \otimes_{\mathbb{C}[\gamma_0^1, \ldots, \gamma_0^l]} \mathcal{O}(U),$$

which forms a vertex algebra generated by the fields $\beta^i(z), \partial\gamma^i(z), c^i(z), b^i(z)$, and $Y(f, z)$ for $f \in \mathcal{O}(U)$, where

$$Y(f, z) = \sum_{i=0}^{\infty} \frac{\partial^i}{i!} f(\gamma) \left(\sum_{n \neq 0} \gamma_n z^{-n} \right)^i .$$

The generators satisfy the following nontrivial OPEs:

$$\beta^i(z)\partial\gamma^j(w) \sim \frac{\delta_{ij}}{(z-w)^2}, \quad b^i(z)c^j(w) \sim \frac{\delta_{ij}}{z-w},$$
$$\beta^i(z)f(w) \sim \frac{\frac{\partial}{\partial\gamma^i} f(w)}{z-w}.$$

For another coordinate system $\tilde{\gamma}^1, \ldots, \tilde{\gamma}^l$ on U, the generators transform as follows:

$$\begin{aligned}
\tilde{\beta}^i &= \beta^j_{-1} \frac{\partial\gamma^j}{\partial\tilde{\gamma}^i} + c^r_0 b^j_{-1} \frac{\partial^2\gamma^j}{\partial\tilde{\gamma}^i \partial\tilde{\gamma}^m} \frac{\partial\tilde{\gamma}^m}{\partial\gamma^r}, \\
\partial\tilde{\gamma}^i &= \partial\gamma^j \frac{\partial\tilde{\gamma}^i}{\partial\gamma^j}, \\
\tilde{b}^i &= b^j_{-1} \frac{\partial\gamma^j}{\partial\tilde{\gamma}^i}, \\
\tilde{c}^i &= c^j_0 \frac{\partial\tilde{\gamma}^i}{\partial\gamma^j},
\end{aligned}$$

where we use Einstein summation convention.

2. Vertex algebras arising from congruence subgroups

2.1. **Chiral de Rham complex on the upper half plane.** We apply the chiral de Rham complex to the upper half-plane $\mathbb{H}$, defined as

$$\mathbb{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\},$$

and consider the vertex algebra of the global sections of the chiral de Rham complex on $\mathbb{H}$:

$$\Omega^{\mathrm{ch}}(\mathbb{H}) := \Omega_1 \otimes_{\mathbb{C}[\gamma_0]} \mathcal{O}(\mathbb{H}).$$

For simplicity, we omit the upper index and write $\beta = \beta^1$, $\gamma = \gamma^1$, $b = b^1$, and $c = c^1$. The fractional linear transformations on $\mathbb{H}$ induce a right $SL(2, \mathbb{R})$-action on $\Omega^{\mathrm{ch}}(\mathbb{H})$ as vertex algebra automorphisms, given by

$$
\begin{aligned}
\pi(g)\beta &= \beta_{-1}(C\gamma+D)^2 + 2c_0 b_{-1} C(C\gamma+D),\\
\pi(g)\partial\gamma &= \gamma_{-1}(C\gamma+D)^{-2},\\
\pi(g)b &= b_{-1}(C\gamma+D)^2,\\
\pi(g)c &= c_0(C\gamma+D)^{-2},\\
\pi(g)f(\gamma) &= f(g\gamma) = f\left(\frac{A\gamma+B}{C\gamma+D}\right),
\end{aligned}
\tag{2.1}
$$

for any

$$
g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(2,\mathbb{R}). \tag{2.2}
$$

Note that equation (2.1) agrees with the coordinate transformation relations in Section 1.3. Considering the infinitesimal action, the embedding of $\mathfrak{sl}_2$ into $\Omega^{\mathrm{ch}}(\mathbb{H})$ is given by

$$
E \mapsto -\beta,\ F \mapsto :\beta\gamma^2: + 2:\gamma:cb::,\ H \mapsto -2:\beta\gamma: - 2:cb:, \tag{2.3}
$$

which determines a representation of the affine Kac-Moody algebra $\widehat{\mathfrak{sl}}_2$ of level 0 [W, FF, F].

Let Ω' be the vertex subalgebra of Ω_1 generated by $\beta, \partial\gamma, c$, and b, which is also a topological vertex algebra. The vertex algebra Ω' admits a grading determined by the part of the four-tuple of partitions, i.e.,

$$
\Omega' = \bigoplus_{n\in\mathbb{Z}} \Omega'[n], \tag{2.4}
$$

where $\Omega'[n]$ is the subspace of Ω' spanned by elements of part n. Since β_0 acts as zero on Ω', the operator $H_{(0)}$ is well-defined on Ω'. In fact, it is a semisimple operator, whose $-2n$-eigenspace is isomorphic to $\Omega'[n]$.

2.2. **Cuspidal conditions.** For convenience, we introduce the following notation. For any partition $\lambda = (\lambda_1, \dots, \lambda_d)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 1$, we define $p(\lambda) = d$. For any symbol $X \in \{\beta, \gamma, b\}$, we denote by $X_{-\lambda}$ the expression $X_{-\lambda_1} \cdots X_{-\lambda_d}$, and set $c_{-\lambda} := c_{-\lambda_1+1} \cdots c_{-\lambda_d+1}$. We call $w = (\lambda, \mu, \nu, \chi)$ a four-tuple of partitions if λ and χ are partitions, while μ and ν are partitions with distinct parts. The part of the four-tuple w is defined as

$$
p(w) := -p(\lambda) + p(\mu) - p(\nu) + p(\chi).
$$

For any congruence subgroup $\Gamma \subset SL(2,\mathbb{Z})$, we now define the cuspidal conditions on the Γ-invariant global sections. Let v be an arbitrary Γ-invariant global section. For any rational number a, we can choose $\rho \in SL(2,\mathbb{Z})$ such that $\rho(a) = \infty$. Since the group $\rho^{-1}\Gamma\rho$ acts from the right, the element

$$
\pi(\rho)v = \sum_{w=(\lambda,\mu,\nu,\chi)} \beta_{-\lambda} c_{-\mu} b_{-\nu} \gamma_{-\chi} f_w
$$

is invariant under its action. Furthermore, since $\rho^{-1}\Gamma\rho$ includes the translation matrix $\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}$ for some positive integer N, the element $\pi(\rho)v$ is fixed by $\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}$. Using the following formula

$$
\pi\left(\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}\right) \sum_{w=(\lambda,\mu,\nu,\chi)} \beta_{-\lambda} c_{-\mu} b_{-\nu} \gamma_{-\chi} f_w(\gamma) = \sum_{w=(\lambda,\mu,\nu,\chi)} \beta_{-\lambda} c_{-\mu} b_{-\nu} \gamma_{-\chi} f_w(\gamma+N),
$$

we conclude that $f_w(\gamma+N) = f_w(\gamma)$ for any four-tuple w. Thus, we have the Fourier expansion

$$f_w(\gamma) = \sum_{m\in\mathbb{Z}} u_w(m)e^{2\pi i m\gamma/N}.$$

We say that v is holomorphic (resp. meromorphic) at the cusp a if, for any four-tuple w, $u_w(m) = 0$ for $m < 0$ (resp. $u_w(m) = 0$ for $m < -N$, for some positive integer N).

We denote by $\Omega^{\mathrm{ch}}(\mathbb{H},\Gamma)$, resp. $\mathcal{M}(\mathbb{H},\Gamma)$, the space spanned by the Γ-invariant vectors in $\Omega^{\mathrm{ch}}(\mathbb{H})$ that are holomorphic, resp. meromorphic at all cusps. Both $\Omega^{\mathrm{ch}}(\mathbb{H},\Gamma)$ and $\mathcal{M}(\mathbb{H},\Gamma)$ are vertex subalgebras of $\Omega^{\mathrm{ch}}(\mathbb{H})$. Similarly, by applying the purely even subsheaf of the chiral de Rham complex (i.e., the chiral differential operators), we obtain the bosonic version of the construction, denoted $\mathscr{D}^{\mathrm{ch}}(\mathbb{H},\Gamma)$.

2.3. **Filtration induced by a semiorder.** Recall that the set of four-tuples of partitions is equipped with a semiorder determined by their part, given by

$$(\lambda,\mu,\nu,\chi) > (\lambda',\mu',\nu',\chi') \quad \text{if} \quad p(\lambda,\mu,\nu,\chi) < p(\lambda',\mu',\nu',\chi').$$

Using this, we define a family of free $\mathcal{O}(\mathbb{H})$-submodules:

$$W_m := \mathrm{Span}_{\mathbb{C}}\{\beta_{-\lambda}c_{-\mu}b_{-\nu}\gamma_{-\chi}f(\gamma) \in \Omega^{\mathrm{ch}}(\mathbb{H}) \mid p(\lambda,\mu,\nu,\chi) \geq m\}.$$

This defines an $SL(2,\mathbb{R})$-invariant decreasing filtration on $\Omega^{\mathrm{ch}}(\mathbb{H})$. The induced $SL(2,\mathbb{R})$-action on the associated graded algebra $\mathrm{gr}\, W$ is significantly simplified. Specifically, for any $g \in SL(2,\mathbb{R})$ as in (2.2), any function $f \in \mathcal{O}(\mathbb{H})$, and any four-tuple (λ,μ,ν,χ) with part n, we have

$$\pi(g)\big(\beta_{-\lambda}c_{-\mu}b_{-\nu}\gamma_{-\chi}f(\gamma)\big) = \beta_{-\lambda}c_{-\mu}b_{-\nu}\gamma_{-\chi}(C\gamma+D)^{-2n}f(g\gamma) \mod W_{n+1}.$$

We denote the induced filtration on $\mathcal{M}(\mathbb{H},\Gamma)$ by $\{W_n(\Gamma)\}_{n\in\mathbb{Z}}$.

We recall that a weakly holomorphic modular form $f(\tau)$ of weight k for Γ is a holomorphic function on $\mathbb{H}$ that satisfies the modular transformation property

$$f(g\tau) = (C\tau+D)^k f(\tau)$$

for all $g \in \Gamma$ and is meromorphic at all cusps. It follows that for any element in $\mathcal{M}(\mathbb{H},\Gamma)$, the function determined by its leading term must be a weakly holomorphic modular form. In particular, when considering the vertex algebras $\mathscr{D}^{\mathrm{ch}}(\mathbb{H},\Gamma)$ and $\Omega^{\mathrm{ch}}(\mathbb{H},\Gamma)$, we obtain modular forms.

3. Main Theorem

We will summarize main theorems established in [D1, D2, DS].

Theorem 3.1. *The character formulas of $\mathscr{D}^{ch}(\mathbb{H},\Gamma)$ and $\Omega^{ch}(\mathbb{H},\Gamma)$ are given as follows*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \dim M_{2m}(\Gamma) q^{2n+m} \prod_{i=1}^{n}\frac{1}{1-q^i}\prod_{j=1}^{m+n}\frac{1}{1-q^j},$$

$$\sum_{m,n=0}^{\infty}\sum_{u=0}^{n}\sum_{v=0}^{m+n} \dim M_{2m}(\Gamma) q^{m+2n+\frac{1}{2}u(u-1)+\frac{1}{2}v(v-3)} \prod_{i=1}^{u}\frac{1}{1-q^i}\prod_{j=1}^{v}\frac{1}{1-q^j}\prod_{k=1}^{n-u}\frac{1}{1-q^k}\prod_{l=1}^{m+n-v}\frac{1}{1-q^l}.$$

Theorem 3.1 presents character formulas for the vertex algebras $\mathscr{D}^{\mathrm{ch}}(\mathbb{H},\Gamma)$ and $\Omega^{\mathrm{ch}}(\mathbb{H},\Gamma)$, which depend on the dimensions of spaces of modular forms of even weights. However, computing the character in the meromorphic setting is not meaningful, as it includes the ring of modular functions for the modular group—specifically, the polynomial ring generated by the Klein j-function—as a subspace of the weight 0 component, which is already infinite-dimensional. The following theorem suggests a quantization of (weakly holomorphic) modular forms.

Theorem 3.2. *Let $M^!_n(\Gamma)$ denote the space of weakly holomorphic modular forms of weight n for Γ. There exists a short exact sequence*

$$0 \longrightarrow W_{n+1}(\Gamma) \longrightarrow W_n(\Gamma) \longrightarrow \Omega'[n] \otimes M^!_{2n}(\Gamma) \longrightarrow 0,$$

where the second map is an embedding, and the third map is a quotient projection.

We introduce a locally nilpotent operator which acts on the space of operators on the global sections determined by

$$D(T) = [F_{(0)}, T] + [H_{(0)}, T]\gamma_0, \quad \text{for any operator } T.$$

The following theorem gives a construction of a linear basis of $\Omega^{\mathrm{ch}}(\mathbb{H}, \Gamma)$.

Theorem 3.3. *Let $M_n(\Gamma)$ denote the space of modular forms of weight n for Γ. For any four-tuple of partitions $w = (\lambda, \mu, \nu, \chi)$ with $p(w) = n$ and any $f \in M_{2n}(\Gamma)$, we define the element $L(w, f)$ as follows:*

(1) Case $n > 0$:

$$L(w, f) := \sum_{m=0}^{\infty} \frac{(2n-1)!}{m!(m+2n-1)!} D^m(\beta_{-\lambda} c_{-\mu} b_{-\nu} \gamma_{-\chi}) f^{(m)}(\gamma) \in \Omega^{ch}(\mathbb{H}, \Gamma). \tag{3.1}$$

(2) Case $n = 0$:

$$L(w, 1) := \beta_{-\lambda} c_{-\mu} b_{-\nu} \gamma_{-\chi} 1 + \frac{\pi i}{6} \sum_{m=1}^{\infty} \frac{1}{m!(m-1)!} D^m(\gamma_{-\lambda} c_{-\mu} b_{-\nu} \gamma_{-\chi}) E_2^{(m-1)}(\gamma) \in \Omega^{ch}(\mathbb{H}, SL(2, \mathbb{Z})). \tag{3.2}$$

(3) Basis Property: The set of elements $L(w, f)$, where n runs over all non-negative integers, f runs through a basis of $M_{2n}(\Gamma)$, and w runs through all four-tuples with $p(w) = n$, forms a linear basis of $\Omega^{ch}(\mathbb{H}, \Gamma)$.

With the basis provided in Theorem 3.3, we can demonstrate that the vertex operations on the basis elements are closely related to the Rankin-Cohen brackets of modular forms. For modular forms f_1 of weight k and f_2 of weight l, the n-th Rankin-Cohen bracket $[f_1, f_2]_n$ (with $n \geq 0$) is a modular form of weight $k + l + 2n$. While the Rankin-Cohen brackets for modular forms f_1 and f_2 of positive weights correspond well to the vertex operations $L(w, f_1)_{(n)} L(v, f_2)$, when one of f_1 or f_2 has weight 0, we must modify the Rankin-Cohen brackets as follows:

For $n > 0$ and $f \in M_k(\Gamma)$ with $k > 0$, we define

$$[1, f]^{\sim}_n := \frac{1}{12(2\pi i)^{n-1}} \sum_{\substack{r+s=n \\ 1 \leq r \leq n}} (-1)^r \binom{n-1}{s} \binom{n+k-1}{r} E_2^{(r-1)}(\tau) f^{(s)}(\tau) + \frac{1}{n(2\pi i)^n} f^{(n)}(\tau), \tag{3.3}$$

and for $[1, 1]^{\sim}_n$, we define

$$[1, 1]^{\sim}_n := \frac{1}{144(2\pi i)^{n-2}} \sum_{\substack{r+s=n \\ 1 \leq r \leq n-1}} (-1)^r \binom{n-1}{s} \binom{n-1}{r} E_2^{(r-1)}(\tau) E_2^{(s-1)}(\tau) + \frac{(-1)^n + 1}{12n(2\pi i)^{n-1}} E_2^{(n-1)}(\tau). \tag{3.4}$$

With these modifications to the Rankin-Cohen brackets as in (3.3) and (3.4), we obtain the following theorem, which provides a precise description of the relationship between Rankin-Cohen brackets and vertex algebras, as conjectured by Eholzer, Manin, and Zagier in 1994 [Z].

Theorem 3.4. *For modular forms $f_1 \in M_{2k}(\Gamma)$, $f_2 \in M_{2l}(\Gamma)$, and any four-tuples $w = (\lambda_1, \mu_1, \nu_1, \chi_1)$, $v = (\lambda_2, \mu_2, \nu_2, \chi_2)$ with parts k and l, respectively, we have*

$$L(w, f_1)_{(n)} L(v, f_2) = \sum_{u=(\lambda,\mu,\nu,\chi)} c^n_{w,v,u} L(u, [f_1, f_2]^{\sim}_{p(u)-p(w)-p(v)}), \tag{3.5}$$

where $c^n_{w,v,u}$ is a constant determined by n and the four-tuples w, v, and u.

By Theorem 3.4, we can show that the vertex algebra $\Omega^{\mathrm{ch}}(\mathbb{H}, \Gamma)$ is generally not simple, as the subspace spanned by the liftings of modular forms of positive weight forms a vertex algebra ideal. However, by relaxing the cuspidal conditions, we demonstrate that the meromorphic analogue of our construction always yields a simple vertex algebra.

Theorem 3.5. *The vertex algebra $\mathcal{M}(\mathbb{H}, \Gamma)$ is simple for any congruence subgroup Γ.*

4. Acknowledgements

This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University. We would like to express our gratitude to Tomoyuki Arakawa for recommending the presentation at the research conference, as well as to the organizer, Sho Suda.

References

[D1] X.Dai, Chiral differential operators on the upper half plane and modular forms, Adv. Math. 390 (2021), 107957.

[D2] X.Dai, Chiral de Rham complex on the upper half plane and modular forms, Int. Math. Res. Not. (IMRN) (2022), no. 24, 19258-19299.

[DS] X.Dai, B. Song, Simple vertex algebras arising from congruence subgroups, Adv. Math. 456 (2024), 109900.

[F] E.Frenkel, Wakimoto modules, opers and the center at the critical level. Adv. Math. 195 (2005), no. 2, 297-404.

[FF] B.Feigin, E.Frenkel, Representations of affine Kac-Moody algebras and bosonization. Physics and mathematics of strings, 271-316, World Sci. Publ., Teaneck, NJ, (1990).

[MSV] F. Malikov, V. Schectman, and A. Vaintrob, Chiral de Rham complex. Comm. Math. Phys. 204 (1999), no. 2, 439-473.

[W] M. Wakimoto, Fock representations of the affine Lie algebra A(1)1. Comm. Math. Phys. 104 (1986), no. 4, 605-609.

[Z] D. Zagier, Modular forms and differential operators. K. G. Ramanathan memorial issue. Proc. Indian Acad. Sci. Math. Sci. 104 (1994), no. 1, 57-75.

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502 JAPAN

Email address: xzdai@kurims.kyoto-u.ac.jp