On a series of simple affine VOAs at non-admissible level arising from rank one 4D SCFTs

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Abstract

We study the representations of some simple affine vertex algebras at non-admissible level arising from rank one 4D SCFTs. In particular, we classify the irreducible highest weight modules of $L_{-2}(G_2)$ and $L_{-2}(B_3)$. It is known by the works of Adamović and Perše that these vertex algebras can be conformally embedded into $L_{-2}(D_4)$. We also compute the associated variety of $L_{-2}(G_2)$, and show that it is the orbifold of the associated variety of $L_{-2}(D_4)$ by the symmetric group of degree 3 which is the Dynkin diagram automorphism group of D_4. This provides a new interesting example of associated variety satisfying a number of conjectures in the context of orbifold vertex algebras.

1 Introduction

For any four-dimensional $\mathcal{N} = 2$ superconformal field theory (SCFT), there is a subsector which can be described by a two-dimensional vertex operator algebra (VOA) discovered in [BLL$^+$]. The normalized character of the corresponding VOA reproduces the special limit of the superconformal index, called the *Schur index*. On the one hand, four-dimensional SCFTs lead to some interesting conjectures for large classes of VOAs. For example, it is expected [BR] that the Higgs branch of such a 4D theory is the associated variety X_V of the corresponding VOA V. This in particular implies all the VOAs coming from 4D theories are quasi-lisse. On the other hand, the representation theory of the VOA produces new physical observables of the 4D SCFT, such as the ordinary Schur index and the Schur index in the presence of boundary conditions, line defects and surface defects.

Recently, it was noticed by Li, Xie and Yan [LXY] that the vertex algebras $L_k(\mathfrak{g})$ for $\mathfrak{g}$ belonging to the series,

$$A_1 \subset A_2 \subset G_2 \subset B_3 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8,$$

and $k = -h/6 - 1$, where h is the Coxeter number, also come from rank one four-dimensional $\mathcal{N} = 2$ superconformal field theories in the context of the Argyres–Douglas theory. Here the rank of a SCFT refers to the dimension of the Coulomb branch. In particular, it is natural to focus on the following two series

$$L_{-2}(G_2) \hookrightarrow L_{-2}(B_3) \hookrightarrow L_{-2}(D_4) \quad \text{and} \quad L_{-3}(F_4) \hookrightarrow L_{-3}(E_6),$$

which involve non simply-laced cases. Note that the above embeddings are conformal by [AP]. The associated varieties and the representations in the category $\mathcal{O}$ of $L_{-2}(D_4)$ and $L_{-3}(E_6)$ have been determined in [AM1].

In this context, studying the representation theory of these simple affine vertex algebras becomes very important. We consider in this article the simple affine vertex algebras $L_{-2}(G_2)$ and $L_{-2}(B_3)$ that appear as orbifold vertex algebras of the simple affine vertex algebra $L_{-2}(D_4)$. We are interested in their representations in the category $\mathcal{O}$. We also compute the associated variety of $L_{-2}(G_2)$; that of $L_{-2}(B_3)$ and of $L_{-2}(D_4)$ were described in [AM1, AM3].

2 Preliminaries

Throughout the article, all Lie algebras are defined over $\mathbb{C}$ and all topological terms refer to the Zariski topology.

Let $\mathfrak{g}$ be a simple Lie algebra with Killing form $\kappa_{\mathfrak{g}}$ as in the introduction, and let $\widetilde{\mathfrak{g}} = \mathfrak{g}[t,t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D$ be the extended affine Kac-Moody Lie algebra associated with $\mathfrak{g}$ and the inner product

$$(-|-) = \frac{1}{2h_{\mathfrak{g}}^{\vee}} \times \kappa_{\mathfrak{g}},$$

with the commutation relations

$$[x(m), y(n)] = [x,y](m+n) + m(x|y)\delta_{m+n,0}K, \quad [D, x(m)] = mx(m), \quad [K, \widetilde{\mathfrak{g}}] = 0,$$

for $m,n \in \mathbb{Z}$ and $x, y \in \mathfrak{g}$, where $x(m) = x \otimes t^m$.

Let $\widehat{\mathfrak{g}} = [\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}] = \mathfrak{g}[t,t^{-1}] \oplus \mathbb{C}K$. Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ so that

$$\widetilde{\mathfrak{g}} = \widehat{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_+ \quad \text{and} \quad \widehat{\mathfrak{g}} = \widehat{\mathfrak{n}}_- \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_+$$

are triangular decompositions for $\widetilde{\mathfrak{g}}$ and $\widehat{\mathfrak{g}}$, respectively, with $\widehat{\mathfrak{n}}_- = \mathfrak{n}_- + t^{-1}\mathfrak{g}[t^{-1}]$, $\widehat{\mathfrak{n}}_+ = \mathfrak{n}_+ + t\mathfrak{g}[t]$, $\widetilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}D$ and $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K$. The Cartan subalgebra $\widetilde{\mathfrak{h}}$ is equipped with a bilinear form extending that on $\mathfrak{h}$ given by

$$(K|D) = 1, \quad (\mathfrak{h}|\mathbb{C}K \oplus \mathbb{C}D) = (K|K) = (D|D) = 0.$$

We write Λ_0 and δ for the elements of $\widetilde{\mathfrak{h}}^*$ orthogonal to $\mathfrak{h}^*$ and dual to K and D, respectively. Let Δ be the root system of $(\mathfrak{g}, \mathfrak{h})$ with basis $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$, and denote by θ the highest positive root. We write $\varpi_1, \ldots, \varpi_\ell$ for the fundamental weights of $\mathfrak{g}$ with respect to $\alpha_1, \ldots, \alpha_\ell$, and $\Lambda_0, \Lambda_1, \ldots, \Lambda_\ell$ for those of $\widetilde{\mathfrak{g}}$.

For $k \in \mathbb{C}$, set

$$V^k(\mathfrak{g}) = U(\widetilde{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}K \oplus \mathbb{C}D)} \mathbb{C}_k,$$

where $\mathbb{C}_k$ is the one-dimensional representation of $\mathfrak{g}[t] \oplus \mathbb{C}K \oplus \mathbb{C}D$ on which $\mathfrak{g}[t] \oplus \mathbb{C}D$ acts trivially and K acts as multiplication by k. The space $V^k(\mathfrak{g})$ is naturally a vertex algebra, called the *universal affine vertex algebra associated with* $\mathfrak{g}$ *at level* k. By the PBW theorem, we have $V^k(\mathfrak{g}) \cong U(\mathfrak{g}[t^{-1}]t^{-1})$ as $\mathbb{C}$-vector spaces.

The vertex algebra $V^k(\mathfrak{g})$ is graded by D:

$$V^k(\mathfrak{g}) = \bigoplus_{d \in \mathbb{Z}_{\geqslant 0}} V^k(\mathfrak{g})_d, \quad V^k(\mathfrak{g})_d = \{a \in V^k(\mathfrak{g}) \colon Da = -da\}.$$

This grading gives a conformal structure provided that k is not critical, that is, $k \neq -h_{\mathfrak{g}}^{\vee}$. A $V^k(\mathfrak{g})$-module is the same as a smooth $\widetilde{\mathfrak{g}}$-module of level k, where a $\widetilde{\mathfrak{g}}$-module M is called smooth if $x(n)m = 0$ for n sufficiently large for all $x \in \mathfrak{g}$, $m \in M$.

2.1 Singular vectors and highest-weight modules

For each $\alpha \in \Delta$, fix a nonzero root vector e_α. Recall that a vector $v \in V^k(\mathfrak{g})$ is called *singular* if $e_\alpha(0)v = 0$ for all $\alpha \in \Pi$ and $e_{-\theta}(1)v = 0$. In other words, v is a singular vector if v is singular for $\widetilde{\mathfrak{g}}$ with respect to $\widehat{\mathfrak{n}}_+$. If v is singular for $V^k(\mathfrak{g})$, denote by $\langle v \rangle$ the ideal in $V^k(\mathfrak{g})$ generated by v, that is, $\langle v \rangle = U(\widetilde{\mathfrak{g}})v$. We set

$$\widetilde{V}_k(\mathfrak{g}) = V^k(\mathfrak{g})/\langle v \rangle, \tag{1}$$

the associated quotient vertex algebra.

Let $L_k(\mathfrak{g})$ be the unique simple graded quotient of $V^k(\mathfrak{g})$. As a $\widetilde{\mathfrak{g}}$-module, $L_k(\mathfrak{g})$ is isomorphic to the irreducible highest-weight representation of $\widetilde{\mathfrak{g}}$ with highest-weight $k\Lambda_0$. If N_k denotes the unique maximal ideal of $V^k(\mathfrak{g})$, then

$$L_k(\mathfrak{g}) = V^k(\mathfrak{g})/N_k,$$

and $L_k(\mathfrak{g})$ is a quotient of $\widetilde{V}_k(\mathfrak{g})$. We will also make use of the notion of *subsingular vector.*

Definition 2.1. *A vector $v_{\text{sub}} \in N_k$ is* subsingular *if there exists a proper submodule N_k' of N_k such that the following conditions hold:*

$$v_{\text{sub}} \notin N_k', \quad e_\alpha(0)v_{\text{sub}} \in N_k' \quad \textit{for all } \alpha \in \Pi, \quad e_{-\theta}(1)v_{\text{sub}} \in N_k'.$$

Note that the image of a subsingular vector in $V^k(\mathfrak{g})/N_k'$ is a singular vector of $V^k(\mathfrak{g})/N_k'$.

For $\lambda \in \mathfrak{h}^*$, we denote by $L_{\mathfrak{g}}(\lambda)$ the irreducible highest-weight representation of $\mathfrak{g}$ with highest-weight λ. Similarly, for $\tilde{\lambda} \in \widetilde{\mathfrak{h}}^*$ we denote by $L_{\widetilde{\mathfrak{g}}}(\tilde{\lambda})$ the irreducible highest-weight representation of $\widetilde{\mathfrak{g}}$. In the case where $\tilde{\lambda} = \lambda + k\Lambda_0$, we shall sometimes write $L_{\mathfrak{g}}(k, \lambda)$ instead of $L_{\widetilde{\mathfrak{g}}}(\tilde{\lambda})$. In this way, we have

$$L_k(\mathfrak{g}) = L_{\widetilde{\mathfrak{g}}}(k\Lambda_0) = L_{\mathfrak{g}}(k, 0).$$

A finitely generated module M over a conformal vertex algebra V is called *ordinary* if L_0 acts semisimply, M_d being finite-dimensional for all d, where

$$M_d = \{m \in M \colon L_0 m = dm\},$$

and the conformal weights of M are bounded from below, i.e. there exists d_0 so that $M_d = 0$ for $d \leqslant d_0$. Call the *conformal dimension* of a simple ordinary V-module M the minimum conformal weight of M. More generally, a V-module M is said to be of *positive energy* if it is $\mathbb{Z}_{\geqslant 0}$-graded, $M = \bigoplus_{d \in \mathbb{Z}_{\geqslant 0}} M_{d_0+d}$, with $M_{d_0} \neq 0$, such that $a(n)M_k \subset M_{k-n}$, where for $a \in V$ of conformal weight Δ we write $a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-\Delta}$.

The highest-weight $\widetilde{\mathfrak{g}}$-module $L_{\mathfrak{g}}(k, \lambda)$, regarded as a $V^k(\mathfrak{g})$-module, has conformal dimension

$$h_{L(\lambda)} = \frac{(\lambda|\lambda + 2\rho)}{2(k + h_{\mathfrak{g}}^\vee)}, \tag{2}$$

where ρ is the half–sum of positive roots.

2.2 Zhu's algebra and the characteristic variety

For a positively $\mathbb{Z}$-graded vertex algebra $V = \bigoplus_d V_d$, let $A(V)$ be the Zhu algebra of V,

$$A(V) = V/V \circ V,$$

where $V \circ V$ is the $\mathbb{C}$-span of the vectors

$$a \circ b := \sum_{i \geqslant 0} \binom{\Delta}{i} a_{(i-2)} b$$

for $a \in V_\Delta$, $\Delta \in \mathbb{Z}_{\geqslant 0}$, $b \in V$, and $V \to (\mathrm{End} V)[\![z, z^{-1}]\!]$, $a \mapsto \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$, denotes the state-field correspondence. The space $A(V)$ is a unital associative algebra with respect to the multiplication defined by

$$a * b := \sum_{i \geqslant 0} \binom{\Delta}{i} a_{(i-1)} b$$

for $a \in V_\Delta$, $\Delta \in \mathbb{Z}_{\geqslant 0}$, $b \in V$. Denote by $[a]$ the image of $a \in V$ in $A(V)$.

Let $M = \bigoplus_{d \in \mathbb{Z}_{\geqslant 0}} M_{d_0+d}$, with $M_{d_0} \neq 0$, be a positive energy representation of V. Then $A(V)$ naturally acts on its top weight space $M_{\text{top}} := M_{d_0}$, and the correspondence $M \mapsto M_{top}$ defines a bijection between isomorphism classes of simple positive energy representations of V and simple $A(V)$-modules [Z].

The Zhu algebra $A(V^k(\mathfrak{g}))$ is naturally isomorphic to the universal enveloping algebra $U(\mathfrak{g})$ [FZ], where the isomorphism $F\colon A(V^k(\mathfrak{g})) \to U(\mathfrak{g})$ is given by

$$F([a_1(-n_1-1)\dots a_m(-n_m-1)\mathbb{1}]) = (-1)^{n_1+\dots+n_m} a_m \dots a_1, \tag{3}$$

for $a_1, \dots, a_m \in \mathfrak{g}$ and $n_1, \dots, n_m \in \mathbb{Z}_{\geqslant 0}$.

We have an exact sequence

$$A(N_k) \to U(\mathfrak{g}) \to A(L_k(\mathfrak{g})) \to 0$$

since the functor $A(-)$ is right exact, and thus $A(L_k(\mathfrak{g}))$ is the quotient of $U(\mathfrak{g})$ by the image J_k of the maximal ideal N_k in $A(V^k(\mathfrak{g})) = U(\mathfrak{g})$:

$$A(L_k(\mathfrak{g})) = U(\mathfrak{g})/J_k.$$

In particular, if v is a singular vector,

$$A(\widetilde{V}_k(\mathfrak{g})) \cong U(\mathfrak{g})/\langle v' \rangle,$$

where $\langle v' \rangle$ is the two-sided ideal in $U(\mathfrak{g})$ generated by the vector

$$v' := F([v]).$$

The top degree component of $L_{\widehat{\mathfrak{g}}}(\lambda)$ is $L_{\mathfrak{g}}(\bar{\lambda})$, where $\bar{\lambda}$ is the restriction of λ to $\mathfrak{h}$. Hence, by Zhu's correspondence, a level k representation $L_{\widehat{\mathfrak{g}}}(\lambda)$, that is $\lambda(K) = k$, is an $L_k(\mathfrak{g})$-module if and only if $J_k L_{\mathfrak{g}}(\bar{\lambda}) = 0$.

Set $U(\mathfrak{g})^{\mathfrak{h}} := \{u \in U(\mathfrak{g}) \colon [h, u] = 0 \text{ for all } h \in \mathfrak{h}\}$ and let

$$\Upsilon\colon U(\mathfrak{g})^{\mathfrak{h}} \to U(\mathfrak{h}) \tag{4}$$

be the *Harish–Chandra projection map* which is the restriction of the projection map $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}_- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_+) \to U(\mathfrak{h})$ to $U(\mathfrak{g})^{\mathfrak{h}}$. It is known that Υ is an algebra homomorphism. For a two-sided ideal I of $U(\mathfrak{g})$, the *characteristic variety of I* is defined as [J]:

$$\mathscr{X}(I) = \{\lambda \in \mathfrak{h}^* \colon p(\lambda) = 0 \text{ for all } p \in \Upsilon(I^{\mathfrak{h}})\},$$

where $I^{\mathfrak{h}} = I \cap U(\mathfrak{g})^{\mathfrak{h}}$. Identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ through $(-|-)$, and thus $\mathfrak{h}^*$ with $\mathfrak{h}$, we view $\mathscr{X}(I)$ as a subset of $\mathfrak{h}$. Then using [J] (see also [Ar3, Lemma 2.1]), it is easy to see that for $\lambda \in \mathfrak{h}^*$, $\lambda \in \mathscr{X}(I)$ if and only if $IL_{\mathfrak{g}}(\lambda) = 0$. In other words, the characteristic variety $\mathscr{X}(I)$ classifies the simple $U(\mathfrak{g})/I$-modules in category $\mathcal{O}^{\mathfrak{g}}$, where $\mathcal{O}^{\mathfrak{g}}$ is the BGG category $\mathcal{O}$ of $\mathfrak{g}$.

According to [Ad, AM, Ar3], we have the following result.

Proposition 2.2. *Let $v \in V^k(\mathfrak{g})$ be a singular vector, $\widetilde{V}_k(\mathfrak{g}) = V^k(\mathfrak{g})/\langle v\rangle$ as in (1), $v' := F([v])$ the corresponding image in $U(\mathfrak{g})$ and R the $U(\mathfrak{g})$-submodule of $U(\mathfrak{g})$ generated by the vector v'. The following statements are equivalent:*

(i) *$L_{\mathfrak{g}}(\mu)$ is an $A(\widetilde{V}_k(\mathfrak{g}))$-module,*

(ii) *$RL_{\mathfrak{g}}(\mu) = 0$,*

(iii) *$R^{\mathfrak{h}} v_\mu = 0$, where $R^{\mathfrak{h}} := R \cap U(\mathfrak{g})^{\mathfrak{h}}$,*

where v_μ is a highest-weight vector of $L_{\mathfrak{g}}(\mu)$.

In the notation of the Proposition 2.2, given $r \in R^{\mathfrak{h}}$, there exists a unique polynomial $p_r \in \Upsilon(R^{\mathfrak{h}})$ such that $rv_\mu = p_r(\mu)v_\mu$. Define the polynomial set of $\mathfrak{h}$ by

$$\mathscr{P}_v = \{p_r \colon r \in R^{\mathfrak{h}}\}. \tag{5}$$

If v is a subsingular vector, one can define similarly $\mathscr{P}_v$ using the $U(\mathfrak{g})$-submodule of $U(\mathfrak{g})$ generated by the vector $v' := F([v])$.

As a consequence of Proposition 2.2, we obtain:

Corollary 2.3. *Let $v \in V^k(\mathfrak{g})$ be a singular vector and $\widetilde{V}_k(\mathfrak{g}) = V^k(\mathfrak{g})/\langle v\rangle$. There is a one-to-one correspondence between the irreducible $A(\widetilde{V}_k(\mathfrak{g}))$-modules in the category $\mathcal{O}^{\mathfrak{g}}$ and the weights $\mu \in \mathfrak{h}^*$ such that $p(\mu) = 0$ for all $p \in \mathscr{P}_v$.*

Define the left-adjoint action on $U(\mathfrak{g})$ by

$$x_L f = [x, f] \text{ for } x \in \mathfrak{g} \text{ and } f \in U(\mathfrak{g}). \tag{6}$$

This action extends to $U(\mathfrak{g})$ and we still denote it by $x_L f$ for $x \in U(\mathfrak{g})$ and $f \in U(\mathfrak{g})$.

2.3 Associated variety

As in the introduction, let X_V be the *associated variety* [Ar1] of a vertex algebra V, that is the reduced scheme associated with the *Zhu C_2-algebra* of V

$$R_V := V/C_2(V),$$

with $C_2(V) = \mathrm{span}_{\mathbb{C}}\{a_{(-2)}b \colon a, b \in V\}$. In the case that V is a quotient of $V^k(\mathfrak{g})$, $V/C_2(V) = V/\mathfrak{g}[t^{-1}]t^{-2}V$ and we have a surjective Poisson algebra homomorphism

$$\mathbb{C}[\mathfrak{g}^*] = S(\mathfrak{g}) \longrightarrow\!\!\!\!\rightarrow R_V = V/\mathfrak{g}[t^{-1}]t^{-2}V, \quad x \mapsto \overline{x(-1)} + \mathfrak{g}[t^{-1}]t^{-2}V, \tag{7}$$

where $\overline{x(-1)}$ denotes the image of $x(-1)$ in the quotient R_V. Then X_V is just the zero locus of the kernel of the above map in $\mathfrak{g}^*$. It is a G-invariant and conic subvariety of $\mathfrak{g}^*$, with G the adjoint group of $\mathfrak{g}$. As for the characteristic variety, identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ through $(-|-)$, we view it as a subset of $\mathfrak{g}$.

For $V = V^k(\mathfrak{g})$, we get

$$R_{V^k(\mathfrak{g})} \cong S(\mathfrak{g})$$

under the algebra isomorphism (7). For $v \in V^k(\mathfrak{g})$, denote by v'' the image of $\overline{v}$ in $S(\mathfrak{g})$ by the above isomorphism. If v is a singular vector of $V^k(\mathfrak{g})$, then

$$R_{\widetilde{V}_k(\mathfrak{g})} \cong S(\mathfrak{g})/I_M,$$

where M is the $\mathfrak{g}$-module generated by v'' under the adjoint action, and I_M is the ideal of $S(\mathfrak{g})$ generated by M.

It will be also useful to consider the *Chevalley projection map*

$$\Psi\colon S(\mathfrak{g})^{\mathfrak{h}} \to S(\mathfrak{h}), \tag{8}$$

where $S(\mathfrak{g})^{\mathfrak{h}} = \{x \in S(\mathfrak{g})\colon [h,x]=0 \text{ for all } h \in \mathfrak{h}\}$. This is the restriction to $S(\mathfrak{g})^{\mathfrak{h}}$ of the projection map from $S(\mathfrak{g})$ to $S(\mathfrak{h})$ with respect to the decomposition $S(\mathfrak{g}) = S(\mathfrak{h}) \oplus (\mathfrak{n}_- S(\mathfrak{g}) + S(\mathfrak{g})\mathfrak{n}_+)$.

2.4 Affine $\mathcal{W}$-algebras

For a nilpotent element f of $\mathfrak{g}$, let $\mathcal{W}^k(\mathfrak{g},f)$ be the universal $\mathcal{W}$-algebra associated with $(\mathfrak{g},f)$ at level k, defined by the generalized quantized Drinfeld–Sokolov reduction [FF, KRW]:

$$\mathcal{W}^k(\mathfrak{g},f) = H^0_{DS,f}(V^k(\mathfrak{g})),$$

where $H^0_{DS,f}(M)$ is the corresponding BRST cohomology with coefficient in a $\widetilde{\mathfrak{g}}$-module M. We have a natural Poisson algebra isomorphism $R_{\mathcal{W}^k(\mathfrak{g},f)} \cong \mathbb{C}[\mathscr{S}_f]$, where $\mathscr{S}_f = f + \mathfrak{g}^e$, with $\mathfrak{g}^e = \{x \in \mathfrak{g}\colon [x,e]=0\}$, is the Slodowy slice associated with an $\mathfrak{sl}_2$-triple (e,h,f) [DK, Ar2]. It follows that

$$X_{\mathcal{W}^k(\mathfrak{g},f)} \cong \mathscr{S}_f.$$

Let $\mathcal{W}_k(\mathfrak{g},f)$ be the unique simple quotient of $\mathcal{W}^k(\mathfrak{g},f)$. Then $X_{\mathcal{W}_k(\mathfrak{g},f)}$ is a $\mathbb{C}^*$-invariant closed Poisson subvariety of $\mathscr{S}_f$. Let $\mathcal{O}_k$ be the category $\mathcal{O}$ of $\widetilde{\mathfrak{g}}$ at level k. We have a functor

$$\mathcal{O}_k \to \mathcal{W}^k(\mathfrak{g},f)\text{-Mod}, \quad M \mapsto H^0_{DS,f}(M),$$

where $\mathcal{W}^k(\mathfrak{g},f)$-Mod denotes the category of $\mathcal{W}^k(\mathfrak{g},f)$-modules. According to [Ar2], for any quotient V of $V^k(\mathfrak{g})$, $X_{H^0_{DS,f}(V)}$ is isomorphic, as a Poisson variety, to the intersection $X_V \cap \mathscr{S}_f$. In particular, $H^0_{DS,f}(V) \neq 0$ if and only if $\overline{G.f} \subset X_V$ and $H^0_{DS,f}(V)$ is lisse if $X_V = \overline{G.f}$.

3 Main results

We have the following theorems:

Theorem A. *There is a singular vector v_{sing} of $V^{-2}(G_2)$ of weight $-2\Lambda_0 + 4\varpi_1 - 6\delta$. In particular, v_{sing} has conformal weight six and there is no singular vector of conformal weight strictly smaller than six. The quotient vertex algerbra $\widetilde{V}_{-2}(G_2) = V^{-2}(G_2)/\langle v_{\text{sing}}\rangle$ is simple.*

Theorem B. *The associated variety of $L_{-2}(G_2)$ is $\overline{\mathbb{O}}_{\text{sreg}}$.*

Theorem C. *The set $\{L_{G_2}(-2,\mu_i)\colon i=1,\dots,20\}$, where the μ_i's are given by Table 1 , provides the complete list of irreducible $L_{-2}(G_2)$-modules from the category $\mathcal{O}$. Among them, $L_{G_2}(-2,0)$, $L_{G_2}(-2,\varpi_1)$ and $L_{G_2}(-2,\varpi_2)$ are precisely the irreducible ordinary modules of $L_{-2}(G_2)$.*

Theorem D. *The set $\{L_{B_3}(-2,\mu_i)\colon i=1,\dots,13\}$, where the μ_i's are given by Table 2 , provides the complete list of irreducible $L_{-2}(B_3)$-modules from the category $\mathcal{O}$. Among them, $L_{B_3}(-2,0)$ and $L_{B_3}(-2,\varpi_1)$ are precisely the irreducible ordinary modules for $L_{-2}(B_3)$.*

Theorem E. *We have the following decomposition*

$$L_{D_4}(-2,-2\varpi_1) = L_{B_3}(-2,-2\varpi_1) \oplus L_{B_3}(-2,-3\varpi_1)$$

as $L_{-2}(B_3)$-modules.

μ_1	0	μ_{11}	$-\frac{1}{3}\varpi_2$
μ_2	ϖ_1	μ_{12}	$-\frac{2}{3}\varpi_2$
μ_3	ϖ_2	μ_{13}	$-\frac{3}{2}\varpi_1 + \frac{1}{2}\varpi_2$
μ_4	$-2\varpi_1$	μ_{14}	$-\frac{1}{2}\varpi_1 - \frac{1}{2}\varpi_2$
μ_5	$-3\varpi_1$	μ_{15}	$\varpi_1 - \frac{3}{2}\varpi_2$
μ_6	$-\varpi_2$	μ_{16}	$\varpi_1 - \frac{4}{3}\varpi_2$
μ_7	$-2\varpi_2$	μ_{17}	$\varpi_1 - \frac{2}{3}\varpi_2$
μ_8	$\varpi_1 - 2\varpi_2$	μ_{18}	$2\varpi_1 - \frac{5}{3}\varpi_2$
μ_9	$-\frac{1}{2}\varpi_1$	μ_{19}	$2\varpi_1 - \frac{4}{3}\varpi_2$
μ_{10}	$-\frac{3}{2}\varpi_1$	μ_{20}	$3\varpi_1 - \frac{5}{2}\varpi_2$

Table 1: The weights μ_i for G_2

μ_1	0	μ_8	$-\frac{5}{2}\varpi_2 + 3\varpi_3$
μ_2	ϖ_1	μ_9	$-\frac{3}{2}\varpi_2 + \varpi_3$
μ_3	$-2\varpi_1$	μ_{10}	$-\frac{1}{2}\varpi_1 - \frac{1}{2}\varpi_2$
μ_4	$-3\varpi_1$	μ_{11}	$-\frac{3}{2}\varpi_1$
μ_5	$-\varpi_2$	μ_{12}	$-\frac{1}{2}\varpi_1$
μ_6	$-2\varpi_3$	μ_{13}	$-\frac{3}{2}\varpi_1 + \frac{1}{2}\varpi_2$
μ_7	$\varpi_1 - 2\varpi_2$		

Table 2: The weights μ_i for B_3

4 future works

Our result is also interesting in the context of orbifold vertex algebras. $\overline{\mathbb{O}}_{\text{sreg}}$ is the orbifold of $\overline{\mathbb{O}}_{\text{min}}$ by the symmetric group $\mathfrak{S}_3$, the group of Dynkin diagram automorphisms of D_4. As mentioned before, according to [AP], we have $L_{-2}(G_2) = L_{-2}(D_4)^{\mathfrak{S}_3}$.Hence, Theorem B can be reformulated as follows:

$$X_{V^{\mathfrak{S}_3}} = X_V/\mathfrak{S}_3,$$

for $V := L_{-2}(D_4)$. In general, there is no reason for an arbitrary vertex algebra V acted by a finite group G that $X_{V^G} = X_V/G$. We have the following conjecture:

Conjecture A. *Let $V = \bigoplus_{n\geqslant 0} V_n$ be a simple positively graded quasi-lisse vertex algebra such that $V_0 \cong \mathbb{C}$ and G a finite solvable automorphism group of V, then V^G is also quasi-lisse.*

It should be helpful to understand this conjecture if we can construct many explicit examples. It is also interesting to study the representation theory of $L_k(\mathfrak{g})$ and $W_k(\mathfrak{g}, f)$ from higher rank $4d$ theories.

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