ON CARTESIAN PRODUCTS OF GRAPHS AND THEIR COHERENT CONFIGURATIONS

JINZHUAN CAI, JIN GUO, ALEXANDER L. GAVRILYUK, AND ILIA PONOMARENKO

Abstract. The coherent configuration $\mathrm{WL}(X)$ of a graph X is the smallest coherent configuration on the vertices of X that contains the edge set of X as a relation. We study $\mathrm{WL}(X)$ when X is a Cartesian product of graphs. The example of a Hamming graph shows that, in general, $\mathrm{WL}(X)$ does not coincide with the tensor product of the coherent configurations of the factors. We prove that if X is "closed" with respect to the 6-dimensional Weisfeiler-Leman algorithm, then $\mathrm{WL}(X)$ is the tensor product of the coherent configurations of certain graphs related to the prime decomposition of X. In addition, we prove that the property of a graph "to be decomposable into a Cartesian product of k connected prime graphs" for some $k \geq 1$ is recognized by the m-dimensional Weisfeiler-Leman algorithm for all $m \geq 6$.

1. Introduction

With every graph X, one can associate a uniquely determined coherent configuration $\mathrm{WL}(X)$ which is the output of the classical Weisfeiler-Leman algorithm [15] and has the same automorphism group as X. One of the main questions in this connection is which natural properties of X can be determined from $\mathrm{WL}(X)$. For example among such properties one can find connectivity, regularity, or acyclicity. On the other hand, whether the planarity of a graph X can be determined from the coherent configuration $\mathrm{WL}(X)$ is currently unknown, see [13].

In order to pose our question more precisely, we will look at it a little more generally. We consider properties of a graph that can be written as a formula in a fragment of the first-order logic with counting quantifiers, see [10]. Those formulas that use at most $m+1$ variables can be seen inside the output[1] $\mathrm{WL}_m(X)$ of the m-dimensional Weisfeiler-Leman algorithm applied to the graph X, see [4]. The output can be thought of as a partition of the Cartesian m-power Ω^m, where Ω is the vertex set of the graph X. For $m = 2$, we have $\mathrm{WL}_m(X) = \mathrm{WL}(X)$, and, moreover, the natural projection $\mathrm{pr}_2 \mathrm{WL}_m(X)$ of $\mathrm{WL}_m(X)$ to the Cartesian square Ω^2 is a coherent configuration which is larger than or equal to $\mathrm{WL}(X)$ (the definition of the partial order of coherent configuration can be found in Subsection 2.3). In fact,

$$\mathrm{WL}(X) = \mathrm{pr}_2 \mathrm{WL}_2(X) < \mathrm{pr}_2 \mathrm{WL}_3(X) < \cdots < \mathrm{pr}_2 \mathrm{WL}_m(X) = \cdots \tag{1}$$

for some positive integer m; here, the final term $\mathrm{pr}_2 \mathrm{WL}_m(X)$ is the coherent configuration associated with the permutation group $\mathrm{Aut}(X)$.

Informally speaking, two graphs X and X' are said to be WL_m-*equivalent* if any formula in the fragment of first-order logic with counting quantifiers, that uses at most $m+1$ variables, is true either for both X and X' or for none of them (an equivalent algebraic definition of this concept is given in Subsection 3.1).

[1] It is an m-ary coherent configuration in the sense of L. Babai, see [14].

Following [2], a property $\mathcal{P}$ of a graph is said to be WL_m-*invariant* if any two WL_m-equivalent graphs satisfy or not the property $\mathcal{P}$ simultaneously, or in other words, the m-dimensional Weisfeiler-Leman algorithm WL_m distinguishes any two graphs if exactly one of them satisfies $\mathcal{P}$. In these terms, our question can be formulated as follows: for a property $\mathcal{P}$ find the minimal m such that $\mathcal{P}$ is WL_m-invariant.

In the present paper, we address the above question for the property "to be decomposable into a nontrivial Cartesian product". Let $X_1 = (\Omega_1, E_1)$ and $X_2 = (\Omega_2, E_2)$ be graphs. The *Cartesian product* $X_1 \,\Box\, X_2$ is a graph with vertex set $\Omega_1 \times \Omega_2$, in which two vertices (α_1, β_1) and (α_2, β_2) are adjacent precisely if

$$\alpha_1 = \alpha_2 \text{ and } (\beta_1, \beta_2) \in E_2 \quad \text{or} \quad (\alpha_1, \alpha_2) \in E_1 \text{ and } \beta_1 = \beta_2.$$

The Cartesian product is one of the basic graph products, which admits a unique prime factorization if the factors are connected. A graph is called prime (under the Cartesian product) if it is not a nontrivial Cartesian product of graphs (see the precise definition in Section 5). The proof of our first main result (Theorem 1.1 below) is based on a well-developed theory of the Cartesian products [11].

Theorem 1.1. *Given a positive integer k, the property of a graph "to be decomposable into a Cartesian product of k connected prime graphs" is* WL_m*-invariant for all $m \geq 6$.*

It is not clear whether the constant 6 in Theorem 1.1 can be reduced. However, the following example shows that the minimal m for which the statement could be true is at least 3. Let X and X' be the Hamming graph $H(2, 4)$, which is the Cartesian product of two complete graphs on 4 vertices, and the Shrikhande graph (see, e.g., [5, Example 2.6.17]), and let $\mathcal{P}$ be the above mentioned property with $k = 2$. Then X satisfies $\mathcal{P}$ whereas X' does not. On the other hand, these graphs are strongly regular and have the same parameters; in particular, they are WL_2-equivalent. Thus, the property $\mathcal{P}$ is not WL_2-invariant.

It would be nice to have a certificate in terms of coherent configurations for the property in Theorem 1.1. An example in Section 5 shows that if X_1 and X_2 are complete graphs of different orders, then

$$\mathrm{WL}(X_1 \,\Box\, X_2) = \mathrm{WL}(X_1) \otimes \mathrm{WL}(X_2).$$

Thus, as such a certificate, one can consider the decomposition of the coherent configuration into a tensor product.

In general, decomposing the coherent configuration of a Cartesian product of graphs into a tensor product is not straightforward. Indeed, assume that we are given $n > 1$ graphs $X_1, \ldots, X_n$ such that any two of them are WL_2-equivalent, and a permutation group $G \leq \mathrm{Sym}(n)$. Then one can define a coherent configuration that we will call the *exponentiation* of the family $\{\mathrm{WL}(X_i)\}_{i=1}^n$ by G, and denote it by $\{\mathrm{WL}(X_i)\}_{i=1}^n \uparrow G$ (see Section 4). If each graph X_i is complete and $G = \mathrm{Sym}(n)$, the exponentiation is equal to the coherent configuration of the Hamming graph, that is indecomposable with respect to the tensor product, though the Hamming graph in our case is just $X_1 \,\Box \cdots \Box\, X_n$.

Our second result gives a sufficient condition for the coherent configuration of a Cartesian product of graphs to be decomposable into tensor product. To formulate the condition, we call the graph X (and also its coherent configuration) WL_m-*closed* ($m \geq 2$) if

$$\mathrm{WL}(X) = \mathrm{pr}_2\, \mathrm{WL}_m(X).$$

Informally, this means that the m-dimensional Weisfeiler-Leman algorithm applied to X results in the same coherent configuration as the 2-dimensional one. By formula (1), this is always true if the coherent configuration $\mathrm{WL}(X)$ is discrete (i.e., the largest coherent configuration on Ω). Since this property is satisfied for almost all graphs (see, e.g., [3]), almost every graph is WL_m-closed for all m. Another natural example of such graphs are the Cai-Fürer-Immerman graphs [4].

Theorem 1.2. *Let $X_1, \ldots, X_n$ be connected prime graphs. Assume that the graph $X_1 \,\square \cdots \square\, X_n$ is WL_6-closed, and denote by $J_1, \ldots, J_a$ the maximal subsets of indices, such that $i, j \in J_k$ for some $1 \le k \le a$ if and only if the graphs X_i and X_j are* WL*-equivalent. Then*

$$\mathrm{WL}(X_1 \,\square \cdots \square\, X_n) = \mathrm{WL}(X_{J_1}) \otimes \cdots \otimes \mathrm{WL}(X_{J_a}) \tag{2}$$

where X_{J_k} is the Cartesian product of the graphs X_j with $j \in J_k$, $k = 1, \ldots, a$. Moreover, for each k,

$$\mathrm{WL}(X_{J_k}) \le \{\mathrm{WL}(X_j)\}_{j \in J_k} \uparrow \mathrm{Sym}(n_k) \tag{3}$$

where $n_k = |J_k|$.

The proof of Theorem 1.2 in Section 7 shows that the condition that the graph $X_1 \,\square \cdots \square\, X_n$ is WL_6-closed can be replaced by a weaker one. Namely, it suffices to assume that the coherent configuration of that graph is 2-closed, see Subsection 3.2.

Two natural questions arising in connection with Theorem 1.2 are as follows: (a) is it possible to replace the condition of the WL_6-closedness by WL_m-closedness with some $m < 6$, and (b) whether the inclusion in (3) is always equality. These questions are also motivated by a problem arising from computing Delsarte's linear programming bound for sum-rank metric codes, see [1]. Unfortunately, in both cases, we have no answer; in particular, [1, Problem 5.1] remains unresolved.

To make the present paper more or less selfcontained, we recall in Section 2 relevant notation and facts about coherent configurations and graphs. In Section 3, we explain in some detail the concept of the WL_m-equivalence and develop some tools to study WL_m-equivalent graphs in an algebraic setting. The exponentiation of a family of coherent configurations by a permutation group is introduced in Section 4. The coherent configurations of the Cartesian products of graphs are studied in Sections 5 and 6. The proofs of Theorems 1.1 and 1.2 are given in Section 7.

2. Coherent configurations and graphs

2.1. **Notation.** Throughout the paper, Ω stands for a finite set. For any $\Delta \subseteq \Omega$, we denote by 1_Δ the diagonal of the Cartesian square $\Delta \times \Delta$. The set of all classes of an equivalence relation e on a subset of Ω is denoted by Ω/e.

For a binary relation $r \subseteq \Omega \times \Omega$, we set $r^* = \{(\beta, \alpha)\colon\ (\alpha, \beta) \in r\}$. The set of all *neighbors* of a point $\alpha \in \Omega$ in the relation r is denoted by $\alpha r = \{\beta \in \Omega\colon\ (\alpha, \beta) \in r\}$. For relations $r, s \subseteq \Omega \times \Omega$, we put

$$r \cdot s = \{(\alpha, \beta)\colon\ (\alpha, \gamma) \in r,\ (\gamma, \beta) \in s \text{ for some } \gamma \in \Omega\},$$

which is called the *dot product* of r and s. For $\Delta, \Gamma \subseteq \Omega$, we set $r_{\Delta,\Gamma} = r \cap (\Delta \times \Gamma)$ (and abbreviate $r_\Delta := r_{\Delta,\Delta}$). For a set S of relations on Ω, we denote by $S^\cup$ the set of all unions of the elements of S and put $S^* = \{s^*\colon s \in S\}$ and $S^f = \{s^f\colon s \in S\}$ for any bijection f from Ω to another set.

2.2. **Coherent configurations: basic definitions.** Let S be a partition of the Cartesian square Ω^2; in particular, the elements of S are treated as binary relations on Ω. A pair $\mathcal{X} = (\Omega, S)$ is called a *coherent configuration* on Ω if the following conditions are satisfied:

(C1) $1_\Omega \in S^\cup$,
(C2) $S^* = S$,
(C3) given $r, s, t \in S$, the number $c_{r,s}^t = |\alpha r \cap \beta s^*|$ does not depend on $(\alpha, \beta) \in t$.

Any relation belonging to S (respectively, $S^\cup$) is called a *basis relation* (respectively, a *relation* of $\mathcal{X}$). The set of all relations is closed with respect to taking the transitive closure and the dot product. A set $\Delta \subseteq \Omega$ is called a *fiber* of $\mathcal{X}$ if the relation 1_Δ is basis. Any union of fibers is called a *homogeneity set* of $\mathcal{X}$.

Let $\mathcal{X} = (\Omega, S)$ and $\mathcal{X}' = (\Omega', S')$ be two coherent configurations. A bijection $f\colon \Omega \to \Omega'$ is called an *isomorphism* from $\mathcal{X}$ to $\mathcal{X}'$ if $S^f = S'$. The isomorphism f induces a natural bijection $\varphi\colon S \to S'$, $s \mapsto s^f$. It preserves the numbers from the condition (C3), namely, the numbers c_{rs}^t and $c_{r^\varphi, s^\varphi}^{t^\varphi}$ are equal for all $r, s, t \in S$. Every bijection $\varphi\colon S \to S'$ having this property is called an *algebraic isomorphism*, written as $\varphi\colon \mathcal{X} \to \mathcal{X}'$. The algebraic isomorphism $\varphi\colon \mathcal{X} \to \mathcal{X}'$ induces a uniquely determined bijection $S^\cup \to S'^\cup$ denoted also by φ. We denote by $\mathrm{id}_\mathcal{X}$ the identity algebraic isomorphism from $\mathcal{X}$ to itself.

2.3. **Coherent configurations: partial order and closure.** There is a partial order $\le$ of the coherent configurations on the same set Ω. Namely, given two such coherent configurations $\mathcal{X}$ and $\mathcal{X}'$, we set $\mathcal{X} \le \mathcal{X}'$ if and only if each basis relation of $\mathcal{X}$ is the union of some basis relations of $\mathcal{X}'$. The minimal and maximal elements with respect to this ordering are the *trivial* and *discrete* coherent configurations: the basis relations of the former one are the reflexive relation 1_Ω and its complement in $\Omega \times \Omega$ (if $|\Omega| \ge 1$), whereas the basis relations of the latter one are singletons.

The *coherent closure* $\mathrm{WL}(r, s, \ldots)$ of some binary relations $r, s, \ldots$ on Ω, is defined to be the smallest coherent configuration on Ω, containing each of them as a relation. When $\{r, s, \ldots\} = S(\mathcal{X}) \cup \{t\}$ for some coherent configuration $\mathcal{X}$ and a relation t, we denote $\mathrm{WL}(r, s, \ldots)$ by $\mathrm{WL}(\mathcal{X}, t)$.

2.4. **Coherent configurations: parabolics.** An equivalence relation e on a set $\Delta \subseteq \Omega$ is called a *partial parabolic* of the coherent configuration $\mathcal{X}$ if e is the union of some basis relations; if, in addition, $\Delta = \Omega$, then e is called a *parabolic* of $\mathcal{X}$. Note that the equivalence closure of any relation of $\mathcal{X}$ is a partial parabolic. If Δ is a class of a partial parabolic, e is a partial parabolic such that $e_\Delta \ne \varnothing$, and $S_{\Delta/e}$ is the set of all non-empty relations

$$s_{\Delta/e} = \{(\Lambda, \Gamma) \in (\Delta/e)^2 : s_{\Delta,\Gamma} \ne \varnothing\},$$

with $s \in S$, then the pair $\mathcal{X}_{\Delta/e} = (\Delta/e, S_{\Delta/e})$ is a coherent configuration called the *restriction* of $\mathcal{X}$ to Δ/e. When $e \subseteq 1_\Omega$, we identify Δ/e with Δ, and put $\mathcal{X}_\Delta = \mathcal{X}_{\Delta/e}$ and $S_\Delta = S_{\Delta/e}$.

A (partial) parabolic e is said to be *decomposable* if e is the union of pairwise disjoint non-empty partial parabolics; we say that e is *indecomposable* if it is not decomposable. Every partial parabolic is a disjoint union of uniquely determined indecomposable partial parabolics, which are called the *indecomposable components* of e.

Any algebraic isomorphism $\varphi\colon \mathcal{X} \to \mathcal{X}'$ induces a bijection between partial parabolics of the coherent configurations $\mathcal{X}$ and $\mathcal{X}'$ that preserves the property of a partial parabolic to be indecomposable [5, Proposition 2.3.25]. Let e be a partial parabolic of $\mathcal{X}$ and $e' = \varphi(e)$. Assume that the classes $\Delta \in \Omega/e$ and $\Delta' \in \Omega'/e'$ are such that φ takes the indecomposable component of e containing Δ as a class to the indecomposable component of e' containing Δ' as a class. Then according to [5, Exercise 2.7.31], φ induces an algebraic isomorphism $\varphi_{\Delta,\Delta'}\colon \mathcal{X}_\Delta \to \mathcal{X}'_{\Delta'}$ such that $\varphi_{\Delta,\Delta'}(s_\Delta) = \varphi(s)_{\Delta'}$ for every $s \in S$. In particular, if $\mathcal{X} = \mathcal{X}'$ and $\varphi = \mathrm{id}_{\mathcal{X}}$, then so is the mapping

$$\mathrm{id}_{\Delta,\Gamma} : S(\mathcal{X}_\Delta) \to S(\mathcal{X}_\Gamma),\ s_\Delta \mapsto s_\Gamma. \tag{4}$$

2.5. **Coherent configurations: tensor product.** Let $\mathcal{X}_1 = (\Omega_1, S_1)$ and $\mathcal{X}_2 = (\Omega_2, S_2)$ be coherent configurations. Denote by $S_1 \otimes S_2$ the set of all tensor products $s_1 \otimes s_2$ with $s_1 \in S_1$ and $s_2 \in S_2$, where

$$s_1 \otimes s_2 = \left\{((\alpha_1,\alpha_2),(\beta_1,\beta_2)) \in (\Omega_1 \times \Omega_2)^2 :\ (\alpha_1,\beta_1) \in s_1,\ (\alpha_2,\beta_2) \in s_2\right\}.$$

The pair $\mathcal{X}_1 \otimes \mathcal{X}_2 = (\Omega_1 \times \Omega_2, S_1 \otimes S_2)$ is a coherent configuration called the *tensor product* of $\mathcal{X}_1$ and $\mathcal{X}_2$. The definition naturally extends to any number of factors. Note that the relations of a tensor product of coherent configurations are just the tensor products of the relations of the factors.

2.6. **Graphs.** By a graph we mean a finite simple undirected graph, i.e., a pair $X = (\Omega, E)$ of a set Ω of vertices and an irreflexive symmetric relation $E \subseteq \Omega \times \Omega$. The elements of $E =: E(X)$ are called *edges*, and E is the *edge set* of the graph X. Two vertices α and β are adjacent (in X) whenever $(\alpha,\beta) \in E$ (equivalently, $(\beta,\alpha) \in E$). The subgraph of X induced by a set $\Delta \subseteq \Omega$ is denoted by $X_\Delta = (\Delta, E_\Delta)$. A graph $X = (\Omega, E)$ is connected if the transitive reflexive closure of E equals Ω^2. The *coherent configuration of a graph* X is just the coherent closure of its edge set: $\mathrm{WL}(X) = \mathrm{WL}(E)$.

3. The Weisfeiler-Leman method

3.1. **The WL_m-equivalence of graphs.** Let $m \geq 1$. Given a coherent configuration $\mathcal{X}$, the m-dimensional Weisfeiler-Leman algorithm constructs a canonical partition $\mathrm{WL}_m(\mathcal{X})$ of the mth Cartesian power Ω^m of the underlying set Ω. The term "canonical" means that for any other coherent configuration $\mathcal{X}'$ on Ω', every isomorphism $f \in \mathrm{Iso}(\mathcal{X},\mathcal{X}')$ naturally extends to a bijection $\Omega^m \to \Omega'^m$ taking every class of $\mathrm{WL}_m(\mathcal{X})$ to a class of $\mathrm{WL}_m(\mathcal{X}')$.

The partition $\mathrm{WL}_m(\mathcal{X})$ is an m-ary coherent configuration in the sense of L. Babai, see [14] and references therein. For $m = 2$, the classes of the partition $\mathrm{WL}_2(\mathcal{X})$ coincide with $S(\mathcal{X})$ and hence ordinary coherent configurations can be treated as 2-ary coherent configurations. In this way, the combinatorial and algebraic isomorphisms naturally generalize to those of m-ary coherent configurations. In the present paper we use m-ary coherent configurations and algebraic isomorphisms between them pure formally, just to define the WL_m-equivalence and to prove Theorem 3.2; all undefined terms and facts about m-ary coherent configurations can be found in [14].

Let $1 \leq i_1 < \ldots < i_k \leq m$ and $K = \{i_1, \ldots, i_k\}$, where $1 \leq k \leq m$. Let

$$\mathrm{pr}_K : \Omega^m \to \Omega^k,\ (x_1, \ldots, x_m) \mapsto (x_{i_1}, \ldots, x_{i_k}),$$

be the natural K-projection. Then $\operatorname{pr}_K \mathrm{WL}_m(\mathcal{X}) = \{\operatorname{pr}_K(\Lambda) : \Lambda \in \mathrm{WL}_m(\mathcal{X})\}$ is a k-ary coherent configuration and also (see [14])

$$\operatorname{pr}_K \mathrm{WL}_m(\mathcal{X}) \geq \mathrm{WL}_k(\mathcal{X}), \tag{5}$$

i.e., each class of the partition on the right-hand side is a union of some classes of the partition on the left-hand side. Furthermore, every algebraic isomorphism $\hat{\varphi} : \mathrm{WL}_m(\mathcal{X}) \to \mathrm{WL}_m(\mathcal{X}')$ induces an algebraic isomorphism

$$\operatorname{pr}_K \hat{\varphi} : \operatorname{pr}_K \mathrm{WL}_m(\mathcal{X}) \to \operatorname{pr}_K \mathrm{WL}_m(\mathcal{X}'),\ \operatorname{pr}_K(\Lambda) \to \operatorname{pr}_K(\hat{\varphi}(\Lambda)), \tag{6}$$

see [14, Lemma 3.4]. When K coincides with $\{1, \ldots, k\}$, we denote pr_K and $\operatorname{pr}_K \varphi$ by pr_k and φ_k, respectively.

Let $m \geq 2$. For any graph X, we put $\mathrm{WL}_m(X) = \mathrm{WL}_m(\mathrm{WL}(X))$. Note that $\mathrm{WL}_2(X) = \mathrm{WL}(X)$. Two graphs X and X' are said to be WL_m-*equivalent* if there exists an algebraic isomorphism $\hat{\varphi} : \mathrm{WL}_m(X) \to \mathrm{WL}_m(X')$ such that $\hat{\varphi}_2(E) = E'$, where $E = E(X)$ and $E' = E(X')$. One can prove that WL_m-equivalent graphs are also WL_k-equivalent for $k \leq m$.

3.2. **The 2-extensions.** Let $\mathcal{X}$ be a coherent configuration on Ω and $\Delta = \mathrm{Diag}(\Omega^2)$. Following [8], the 2-*extension* and 2-*closure* of $\mathcal{X}$ are defined to be, respectively, the coherent configurations

$$\hat{\mathcal{X}} = \mathrm{WL}(\mathcal{X} \otimes \mathcal{X}, 1_\Delta) \quad \text{and} \quad \bar{\mathcal{X}} = (\hat{\mathcal{X}}_\Delta)^\zeta,$$

where $\zeta : \Delta \to \Omega$ is the mapping taking a pair (α, α) to the point α, $\alpha \in \Omega$. It holds true that $\bar{\mathcal{X}} \geq \mathcal{X}$ and if $\bar{\mathcal{X}} = \mathcal{X}$, then $\mathcal{X}$ is said to be 2-*closed*. Examples of 2-closed coherent configurations include all schurian coherent configurations, i.e., those the basis relations of which are the orbits of a permutation group acting on pairs of points. On the other hand, there are many coherent configurations which are not 2-closed, see, e.g., [5, Example 3.5.18].

The basis relations of the coherent configuration $\hat{\mathcal{X}}$ form a partition of the set $\Omega^2 \times \Omega^2$ which is naturally identified with Ω^4. By [5, Theorem 4.6.18], we have

$$\operatorname{pr}_4 \mathrm{WL}_6(\mathcal{X}) \geq S(\hat{\mathcal{X}}). \tag{7}$$

Furthermore, the 2-projection of every reflexive basis relation of $\hat{\mathcal{X}}$ is a basis relation of $\bar{\mathcal{X}}$ (see [5, Theorem 3.5.16]). Since $S(\hat{\mathcal{X}})$ is a partition of $\Omega^2 \times \Omega^2$, this implies that $\operatorname{pr}_2 S(\hat{\mathcal{X}}) \supseteq S(\bar{\mathcal{X}})$. It follows that if the coherent configuration $\mathcal{X}$ is WL_6-closed (see page 2), then inclusion (7) implies that

$$S(\mathcal{X}) = \operatorname{pr}_2 \mathrm{WL}_6(\mathcal{X}) = \operatorname{pr}_2(\operatorname{pr}_4 \mathrm{WL}_6(\mathcal{X})) \geq \operatorname{pr}_2 S(\hat{\mathcal{X}}) \supseteq S(\bar{\mathcal{X}}) \geq S(\mathcal{X}).$$

Thus, $S(\bar{\mathcal{X}}) = S(\mathcal{X})$ and hence the coherent configuration $\mathcal{X} = \bar{\mathcal{X}}$ is 2-closed. Applying this statement to $\mathcal{X} = \mathrm{WL}(X)$, where X is a WL_6-closed graph, we obtain the following lemma.

Lemma 3.1. *Let X be a WL_6-closed graph. Then the coherent configuration $\mathrm{WL}(X)$ is 2-closed.*

In general, it is difficult to determine the basis relations of $\hat{\mathcal{X}}$ explicitly. However, some of its relations are known, for example, if s is a relation of a coherent configuration $\mathcal{X}$ on Ω, then a *cylindrical* relation of the form

$$\mathrm{Cyl}_s(i, j) = \{(x, y) \in \Omega^2 \times \Omega^2 : (x_i, y_j) \in s\} \tag{8}$$

is a relation of $\hat{\mathcal{X}}$ for all indices $i, j \in \{1, 2\}$, see [5, Theorem 3.5.7]. The proof of the following theorem is based on some facts about m-ary coherent configurations; all of them can be found in [14] and [7].

Theorem 3.2. *Let X and X' be two WL_m-equivalent graphs, $m \geq 6$ an integer, and let $\mathcal{X} = \mathrm{WL}(X)$ and $\mathcal{X}' = \mathrm{WL}(X')$. Then there are algebraic isomorphisms $\varphi : \mathcal{X} \to \mathcal{X}'$ and $\hat{\varphi} : \hat{\mathcal{X}} \to \hat{\mathcal{X}}'$ such that*

$$\varphi(E(X)) = E(X') \quad \text{and} \quad \hat{\varphi}(\mathrm{Cyl}_s(i,j)) = \mathrm{Cyl}_{\varphi(s)}(i,j) \tag{9}$$

for all $s \in S(\mathcal{X})^{\cup}$ and all $i, j \in \{1, 2\}$.

4. A generalization of exponentiation

Let $n \geq 1$ be an integer, $\mathcal{X}_i$ a coherent configuration on Ω_i, and $\varphi_{ij} : \mathcal{X}_i \to \mathcal{X}_j$ an algebraic isomorphism, $1 \leq i, j \leq n$. We assume that for all i, j, k,

$$\varphi_{ii} = \mathrm{id}_{\mathcal{X}_i} \quad \text{and} \quad \varphi_{ij}\varphi_{jk} = \varphi_{ik}. \tag{10}$$

In particular, $\varphi_{ij} = \varphi_{ji}^{-1}$ for all i, j. Let $G \leq \mathrm{Sym}(n)$ be a permutation group. For each basis relation $s_1 \otimes \ldots \otimes s_n$ of the coherent configuration $\mathcal{X} = \mathcal{X}_1 \otimes \ldots \otimes \mathcal{X}_n$ and for each permutation $g \in G$, denote by φ_g a permutation of the basis relations of $\mathcal{X}$, such that

$$(s_1 \otimes \cdots \otimes s_n)^{\varphi_g} = \varphi_{j_1 1}(s_{j_1}) \otimes \cdots \otimes \varphi_{j_n n}(s_{j_n}), \tag{11}$$

where $j_1 = 1^{g^{-1}}, \ldots, j_n = n^{g^{-1}}$.

Lemma 4.1. *For each $g \in G$, the bijection φ_g is an algebraic automorphism of the coherent configuration $\mathcal{X}$.*

Now we define the *exponentiation* $\{\mathcal{X}_i\}_{i=1}^n \uparrow G$ of the family of the coherent configurations $\mathcal{X}_i$, $1 \leq i \leq n$, by a permutation group $G \leq \mathrm{Sym}(n)$ to be the algebraic fusion of the tensor product $\mathcal{X}_1 \otimes \ldots \otimes \mathcal{X}_n$ with respect to the group $\{\varphi_g : g \in G\}$. Thus the exponentiation is a coherent configuration each basis relation of which is of the form

$$(s_1 \otimes \cdots \otimes s_n)^G = \bigcup_{g \in G} (s_1 \otimes \cdots \otimes s_n)^{\varphi_g} \tag{12}$$

for some basis relations $s_1, \ldots, s_n$ of the coherent configurations $\mathcal{X}_1, \ldots, \mathcal{X}_n$, respectively. In the special case $\mathcal{X}_i = \mathcal{X}_j$ and $\varphi_{ij} = \mathrm{id}_{\mathcal{X}_i}$ for all i, j, our construction is just the exponentiation of $\mathcal{X}_1$ by G, see [5, Subsection 3.4.2].

5. Cartesian product of graphs

Let $I = \{1, \ldots, n\}$, and let $X_i = (\Omega_i, E_i)$ be a graph, $i \in I$. Following [11], the *Caresian product*

$$X = X_1 \square \cdots \square X_n \tag{13}$$

of $X_1, \ldots, X_n$ is defined to be the graph with vertex set $\Omega = \Omega_1 \times \ldots \times \Omega_n$ and edge set $E = c_1 \cup \cdots \cup c_n$, where for each $i \in I$ we set

$$c_i = c_i(X) = 1_{\Omega_1} \otimes \ldots \otimes 1_{\Omega_{i-1}} \otimes E_i \otimes 1_{\Omega_{i+1}} \otimes \ldots \otimes 1_{\Omega_n}. \tag{14}$$

In what follows, we always assume that the graph X_i is connected for each i. In this case, in a sense of [6], the equivalence relations $\langle c_i \rangle$, $i \in I$, form the standard atomic Cartesian decomposition of a set Ω.

The graph X is called *prime* if whenever equality (13) holds for some graphs $X_1, \ldots, X_n$, each with at least two vertices, it follows that $n = 1$. If the graphs $X_1, \ldots, X_n$ are prime, then we refer to equality (13) as a *prime decomposition* of X. Every connected graph admits a unique prime decomposition up to isomorphisms and order of the factors [11, Theorem 6.6].

Clearly, c_i is a relation of the coherent configuration $\mathrm{WL}(X_1) \otimes \ldots \otimes \mathrm{WL}(X_n)$ for each $i \in I$, Therefore so is the relation $E = c_1 \cup \cdots \cup c_n$. By the minimality of the coherent closure $\mathrm{WL}(X) = \mathrm{WL}(E)$, we conclude that

$$\mathrm{WL}(X) \le \mathrm{WL}(X_1) \otimes \ldots \otimes \mathrm{WL}(X_n). \tag{15}$$

In the following statement, we give a necessary and sufficient condition for equality in this formula.

Lemma 5.1. *Let X be the Cartesian product* (13) *of connected graphs. Then*

$$\mathrm{WL}(X) = \mathrm{WL}(X_1) \otimes \ldots \otimes \mathrm{WL}(X_n) \tag{16}$$

if and only if the equivalence relation $\langle c_i \rangle$ is a parabolic of the coherent configuration $\mathrm{WL}(X)$ *for each $i \in I$.*

Example.[2] Let X_1 and X_2 be complete graphs of different orders $n_1 \ge 3$ and $n_2 \ge 3$, respectively. Then $\mathrm{WL}(X_1 \,\square\, X_2) = \mathrm{WL}(X_1) \otimes \mathrm{WL}(X_2)$. To prove this equality, denote by A_i the adjacency matrix of the relation c_i, $i = 1, 2$. Then

$$(A_i)^2 = (n_i - 2)A_i + (n_i - 1)A_0 \quad \text{and} \quad A_i A_j = A_j A_i, \qquad i, j = 1, 2,$$

where A_0 is the identity matrix of order $n_1 n_2$. Furthermore, the adjacency matrix of the graph X is equal to $A = A_1 + A_2$ and

$$A^2 = A_1^2 + 2A_1A_2 + A_2^2 = (n_1 - 2)A_1 + (n_2 - 2)A_2 + A',$$

where A' is a matrix such that $A' \circ A = 0$ (here, $\circ$ denotes the Hadamard multiplication). According to the Wielandt principle (see [5, Theorem 2.3.10]), the set s_i of all pairs $(\alpha, \beta) \in \Omega^2$ such that $(A^2 \circ A)_{\alpha,\beta} = n_i - 2$ is a relation of the coherent configuration $\mathrm{WL}(X_1 \,\square\, X_2)$ for each i. On the other hand, $s_i = c_i$, because $n_1 \ne n_2$ and $n_1, n_2 \ge 3$ by assumption. Thus, $\langle c_i \rangle$ is a parabolic of $\mathrm{WL}(X_1 \,\square\, X_2)$ and we are done by Lemma 5.1. □

When the graphs in the Cartesian product (13) are pairwise WL-equivalent, the coherent configuration $\mathrm{WL}(X)$ cannot be a tensor product. An easy example is given by the Cartesian product of complete graphs of the same order k. In this case, X is a Hamming graph and the coherent configuration $\mathrm{WL}(X)$ is a Hamming scheme. If $k \ge 3$, then the latter scheme has only trivial parabolics and hence cannot be a nontrivial tensor product.

Lemma 5.2. *Let X be the Cartesian product* (13) *of connected graphs. Assume that the graphs $X_1, \ldots, X_n$ are pairwise* WL*-equivalent. Then*

$$\{\mathrm{WL}(X_1), \ldots, \mathrm{WL}(X_n)\} \uparrow \mathrm{Sym}(n) \ge \mathrm{WL}(X). \tag{17}$$

[2] A more general statement was proved in [1].

6. The 2-extension of Cartesian product

Let X_i, $i \in I$, be connected graphs. The *product relation* $c(X)$ associated with their Cartesian product (13) is a relation on E that consists of all pairs $(u,v) \in E \times E$ for which there exists $i \in I$ such that $u, v \in c_i(X)$, or, equivalently,

$$c(X) = c_1(X)^2 \cup \ldots \cup c_n(X)^2.$$

Thus, the number of classes of the equivalence relation $c(X)$ is equal to the number of factors of the Cartesian product (13).

The product relation admits an explicit description in the case when decomposition (13) is prime. Namely, in accordance with [11, Theorem 23.2], we have

$$c(X) = \langle \Theta(X) \cup \tau(X) \rangle \tag{18}$$

where $\Theta = \Theta(X)$ and $\tau = \tau(X)$ are relations on the edge set E of the graph X that are defined as follows (see [11, Section 23.1]): two pairs $(x,y), (x',y') \in E$ belong to the relation

(1) Θ if and only if $\partial(x,y') + \partial(y,x') \neq \partial(x,x') + \partial(y,y')$,
(2) τ if and only if $\{x\} = \{x'\} = yE \cap y'E$,

where $\partial(\cdot,\cdot)$ is the distance function of the graph X. In turn, these relations can easily be described in terms of the relations (8) and some relations of the coherent configuration $\mathcal{X} = \mathrm{WL}(X)$. Indeed, denote by $s_i = s_i(X)$ the set of all pairs (α,β) at distance i in the graph X. Then

$$\Theta = \bigcap_{a+b \neq c+d} s(a,b,c,d), \tag{19}$$

where given nonnegative integers a, b, c, d, we put

$$s(a,b,c,d) = s(a,b,c,d;X) = \mathrm{Cyl}_{s_a}(1,2) \cap \mathrm{Cyl}_{s_b}(2,1) \cap \mathrm{Cyl}_{s_c}(1,1) \cap \mathrm{Cyl}_{s_d}(2,2).$$

Furthermore, set $s' = s'(X)$ to be the union of all relations $t \in S(\mathcal{X})$ for which the sum $c_{r,s}^t$ over all $r, s \in S(\mathcal{X})$ such that $r, s \subseteq E$, is equal to 1. Then

$$\tau = \mathrm{Cyl}_{1_\Omega}(1,1) \cap \mathrm{Cyl}_{s'}(2,2). \tag{20}$$

Lemma 6.1. *If the decomposition* (13) *is prime, then* $\Theta(X)$ *and* $\tau(X)$ *are relations of the coherent configuration* $\hat{\mathcal{X}}$ *and, in particular,* $c(X)$ *is a partial parabolic of* $\hat{\mathcal{X}}$*.*

In what follows, we always assume that the decomposition (13) is prime. Denote by e the equivalence relation $\langle s \rangle \subseteq E^2$, where $s = (e_1 \cap c) \cup (e_2 \cap c)$ with $e_1 = \mathrm{Cyl}_{1_\Omega}(1,1)$, $e_2 = \mathrm{Cyl}_{1_\Omega}(2,2)$, and $c = c(X)$. Clearly, e is a partial parabolic of the coherent configuration $\hat{\mathcal{X}}$. It is not hard to see that each class of the partial parabolic e is of the form

$$\Delta(\bar{x}, i) = 1_{x_1} \otimes \cdots \otimes 1_{x_{i-1}} \otimes E_i \otimes 1_{x_{i+1}} \otimes \cdots \otimes 1_{x_n}, \tag{21}$$

for some $(n-1)$-tuple $\bar{x} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ and $i \in I$.

Lemma 6.2. *Let* $\Delta = \Delta(\bar{x}, i)$ *and* $\Gamma = \Delta(\bar{y}, j)$ *be classes of the partial parabolic* e*. Assume that the graphs* X_i *and* X_j *are not* WL*-equivalent. Then* Δ *and* Γ *are classes of different indecomposable components of* e*.*

7. Proof of Theorems 1.1 and 1.2

7.1. **Proof of Theorem 1.1.** Let X and X' be WL_m-equivalent graphs, $m \geq 6$. In what follows, $\mathcal{X} = \mathrm{WL}(X)$ and $\mathcal{X}' = \mathrm{WL}(X')$, and φ and $\hat{\varphi}$ are the algebraic isomorphisms from Lemma 3.2. We need to verify that if one of the graphs, say X, is a Cartesian product of k connected prime graphs, then so is the other one, or equivalently, that the product relations $c(X)$ and $c(X')$ have the same number of classes.

By the second part of Lemma 6.1, the product relation $c(X)$ is a partial parabolic of the coherent configuration $\hat{\mathcal{X}}$. Since any partial parabolic has the same number of classes as its image under an algebraic isomorphism, it suffices to verify that

$$\hat{\varphi}(c(X)) = c(X'). \tag{22}$$

By the first part of Lemma 6.1 and formula (18), this is true if $\hat{\varphi}(\Theta(X)) = \Theta(X')$ and $\hat{\varphi}(\tau(X)) = \tau(X')$. On the other hand, the first equality in (9) implies that $\varphi(s_i(X)) = s_i(X')$ for all nonnegative integers i, see [5, Exercise 2.7.55(1)]. By the second equality in (9), this yields $\hat{\varphi}(s(a,b,c,d;X)) = s(a,b,c,d;X')$ for all nonnegative integers a,b,c,d. Thus by formula (19), we obtain

$$\hat{\varphi}(\Theta(X)) = \hat{\varphi}(\bigcap_{a+b\neq c+d} s(a,b,c,d;X)) = \bigcap_{a+b\neq c+d} s(a,b,c,d;X') = \Theta(X').$$

Similarly, $\varphi(s'(X)) = s'(X')$ and using again the second equality in (9), and formula (20), we conclude that

$$\hat{\varphi}(\tau(X)) = \hat{\varphi}(\mathrm{Cyl}_{1_\Omega}(1,1) \cap \mathrm{Cyl}_{s'(X)}(2,2)) = \mathrm{Cyl}_{1_\Omega}(1,1) \cap \mathrm{Cyl}_{s'(X')}(2,2) = \tau(X'),$$

as required.

7.2. **Proof of Theorem 1.2.** We keep the notation of Section 6. To prove equality (2), put $Y_k = X_{J_k}$, $1 \leq k \leq a$. Then $X = Y_1 \,\square \ldots \square\, Y_a$. Denote by $\hat{e}_k$ a partial equivalence relation on Ω^2 with classes $\Delta(\bar{x}, j)$, where $\bar{x}$ is as in formula (21) and $j \in J_k$. Then obviously

$$e = \hat{e}_1 \cup \cdots \cup \hat{e}_a.$$

Let $1 \leq k \leq a$. By the definition of the set J_k, for no two indices $j \in J_k$ and $j' \notin J_k$, the graphs X_j and $X_{j'}$ are WL-equivalent. By Lemma 6.2, this implies that $\hat{e}_k$ is a union of indecomposable components of the parabolic e. Consequently, the support $\Omega(\hat{e}_k)$ of $\hat{e}_k$ is a homogeneity set of the coherent configuration $\hat{\mathcal{X}}$. By [5, Theorem 3.5.16], this implies that $\Omega(\hat{e}_k)$ is a relation of the coherent configuration $\bar{\mathcal{X}}$, and hence of the coherent configuration $\mathcal{X} = \bar{\mathcal{X}}$ (recall that $\mathcal{X}$ is 2-closed by Lemma 3.1). On the other hand,

$$\Omega(\hat{e}_k) = \bigcup_{j\in J_k} c_j(X) := c_{J_k}(X).$$

Thus, $e'_k := \langle c_{J_k}(X)\rangle$ is a parabolic of the coherent configuration $\mathcal{X}$. It is not hard to see that the parabolic e'_k coincides with the parabolic $\langle c_k(X)\rangle$ with respect to the Cartesian product $X = Y_1 \,\square \cdots \square\, Y_a$. Since this is true for all k, we conclude by Lemma 5.1 that

$$\mathrm{WL}(X) = \mathrm{WL}(Y_1) \otimes \ldots \otimes \mathrm{WL}(Y_a).$$

as required. The inclusion in (3) immediately follows from Lemma 5.2.

Acknowledgements. This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

References

1. A. Abiad, A. L. Gavrilyuk, A. P. Khramova, and I. Ponomarenko, *A linear programming bound for sum-rank metric codes*, IEEE Transactions on Information Theory, **71(1)** (2025), 317–329.
2. V. Arvind, F. Fuhlbrück, J. Köbler, and O. Verbitsky, *On Weisfeiler-Leman invariance: subgraph counts and related graph properties*, J. Comput. System Sci., **113** (2020), 42–59.
3. L. Babai, P. Erdös, and S. M. Selkow, *Random graph isomorphism*, SIAM J. Comput., **9** (1980), 628–635.
4. J. Y. Cai, M. Fürer, and N. Immerman, *An optimal lower bound on the number of variables for graph identification*, Combinatorica, **12** (1992), 389–410.
5. G. Chen and I. Ponomarenko, *Coherent configurations*[3], Central China Normal University Press, 2019.
6. G. Chen and I. Ponomarenko, *Tensor products of coherent configurations*, Front. Math. China, **17** (2021), no. 5, 829–852.
7. G. Chen, Q. Ren, and I. Ponomarenko, *On multidimensional Schur rings of finite groups*, J. Group Theory, **27** (2024), no. 1, 61–88.
8. S. Evdokimov and I. Ponomarenko, *On highly closed cellular algebras and highly closed isomorphisms*, Electronic J. Combin., **6** (1999), no. #R18, 1–31.
9. S. Evdokimov and I. Ponomarenko, *Permutation group approach to association schemes*, European J. Combin., **30** (2009), no. 1, 1456–1476.
10. M. Grohe, *Descriptive complexity, canonisation, and definable graph structure theory*, Cambridge University Press, 2017.
11. R. Hammack, W. Imrich, and S. Klavžar, *Handbook of product graphs*, second ed., Discrete Mathematics and its Applications (Boca Raton), CRC Press, Boca Raton, FL, 2011.
12. D. Higman, *Coherent configurations 1*, Rend. Mat. Sem. Padova **44** (1970), 1–25.
13. S. Kiefer, I. Ponomarenko, and P. Schweitzer, *The Weisfeiler–Leman Dimension of Planar Graphs Is at Most* 3, Journal of the ACM, **66** (2019), no. 6, 1–31.
14. I. Ponomarenko, *On the WL-dimension of circulant graphs of prime power order*, Algebraic Combinatorics, **6** (2023), no. 6, 1469–1490.
15. B. Yu. Weisfeiler and A. A. Leman, *The reduction of a graph to canonical form and the algebra which appears therein*, NTI, Ser. 2, **9** (1968), 12–16, English translation is available at https://www.iti.zcu.cz/wl2018/pdf/wl_paper_translation.pdf.

Hainan University, Haikou, China
Email address: caijzh12@163.com

Hainan University, Haikou, China
Email address: guojinecho@163.com

University of Memphis, Tennessee, U.S.A.
Email address: a.gavrilyuk@memphis.edu

Hainan University, Haikou, China; Steklov Institute of Mathematics at St. Petersburg, Russia
Email address: inp@pdmi.ras.ru

[3]An updated electronic version, called "Lectures on Coherent Configurations", is available at http://www.pdmi.ras.ru/~inp/ccNOTES.pdf (Retrieved 2024-11-14).