Bounding the parameter β of a distance-regular graph with classical parameters

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Abstract

Let Γ be a distance-regular graph with classical parameters (D, b, α, β) satisfying $b \geq 2$ and $D \geq 3$. Let $r = 1 + b + b^2 + \cdots + b^{D-1}$. In 1999, K. Metsch showed that there exists a positive constant $C(\alpha, b)$ only depending on α and b, such that if $\beta \geq C(\alpha, b)r^2$, then either Γ is a Grassmann graph or a bilinear forms graph.

In this work, we give an outline for that for $b \geq 2$ and $D \geq 3$, then there exists a constant $C_1(\alpha, b)$ only depending on α and b, such that if $\beta \geq C_1(\alpha, b)r$, then either Γ is a Grassmann graph, or a bilinear forms graph.

Key Words: distance-regular graphs, classical parameters, geometric, eigenvalues, cliques

1 Introduction

For undefined notions and more details see [1] and [5]. This report is based on arXiv:2410.22994. The second author gave a lecture on this material on December 17, 2024 in Kyoto University. We introduce distance-regular graphs with classical parameters in the next section. In this paper, we study distance-regular graphs with classical parameters (D, b, α, β). It is known that the parameter b is an integer such that $b \neq 0, -1$ (Proposition 3). C. Weng [17] classified the distance-regular

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graphs with classical parameters (D, b, α, β) and $b \leq -2$ under weak conditions, see Remark 2 below. In this paper we consider the distance-regular graphs with classical parameters (D, b, α, β) and $b \geq 1$. The distance-regular graphs with classical parameters (D, b, α, β) and $b = 1$ were classified by Terwilliger, see [1, Theorem 6.1.1]. In [13], Koolen et al. showed that for $b \geq 2$ and D much larger than b, the parameter α is bounded by $b(b+1)^2 + 1$. This was improved in [11], where Koolen et al. showed that for $b \geq 2$ and $D \geq 9$, the parameter α is bounded by $b^2(b+1) + 1$. Moreover, they showed that for $b = 2$ and $D \geq 12$, the parameter α is bounded by 2. Let $r = 1 + b + \cdots + b^{D-1}$. In [14, Corollary 1.3], Metsch showed that if $b \geq 2$ and $\beta \geq C(\alpha, b)r^2$, then the graph must be a bilinear forms graph or a Grassmann graph, where $C(\alpha, b)$ is a constant only depending on α and b. Moreover, in the same paper, he showed that, if $\alpha = b - 1$, then $\beta < \frac{1}{2b-1}(2b^4 + 2b^3 + 2b^2 + b - 1)r$ or the graph is a bilinear forms graph, and if $\alpha = b$, then $\beta < \frac{8}{3}(b^2 + 2b)r$ or the graph is a Grassmann graph.

In this paper we give an outline how to show that if $b \geq 2$ and $\beta \geq C_1(\alpha, b)r$, where $C_1(\alpha, b)$ is a constant only depending on α and b, then the graph must be a bilinear forms graph, or a Grassmann graph.

Our main result is the following.

Theorem 1 *Let Γ be a distance-regular graph with classical parameters (D, b, α, β), such that $b \geq 1$ and $D \geq 3$. Let $r = 1 + b + b^2 + \cdots + b^{D-1}$. Then one of the following holds:*

(1) Γ is a Johnson graph;

(2) Γ is a Hamming or a Doob graph;

(3) Γ is a halved cube;

(4) Γ is the Gosset graph with intersection array $\{27, 10, 1; 1, 10, 27\}$;

(5) Γ is a Grassmann graph;

(6) Γ is a bilinear forms graph;

(7) $b \geq 2$ and $\beta < \max\{2\frac{b+2}{2b+3}r(b+2)(\alpha b + b + \alpha), 2\frac{b+2}{2b+3}r((b+1)(b^2+b+2) - 3)\}$.

Remark 1 *(1) For $\alpha \in \{b - 1, b\}$, [14, Corollary 1.3] gives slightly better bounds.*

(2) For $\alpha = b = 2$, the bound of [14, Corollary 1.3] was slightly improved in [12].

(3) In [7], they showed that if a distance-regular graph with classical parameters (D, b, α, β) satisfies $\alpha = b \geq 2$, $D \geq 9$ and $\beta = b(b^{D-1} + b^{D-2} \ldots + 1)$, then the graph has to be the Grassmann graph $J_b(2D, D)$.

(4) In [6], they showed that if a distance-regular graph with classical parameters (D, b, α, β) satisfies $\alpha + 1 = b = 2$, $D \geq 3$ and $\beta = b^D - 1$, then the graph has to be the bilinear forms graph $H_2(D, D)$.

(5) For $b \geq 2$, the twisted Grassmann graphs as found by Van Dam and Koolen in [4] are not geometric and have $\alpha = b$ and $\beta = b^2r + b$. This shows that for $\beta = b^2r + b$ we can have Item (7) of Theorem 1, and hence the bound in Item (7) is quite tight.

Our proof closely follows the ideas of Metsch as given in [14]. We do not assume that α is an integer. We obtain this once we show that the distance-regular graph is geometric.

Outline of the paper: In the next section we give preliminaries and definitions. In Section 3, we discuss partial linear spaces. In this section we give a sufficient condition for that a distance-regular graph is the point graph of a partial linear space, due to Metsch. For the point graph of a partial linear space, we propose the PLS(γ) and SPLS(c, s) properties in Section 4, and give sufficient conditions for a distance-regular graph to satisfy these properties. Also, in this section, we give some structural results for a distance-regular graph satisfying these properties. In Section 5, we give sufficient conditions for a distance-regular graph to be geometric. In Section 6, we introduce the ELS property and establish the existence of geodetically closed strongly regular subgraphs. In the last section we give the proof of the Theorem 1.

2 Definitions and preliminaries

All the graphs considered in this paper are finite, undirected and simple. The reader is referred to [1, 5] for more information. Let Γ be a connected graph with vertex set $V(\Gamma)$. The *distance* $d(x, y)$ between two vertices $x, y \in V(\Gamma)$ is the length of a shortest path between x and y in Γ. The *diameter* $D = D(\Gamma)$ of Γ is the maximum distance between any two vertices of Γ. For each $x \in V(\Gamma)$, let $\Gamma_i(x)$ be the set of vertices in Γ at distance i from x ($0 \leq i \leq D$). In addition, define $\Gamma_{-1}(x) = \Gamma_{D+1}(x) = \emptyset$. For the sake of simplicity, we denote $\Gamma_1(x)$ by $\Gamma(x)$. For any vertex x of Γ, the subgraph induced on $\Gamma(x)$ is called the *local graph* of Γ at x, and we denote it by Δ_x. For a vertex x of Γ, the cardinality $|\Gamma(x)|$ of $\Gamma(x)$ is called the *valency* of x in Γ. In particular, Γ is *regular* with valency k if $k = |\Gamma(x)|$ holds for all $x \in V(\Gamma)$. We also define for vertices x, y of Γ at distance i ($i = 1, 2, \ldots, D$) the set $C_i(x, y)$ by $C_i(x, y) := \Gamma_{i-1}(x) \cap \Gamma(y)$. For vertices u, v of Γ at distance j ($j = 0, 1, \ldots, D - 1$), we define the set $B_j(u, v)$ by $B_j(u, v) := \Gamma_{j+1}(u) \cap \Gamma(v)$.

The *adjacency matrix* $A = A(\Gamma)$ of Γ is the matrix whose rows and columns are indexed by vertices of Γ and the (x, y)-entry is 1 whenever x and y are adjacent and 0 otherwise. The *eigenvalues* of Γ are the eigenvalues of its adjacency matrix A.

A *clique* of a graph Γ is a set of mutually adjacent vertices of Γ. We sometimes also refer to a complete subgraph of Γ as a clique. Let $K_{m,n}$ be the complete bipartite graph with m and n be the number of vertices in each partition.

A connected graph Γ is called *edge-regular* with parameters (n, k, a) if Γ has n vertices, is k-regular and any two adjacent vertices have exactly a common neighbours.

A connected graph Γ is called *amply-regular* with parameters (n, k, a, c) if Γ is edge-regular with parameters (n, k, a) and any two vertices at distance 2 have exactly c common neighbours.

2.1 Distance-regular graphs

A graph Γ is called *distance-regular* if there exist integers b_i, c_i $(0 \leq i \leq D)$ such that for any two vertices $x, y \in V(\Gamma)$ with $d(x, y) = i$, there are precisely c_i neighbors of y in $\Gamma_{i-1}(x)$ and b_i neighbors of y in $\Gamma_{i+1}(x)$, where we define $b_D = c_0 = 0$. Note that in this case $C_i(x, y)$ contains exactly c_i vertices for $i = 1, 2, \ldots, D$ and $B_i(x, y)$ contains exactly b_i vertices for $i = 0, 1, \ldots, D-1$. In particular, any distance-regular graph is regular with valency $k := b_0$. We define $a_i := k - b_i - c_i$ for notational convenience. Note that $a_i = |\Gamma(y) \cap \Gamma_i(x)|$ holds for any two vertices x, y with $d(x, y) = i$ $(0 \leq i \leq D)$ and that the numbers a_i, b_i and c_i $(0 \leq i \leq D)$ are called the *intersection numbers* of Γ. For an eigenvalue θ of Γ, the sequence $(u_i)_{i=0,1,\ldots,D} = (u_i(\theta))_{i=0,1,\ldots,D}$ satisfying $u_0 = u_0(\theta) = 1$, $u_1 = u_1(\theta) = \theta/k$, and

$$c_i u_{i-1} + a_i u_i + b_i u_{i+1} = \theta u_i \ (i = 1, 2, \ldots, D-1) \tag{1}$$

is called the *standard sequence* corresponding to the eigenvalue θ (see [1, p.128]).

The following lemma is called the Delsarte bound. For a set T of vertices of a connected graph and a vertex x, we define the distance $d(x, T)$ by $d(x, T) = \min\{d(x, y) \mid y \in T\}$.

Lemma 2 (cf. [1, Proposition 4.4.6]) *Let Γ be a distance-regular graph with diameter $D \geq 2$, valency k and smallest eigenvalue $\theta_{\min}$, say with multiplicity m. Let $(u_i)_i$ be the standard sequence corresponding to $\theta_{\min}$. Then the order c of a clique C in Γ is bounded by $c \leq 1 + \frac{k}{-\theta_{\min}}$.*

Assume that $c = 1 + \frac{k}{-\theta_{\min}}$. Then any vertex of Γ has distance at most $D-1$ to C. Let x be a vertex with $d(x, C) = j$ for some $1 \leq j \leq D-1$. Then the number of vertices $\phi_j(x)$ in C at distance j from x satisfies:

$$(u_{j+1} - u_j)\phi_j(x) = (1 + \frac{k}{-\theta_{\min}})u_{j+1}, \tag{2}$$

and hence $\phi_j(x)$ does not depend on x, only on j.

A clique of Γ with $1 + \frac{k}{-\theta_{\min}}$ vertices is called a *Delsarte clique*.

A distance-regular graph Γ is called *geometric* if there exists a set of Delsarte cliques $\mathcal{C}$ in Γ such that each edge lies in a unique $C \in \mathcal{C}$. Let Γ be a geometric distance-regular graph with respect to $\mathcal{C}$. Let C be a Delsarte clique and x be a vertex at distance j from C for $0 \leq j \leq D-1$. Define ϕ_j be the number of vertices in C at distance j from x. For two vertices x and y at distance j for $1 \leq j \leq D$, let τ_j be the number of cliques in $\mathcal{C}$ containing x at distance $j-1$ from y. The number τ_j $(j = 1, 2, , \ldots, D)$ does not depend on the pair x, y, only at their distance.

We can express the intersection numbers of Γ in terms of the ϕ_js and τ_js as follows (see [10, Lemma 4.1]):

$$c_i = \tau_i \phi_{i-1} \ (i = 1, 2, \ldots, D), \quad b_i = -(\theta_{\min} + \tau_i)(1 + \frac{k}{-\theta_{\min}} - \phi_i) \ (i = 1, 2, \ldots, D - 1). \tag{3}$$

2.2   Distance-regular graphs with classical parameters

A distance-regular graph Γ is said to have *classical parameters* (D, b, α, β) if the diameter of Γ is D and the intersection numbers of Γ can be expressed as follows:

$$b_i = ([D] - [i])(\beta - \alpha[i]), \quad 0 \le i \le D - 1,$$

$$c_i = [i](1 + \alpha[i - 1]), \quad 1 \le i \le D,$$

where

$$[j] = \begin{cases} \frac{b^j - 1}{b - 1} & \text{if } b \neq 1, \\ \binom{j}{1} & \text{if } b = 1. \end{cases}$$

From [1, Corollary 8.4.2], we know that the eigenvalues of Γ are

$$[D - i](\beta - \alpha[i]) - [i] = \frac{b_i}{b^i} - [i], \quad 0 \le i \le D.$$

We note that $c_2 = (b+1)(\alpha+1)$ and that if $b \ge 1$, then the eigenvalues $\theta_i = \frac{b_i}{b^i} - [i](0 \le i \le D)$ of Γ are in the natural ordering, i.e., $k = \theta_0 > \theta_1 > \cdots > \theta_D$.

From the following result we know that the parameter b of a distance-regular graph with classical parameters (D, b, α, β) is an integer.

Proposition 3 (cf. [1, Proposition 6.2.1]) *Let Γ be a distance-regular graph with classical parameters (D, b, α, β) and diameter $D \ge 3$. Then b is an integer such that $b \neq 0, -1$.*

The following lemma is an easy consequence of Proposition 3.

Lemma 4 *Let Γ be a distance-regular graph with classical parameters (D, b, α, β) and diameter $D \ge 3$, such that $b \ge 1$. Then $\alpha \ge 0$.*

The members of many of the infinite families of the known distance-regular graphs with unbounded diameter have classical parameters, like the Johnson graphs, Hamming graphs, Grassmann graphs and the bilinear forms graphs where the parameter b is positive.

Remark 2 *(1) For $b \le -2$, C. Weng [17] classified the distance-regular graphs with classical parameters (D, b, α, β) and $b \le -2$. His result states: If $D \ge 4$, $a_1 \neq 0$ and $c_2 > 1$, then one of the following holds: (1) Γ is the dual polar graph ${}^2A_{2D-1}(-b)$, (2) Γ is the Hermitian forms graph $Her_{-b}(D)$, (3) $\alpha = \frac{b-1}{2}$, $\beta = -\frac{1+b^D}{2}$ and $-b$ is a power of an odd prime.*

(2) Tian et al. [16] give the current state of art on distance-regular graphs with classical parameters with $b \leq -2$.

Before we look at the Delsarte bound for distance-regular graphs with classical parameters, we will determine the standard sequence for the smallest eigenvalue of such a distance-regular graph. Let $b \geq 2$ be an integer and let $[i] = \frac{b^i-1}{b-1}$ and $r = [D]$, as before.

Now we can calculate the ϕ_j's and τ_j's of Equation (3) for a geometric distance-regular graph with classical parameters.

Lemma 5 *Let Γ be a distance-regular graph with classical parameters (D, b, α, β) such that $b \geq 2$ and $D \geq 3$, and assume that Γ is geometric. Then the numbers ϕ_j's and τ_j's of Equation (3) satisfy*

$$\phi_j = 1 + \alpha[j], \quad (j = 0, 1, \ldots, D - 1), \quad \tau_j = [j] \quad (j = 1, 2, \ldots, D),$$

where $[j] = \frac{b^j-1}{b-1}$.

For the case $\alpha = 0$ we have the following.

Proposition 6 *Let Γ be a distance-regular graph with classical parameters (D, b, α, β) such that $b \geq 1$, $\alpha = 0$, $\beta > 1$ and $D \geq 3$. If Γ is geometric, then Γ is a dual polar graph or a Hamming graph.*

2.3 Strongly regular graphs

A graph Γ is called a *strongly regular graph* with parameters (n, k, λ, μ) if Γ has $n \geq 2$ vertices, is k-regular and any two distinct vertices have λ (resp. μ) common neighbours depending on whether they are adjacent or not. For more information and for proofs of some the results we mention without proof see [9, Chapter 10] and [2].

If Γ or its complement is disconnected, we say Γ is *imprimitive*, and *primitive* otherwise. Note that the only connected imprimitive non-complete strongly regular graphs are the complete multipartite graphs $K_{m \times s}$ with $m \geq 2, s \geq 2$. It is known that a connected non-complete k-regular graph is strongly regular if and only if it has exactly three distinct eigenvalues $k > \sigma_1 > \sigma_2$. Note that a non-complete connected strongly regular graph with parameters (n, k, λ, μ) has non-integral eigenvalues only if $n = 4\mu + 1, k = 2\mu$, and $\lambda = \mu - 1$.

We say that a connected non-complete strongly regular graph with parameters (n, k, λ, μ) has *classical parameters* (b, α, β) if $k = \beta(b + 1)$, $\lambda = \beta - 1 + \alpha b$ and $\mu = (b + 1)(\alpha + 1)$. Note that a distance-regular graph with classical parameters $(2, b, \alpha, \beta)$ is just a strongly regular graph with classical parameters (b, α, β). Also note that any connected non-complete strongly regular graph with distinct eigenvalues $k > \sigma_1 > \sigma_2$ has classical parameters (b, α, β) where $b + 1 \in \{-\sigma_1, -\sigma_2\}$.

We now reformulate Theorem 8.6.3 of [2] in terms of terms of (b, α, β). Note that, if $b > 0$, then the smallest eigenvalue is equal to $-b - 1$. We refer to [2, Sections 8.4.2, 8.6] for the definition of the block graph of a 2-$(v, k, 1)$-design and the Latin square graph $LS_m(n)$.

Theorem 7 *Let b be an integer at least 2. Let Γ be a primitive strongly regular graph with classical parameters (b, α, β). Let $f(b, \alpha) = \frac{1}{2}b(b+1)((\alpha+1)(b+1)+1) + \alpha$. Then $\beta \leq f(b, \alpha)$ or one of the following hold:*

(1) $\alpha = b$ and Γ is the block graph of a 2-$(b\beta + b + 1, b + 1, 1)$-design;

(2) $\alpha = b - 1$ and Γ is a Latin square graph $LS_{b+1}(\beta + 1)$.

3 Partial linear spaces

An *incidence structure* is a tuple $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ where $\mathcal{P}$ and $\mathcal{L}$ are non-empty disjoint sets and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{L}$. The elements of $\mathcal{P}$ and $\mathcal{L}$ are called *points* and *lines*, respectively. If $(p, \ell) \in \mathcal{I}$, then we say that p is *incident* with ℓ, or that p is on the line ℓ. The *order of a point* is the number of lines it is incident with and similarly for lines. The *point-line incidence matrix* of $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is the $|\mathcal{P}| \times |\mathcal{L}|$-matrix such that the (p, ℓ)-entry is 1 if p is incident with ℓ and 0 otherwise. If it is clear from the context what the incidence relation is, then we omit it.

A *partial linear space* is an incidence structure such that each pair of distinct points are both incident with at most one line.

Let $X = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a partial linear space. For a non-incident pair (x, ℓ) of a point x and line ℓ, let $\tau(x, \ell)$ be the number of points on ℓ that are collinear with x. We define $\tau(X)$ for a partial linear space $X = (\mathcal{P}, \mathcal{L}, \mathcal{I})$, as the number $\tau(X) = \max\{\tau(x, \ell) \mid x \in \mathcal{P}, \ell \in \mathcal{L}, \text{ and } (x, \ell) \notin \mathcal{I}\}$.

The *point graph* Γ of an incidence structure $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ is the graph with vertex set $\mathcal{P}$ and two distinct points are adjacent if and only if they are on a common line. Note that lines are cliques in Γ. If the point graph is connected, then for a vertex x and a line ℓ, we define the distance $d(x, \ell)$ by $d(x, \ell) = \min\{d(x, y) \mid y \in \ell\}$. For two vertices x, y at distance 2 in Γ, we denote by $[x, y]$ the set of lines ℓ through x such that $d(y, u) \leq 2$ for all $u \in \ell$. By Lemma 2, the set of lines $[x, y]$ are exactly the lines through x at distance 1 from y for a geometric distance-regular graph.

Now we recall the following result of Metsch.

Theorem 8 (cf. [14, Result 2.1]) *Let Γ be an amply-regular graph with parameters (n, k, λ, μ). Assume that there exists a positive integer s such that the following two conditions are satisfied:*

(1) $(s+1)(\lambda+1) - k > (\mu-1)\binom{s+1}{2}$;

(2) $\lambda + 1 > (\mu - 1)(2s - 1)$.

Define a line as a maximal clique with at least $\lambda + 2 - (\mu - 1)(s - 1)$ vertices. Then $X = (V(\Gamma), \mathcal{L}, \in)$ is a partial linear space, where $\mathcal{L}$ is the set of all lines, and Γ is the point graph of X. Moreover, each vertex lies on at most s lines.

4 The SPLS(s) Property

We say that a distance-regular graph Γ has the *PLS(γ)* property for some $\gamma \geq 3$ if Γ is the point graph of a partial linear space $X = (V(\Gamma), \mathcal{L}, \in)$, where the set of lines $\mathcal{L}$ are the maximal cliques with at least γ vertices.

We have $\gamma \leq 1 + \frac{k}{-\theta_{\min}}$ where $\theta_{\min}$ is the smallest eigenvalue of Γ, and equality if and only if Γ is geometric.

Let Γ be a distance-regular graph that is the point graph of a partial linear space $X = (V(\Gamma), \mathcal{L}, \in)$. A vertex x of Γ is called *Delsarte vertex* if all the lines in $\mathcal{L}$ through x are Delsarte cliques.

Lemma 9 *Let Γ be a distance-regular graph with classical parameters (D, b, α, β) such that $b \geq 2$ and $D \geq 3$. Assume that Γ satisfies the PLS(γ) property for some $\gamma \geq 3$. Let $r = [D]$. Then a vertex x lies on at least r lines and it is on exactly r lines if and only if it is a Delsarte vertex.*

Now we prepare a lemma that will bound the parameter α of a distance-regular graph with classical parameters (D, b, α, β) such that $b \geq 2$.

Lemma 10 *Let Γ be a distance-regular graph with classical prameters (D, b, α, β) such that $b \geq 2$ and $D \geq 3$. Assume that Γ satisfies the PLS(γ) property for some $\gamma \geq 3$. Assume that Γ contains two distinct Delsarte vertices at distance at most two. Then $\alpha \leq b$ and α is a non-negative integer.*

Now we will strengthen the PLS(γ) property as follows:
We say that a distance-regular graph Γ has the *SPLS(c, s) property* for some integers $c \geq 1$ and $s \geq 2$, if Γ is the point graph of a partial linear space $X = (V(\Gamma), \mathcal{L}, \in)$, where the set of lines $\mathcal{L}$ are the maximal cliques with at least $a_1 + 2 - (c-1)(s-1)$ vertices, $a_1 \geq (2s-1)(c-1)$, each vertex lies on at most s lines and $\tau(X) \leq c$.

If Γ has the SPLS(c, s) property for some integers $c \geq 1$ and $s \geq 2$, then each line has at least $a_1 + 2 - (c-1)(s-1) \geq s(c-1) + 2$ vertices, and clearly $c \leq c_2$ holds. If $c = c_2$, then we write SPLS(s) instead of SPLS(c_2, s).

Note that if a distance-regular graph Γ satisfies the conditions of Theorem 8 for a positive integer s, then it has the SPLS(s) property. Note that the twisted Grassmann graphs $\tilde{J}_q(2D+1, D)$ have the PLS($\frac{q^{D+1}-1}{q-1}$) property (see [4]) , but do not have the SPLS(s) property for any $s \geq 2$. This follows from the Proposition 12, as the twisted Grassmann graphs do not have Delsarte vertices.

Now we give some bounds for s for a distance-regular graph satisfying the SPLS(s) property.

Lemma 11 *Let Γ be a distance-regular graph with valency k and smallest eigenvalue $\theta_{\min}$. If Γ has the SPLS(s) property, then $-\theta_{\min} \leq s$, and if equality holds, then Γ is a geometric distance-regular graph. Let σ be the smallest integer such that Γ has the SPLS(σ) property. Then $\sigma(a_1 + 1) - k \leq (c_2 - 1)\binom{\sigma}{2}$, and $\sigma \geq -\theta_{\min}$.*

Now we give an extra condition on distance-regular graphs with classical parameters that have the SPLS(s) property.

Proposition 12 *Let Γ be a distance-regular graph with classical parameters (D, b, α, β) such that $b \geq 2$ and $D \geq 3$. Let s be a positive integer and assume Γ has the SPLS(s) property. Let $r = [D]$. Then the following hold:*

(1) $s \geq r$;

(2) more than $\frac{c_2 - 2}{c_2 - 1} |V(\Gamma)|$ vertices are Delsarte vertices;

(3) α is an integer such that $0 \leq \alpha \leq b$ and β is a positive integer;

(4) There exists two vertices x, y at distance 2 such that $|[x, y]| \geq b + 1$.

Now we give a sufficient condition for a distance-regular graph with classical parameters to have the SPLS(s) property.

Theorem 13 *Let Γ be a distance-regular graph with classical parameters (D, b, α, β) such that $b \geq 2$ and $D \geq 3$. If $\beta \geq \frac{1}{3}(8\alpha b + 8b + 5\alpha)r$, then Γ has the SPLS($\lfloor \frac{4r}{3} \rfloor$) property. In particular, in this case, α is an integer such that $0 \leq \alpha \leq b$.*

5 Geometric distance-regular graphs

In this section we will give a sufficient condition for a distance-regular graph with classical parameters (D, b, α, β) which has the SPLS(s) property to be geometric.

Our main result in this section is the following and we will give the proof using several claims.

Theorem 14 *Let Γ be a distance-regular graph with classical parameters (D, b, α, β) such that $b \geq 2$ and $D \geq 3$. Let s be a positive integer and assume Γ has the SPLS(s) property. Let $r = [D]$. Assume that β satisfies the following bounds:*

(1) $\beta > (b + 2)(b + 1)(s - 1) - (b(b + 1) + r - 1)\alpha$;

(2) $\beta > ((b + 1)(b^2 + b + 1) - 2)(s - b) - \alpha(b + 1)r + \alpha(b + 1)^2$;

(3) $\beta > ((b + 1)(b^2 + b + 1) - 2 + b)(s - b) - \alpha(r - 1) + b^2 - b$.

Then Γ is geometric.

For geometric distance-regular graphs with classical parameters, we can provide a better bound for β, using the same discussion as before.

Theorem 15 *Let Γ be a geometric distance-regular graph with classical parameters (D, b, α, β) such that $b \geq 2$ and $D \geq 3$. Let $r = [D]$. If $\beta > (r - \alpha - 1)\alpha b + \alpha$, then for each pair of a vertex x and a line ℓ, with $d(x, \ell) = 1$ and vertices y_1, y_2 on ℓ with $d(x, y_1) = d(x, y_2) = 2$, we have that the sets of lines $[x, y_1]$ and $[x, y_2]$ are equal.*

In the next theorem we give a bound on β to insure that a distance-regular graph with classical parameters is geometric.

Theorem 16 *Let Γ be a distance-regular graph with classical parameters (D, b, α, β) such that $b \geq 2$ and $D \geq 3$. Let $r = [D]$. If*

$$\beta \geq \max\{\frac{2b+4}{2b+3}r(b+2)(\alpha b + b + \alpha), \frac{2b+4}{2b+3}r((b+1)(b^2+b+2)-3)\},$$

then Γ is geometric.

As a corollary we have the following consequence for distance-regular graph with classical parameters $(D, b, 0, \beta)$.

Corollary 17 *Let Γ be a distance-regular graph with classical parameters (D, b, α, β), such that $b \geq 2$, $\alpha = 0$ and $D \geq 3$. Let $r = [D]$. Then $\beta < 2\frac{b+2}{2b+3}(b^3 + 2b^2 + 3b - 1)r$ holds.*

6   The ELS property and strongly regular subgraphs

Let Γ be a geometric distance-regular graph with respect to the set $\mathcal{C}$ of Delsarte cliques. We call the elements of $\mathcal{C}$ lines. We say that Γ has the *equal line set* property or the *ELS* property if the following condition holds:
For each pair (x, ℓ), where x is a vertex and ℓ a line with $d(x, \ell) = 1$ and y_1, y_2 be two vertices on ℓ at distance 2 from x, the sets of lines $[x, y_1]$ and $[x, y_2]$ are equal.

Lemma 18 *Let Γ be a geometric distance-regular graph with diameter $D \geq 3$, say with respect to the set of Delsarte cliques $\mathcal{C}$. Assume that Γ satisfies the ELS property. Let x, y be two vertices of Γ at distance 2 in Γ. Let $[x, y] = \{\ell_1, \ell_2, \ldots, \ell_{\tau_2}\}$ and $[y, x] = \{m_1, m_2, \ldots, m_{\tau_2}\}$. Let $z \in \Gamma(y) \cap \Gamma_2(x)$. Then $C_2(x, z) \subseteq \ell_1 \cup \ell_2 \cup \cdots \cup \ell_{\tau_2}$ if and only if there exists i with $1 \leq i \leq \tau_2$ such that $z \in m_i$.*

Theorem 19 *Let Γ be a geometric distance-regular graph, with valency k, smallest eigenvalue θ_{min}, diameter $D \geq 3$ and $\phi_1 \geq 2$. Let $r = -\theta_{\min}$ and $\beta = \frac{k}{-\theta_{\min}}$. Let $\mathcal{C}$ be a set of Delsarte cliques of Γ such that Γ is the point graph of the partial linear space $X = (V(\Gamma), \mathcal{C}, \in)$. Assume that Γ satisfies the ELS property. Then the following hold: Let x, y be a pair of vertices at distance 2 in Γ. Assume that $[x, y] = \{\ell_1, \ell_2, \ldots, \ell_{\tau_2}\}$. Let $\Sigma = \Sigma(x, y)$ be the subgraph induced on $\{x\} \cup \ell_1 \cup \ell_2 \cup \cdots \cup \ell_{\tau_2} \cup \{z \in \Gamma_2(x) \mid [x, y] = [x, z]\}$. Then the graph Σ has the following properties:*

(1) Σ has diameter 2;

(2) Σ is geodetically closed;

(3) If Σ contains two vertices of a Delsarte clique $C \in \mathcal{C}$, then Σ contains all the vertices of C;

(4) Σ is a strongly regular graph with parameters $(\frac{\beta(\beta-\phi_1+1)(\tau_2-1)}{\phi_1} + \beta\tau_2 + 1, \beta\tau_2, \beta - 1 + (\phi_1 - 1)(\tau_2 - 1), \phi_1\tau_2)$.

Remark 3 *Note that the idea of finding the kind of subgraphs as in Theorem 19 goes back to Shult and Yanushka [15] and Brouwer and Wilbrink [3].*

7   Proof of Theorem 1

In this section we give the proof of Theorem 1.

Proof of Theorem 1. Terwilliger classified the distance-regular graphs with classical parameters (D, b, α, β) with $D \geq 3$ and $b = 1$, and showed that they are the graphs in Items (1)-(4), see [1, Theorem 6.1.1].

Therefore we may assume that $b \geq 2$. If $\alpha = 0$, then we are done by Corollary 17. So we assume $\alpha > 0$.

We first consider the case that $\alpha \in \{b, b-1\}$. Assume that Γ is not a Grassmann graph or a bilinear forms graph. In this case, by [14, Corollary 1.3], we have: if $\alpha = b - 1$, then

$$\beta < \frac{1}{2b-1}(2b^4 + 2b^3 + 2b^2 + b - 1)r$$
$$< \max\{\frac{2}{2b+3}r(b+2)^2(\alpha b + b + \alpha), 2\frac{b+2}{2b+3}r((b+1)(b^2 + b + 2) - 3)\};$$

if $\alpha = b$, then

$$\beta < \frac{8}{3}(b^2 + 2b)r$$
$$< \max\{\frac{2b+4}{2b+3}r(b+2)(\alpha b + b + \alpha), \frac{2b+4}{2b+3}r((b+1)(b^2 + b + 2) - 3)\}.$$

From now on, we may assume that

$$\beta \geq \max\{\frac{2}{2b+3}r(b+2)^2(\alpha b + b + \alpha), 2\frac{b+2}{2b+3}r((b+1)(b^2 + b + 2) - 3)\}.$$

By Theorem 16, the graph Γ is geometric and hence it has Delsarte clique. This means that $1 \leq \alpha \leq b-2$ is an integer, as we already have considered $\alpha \in \{0, b-1, b\}$.

Note that Γ is geometric. Let $\mathcal{C}$ be a set of Delsarte cliques of Γ such that Γ is the point graph of the partial linear space $X = (V(\Gamma), \mathcal{C}, \in)$.

By Theorem 15, the graph Γ has the ELS property. By Theorem 19, there exists an induced subgraph Σ of Γ such that Σ is a strongly regular graph with classical parameters (b, α, β). Clearly Σ is not a completely multipartite as the valency $k_\Sigma = (b+1)\beta > \mu(\Sigma) = (b+1)(\alpha+1)$ as the diameter of Γ is at least 3. As $1 \leq \alpha \leq b-2$, we see that $\mu(\Sigma) = (b+1)(\alpha+1) < (b+1)b$. Hence, by Theorem 7, we have $\beta \leq \frac{1}{2}(b+1)b((\alpha+1)(b+1)+1) + \alpha$. This gives a contradiction. This shows the theorem. $\square$

Acknowledgments

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