Optimal Contracts for Off-Exchange Trading with Generalized Price Impact Model*

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Abstract

In financial markets, institutional investors often outsource the execution of large block orders to off-exchange dealers to mitigate market impact costs. We consider principal trading contracts where off-exchange trades are executed at predetermined prices, such as TWAP or the closing price. This arrangement involves an agency conflict: the dealer may have an incentive to manipulate the market as they hedges the off-exchange trade through exchange executions that are unobservable to the client(institutional investor). In this context, the optimal contract is characterized by a principal-agent problem, with the client as principal and the dealer as agent. This study generalizes the price impact model in the principal-agent setting from prior research, analyzing the optimal contract that minimizes the client's execution costs while providing appropriate incentives to the dealer.

1 Introduction

When a investor executes a large order, it typically causes a significant price impact on the market, leading to high transaction costs for the investor. Therefore, large investors often split their large orders into smaller ones to reduce their execution costs. This type of execution strategies has been studied in the context of optimal execution problem, where an investor is assumed to execute their orders by themselves on the exchange. In practice, however, large investors—such as pension funds and other institutions—often delegate their execution to some dealers through a block trade contract. Unlike in the traditional optimal execution problem, the execution agent is the dealer, who generally aims to maximize their own profit under the contract. Under typical block trade contracts, clients and dealers are supposed to trade at a guaranteed benchmark price on the execution day, such as Market-on-Close (MOC), Volume-Weighted Average Price (VWAP) and Time-Weighted Average Price (TWAP). As a result, dealers may have an incentive to manipulate the market through the front-running against the benchmark, which has been pointed out as a serious conflict of interest in practice [3]. To address this issue, [3] first extended optimal execution problem to a optimal block trade contracting problem, which is formulated as a principal-agent problem with clients as principals and dealers as agents. This formulation enabled a mathematical analysis of block trade contracts

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and allowed for optimizing contracts to minimize the client's costs while properly controlling the dealer's incentive.

In [3], the optimality of guaranteed VWAP contracts was proved under the Almgren-Chriss price impact model [2] without permanent price impact. Subsequently,[4] incorporated both temporary and permanent impacts into the price model and restricted the class of contracts to weighted-average-price contracts, thereby reducing the optimal contracting problem to a mathematical program. In their framework, both the dealer's optimal strategy and the optimal contract are derived in a closed form.

In this paper, we further extend the price impact model in [4] to the transient impact model[5], which reflects price resilience effects in real markets. Following the approach in [4], we focused on weighted-average-price contracts and reduce the optimal contracting problem to a mathematical program. We show that the dealer's optimal strategy can still be derived in a closed form, whereas the optimal contracting problem becomes a non-convex quadratic program, which we solve numerically. Through numerical experiments, we investigate how the optimal contract and the dealer's optimal strategy depend on the model parameters including the resilience parameter.

The rest of this paper is organized as follows. In Section 2, we formulate the optimal contracting problem under the transient price impact model. Building on this formulation, we derive the dealer's optimal strategy and reduce the optimal contracting problem to a mathematical program in Section 3. In Section 4, we conduct numerical experiments to examine the influence of each parameter. Finally, we conclude this paper in Section 5.

2 Problem formulation

We consider a client who wishes to sell a large order of size $x_0 > 0$ and assume that the client delegates the execution to a dealer through a block trade contract. The dealer executes the order on the exchange over the period $\{1, 2, \ldots, T\}$ with a deterministic trading strategy $x = (x_1, x_2, \ldots, x_T)^\top \in \mathbb{R}^T$, where x_t represents the size of the order executed at time t. Here, the dealer's trading strategy x is required to satisfy the inventory constraint:

$$\sum_{t=1}^{T} x_t = x_0. \tag{2.1}$$

Let $\mathcal{X}$ denote the set of all admissible trading strategies;that is, all deterministic strategies satisfying the inventory constraint:

$$\mathcal{X} := \left\{ x = (x_1, x_2, \ldots, x_T)^\top \in \mathbb{R}^T ; \sum_{t=1}^{T} x_t = x_0 \right\} \tag{2.2}$$

To capture the effects of large executions, we introduce a price impact model on the exchange. The price dynamics on the exchange is given by the transient price impact model [5] as follows:

$$p_t = p_0 + \sigma \sum_{s=1}^{t} \epsilon_s + \sum_{s=1}^{t} f(x_s) G(|t-s|) \tag{2.3}$$

for $t = 1, 2, \dots, T$, where p_0 is the initial price, σ is the volatility parameter, ϵ_s are i.i.d. standard normal random variables, f is the (instantaneous) impact function, and G is the decay kernel representing price resilience effect. Specifically, we assume a linear impact function $f(x) = \eta x$ with $\eta > 0$ and an exponential decay kernel $G(t) = e^{-\kappa t}$ with $\kappa > 0$.

Similar to [4], the class of contracts we consider is restricted to the weighted-average-price contracts defined by

$$\mathcal{T} := \left\{ \tau = (\tau_t)_{t=1}^T \in \mathbb{R}^T; \sum_{t=1}^T \tau_t = 1, \tau_t \geq 0, t = 1, 2, \dots, T \right\}. \tag{2.4}$$

Additionally, we assume that the dealer has a CARA utility function $u(x) = -e^{-\gamma x}$ with $\gamma > 0$.

Under this setting, the optimal contracting problem is formulated as a principal-agent problem as follows.

Problem 1 (The optimal contracting problem(The client's problem))

$$\begin{aligned} &\min_{\tau \in \mathcal{T}, x \in \mathcal{X}} \mathbb{E}[x_0 \langle \tau, p \rangle] \\ \textit{subject to } &\textit{(IR):} \mathbb{E}[u(x_0\langle \tau, p\rangle - \langle x, p \rangle)] \geq u(0), \\ &\textit{(IC):} x \in \arg\max_{\hat{x} \in \mathcal{X}} \mathbb{E}[u(x_0\langle \tau, p\rangle - \langle \hat{x}, p \rangle)]. \end{aligned}$$

The IR and IC constraints represent the individual rationality constraint, which guarantees the dealer's participation, and the incentive compatibility constraint, which ensures that the dealer chooses the trading strategy that maximizes their expected utility under the contract, respectively. This formulation captures the conflict of interest between the client and the dealer, and aims to construct a contract under which the dealer has no incentive to manipulate prices.

As Problem 1 is referred to as the client's problem, the IC constraint is correspondingly called the dealer's problem.

Problem 2 (The dealer's problem) *For a given contract $\tau \in \mathcal{T}$,*

$$\max_{x \in \mathcal{X}} \mathbb{E}[u(x_0\langle \tau, p\rangle - \langle x, p \rangle)].$$

We can obtain the dealer's optimal trading strategy and the optimal contract as solutions to Problem 2 and 1. The next section provides a theoretical analysis to solve these problems.

3 Theoretical analysis

We begin by solving the dealer's problem. The dealer's optimal strategy can be uniquely derived in a closed form, as established in the following theorem.

Theorem 1 *For a given contract $\tau \in \mathcal{T}$, Problem 2 has a unique solution $x^* = (x_1^*, x_2^*, \dots, x_T^*)^\top \in \mathcal{X}$ given by*

$$x^*(\tau) = \frac{1}{2} Q_{sym}^{-1} \left(q(\tau) - \mathbf{1} \frac{\mathbf{1}^\top Q_{sym}^{-1} q(\tau) - 2x_0}{\mathbf{1}^\top Q_{sym}^{-1} \mathbf{1}} \right), \tag{3.1}$$

for some positive definite matrix $Q_{sym} \in \mathcal{M}_T(\mathbb{R})$ and affine map $q(\tau) \in \mathbb{R}^T$.

Proof. For any $x \in \mathcal{X}$, the dealer's profit $W(x)$ is given by

$$W(x) := x_0\langle \tau, p\rangle - \langle x, p\rangle = \sum_{t=1}^{T}(x_0\tau_t - x_t)\left[p_0 + \sigma\sum_{s=1}^{t}\epsilon_s + \eta\sum_{s=1}^{t}x_s G(|t-s|)\right]. \tag{3.2}$$

Let $m(x)$ and $v(x)$ be the mean and variance of $W(x)$, respectively. Then, we have

$$m(x) := \mathbb{E}[W(x)] = x_0\eta\sum_{t=1}^{T}\sum_{s=1}^{t}\tau_t x_s G(|t-s|) - \eta\sum_{t=1}^{T}\sum_{s=1}^{t}x_s x_t G(|t-s|), \tag{3.3}$$

and

$$v(x) := \mathrm{Var}(W(x)) = \sigma^2\sum_{t=1}^{T}\left\{\sum_{s=t}^{T}(x_0\tau_s - x_s)\right\}^2. \tag{3.4}$$

Since $W(x) \sim \mathcal{N}(m(x), v(x))$ and the dealer has a CARA utility function, we can use the form of the moment generating function of the normal distribution to obtain

$$\mathbb{E}[u(W(x))] = -\exp\left(-\gamma m(x) + \frac{\gamma^2}{2}v(x)\right). \tag{3.5}$$

Hence, by the monotonicity of the exponential function, Problem 2 is equivalent to the following mean-variance optimization problem:

$$\max_{x\in\mathcal{X}}\left\{m(x) - \frac{\gamma}{2}v(x)\right\}. \tag{3.6}$$

Define matrices $G, A, C \in \mathcal{M}_T(\mathbb{R})$ by

$$G_{s,t} := G(|t-s|), A_{s,t} := \begin{cases} G(|t-s|) & \text{if} \quad s \geq t, \\ 0 & \text{otherwise} \end{cases}, C_{s,t} := \begin{cases} 1 & \text{if} \quad s \leq t, \\ 0 & \text{otherwise} \end{cases} \tag{3.7}$$

Then A is lower triangular and C is upper triangular. With these matrices, we can rewrite $m(x)$ and $v(x)$ as

$$m(x) = x_0\eta\tau^\top Ax - \eta x^\top Ax, v(x) = \sigma^2(x_0\tau - x)^\top C^\top C(x_0\tau - x), \tag{3.8}$$

and thus the objective function of Problem (3.6) becomes

$$m(x) - \frac{\gamma}{2}v(x) = x_0\eta\tau^\top Ax - \eta x^\top Ax - \frac{\gamma}{2}\sigma^2(x_0\tau - x)^\top C^\top C(x_0\tau - x). \tag{3.9}$$

This representation can be further simplified by introducing a matrix $Q \in \mathcal{M}_T(\mathbb{R})$ and vectors $q(\tau), c \in \mathbb{R}^T$ defined as

$$Q := \eta A + \frac{\gamma}{2}\sigma^2 C^\top C, \quad q(\tau) := x_0\eta A^\top\tau + \gamma\sigma^2 x_0 C^\top C\tau, \quad c := \frac{\gamma}{2}\sigma^2 x_0^2\tau^\top C^\top C\tau, \tag{3.10}$$

which leads to

$$m(x) - \frac{\gamma}{2}v(x) = -x^\top Qx + q(\tau)^\top x - c. \tag{3.11}$$

Although the matrix Q is not symmetric in general, its symmetrization $Q_{sym} := (Q + Q^\top)/2$ is not only symmetric but also positive definite. Indeed, Q_{sym} can be represented as

$$Q_{sym} = \eta\frac{A + A^\top}{2} + \frac{\gamma}{2}\sigma^2 C^\top C = \eta G + \frac{\gamma}{2}\sigma^2 C^\top C, \tag{3.12}$$

where both G and $C^\top C$ are positive definite matrices. The positive definiteness of G follows from Proposition 2 in [1], which states that convex and non-constant decay kernels are always positive definite, and that of $C^\top C$ is immediate.

Therefore, the dealer's problem can finally be written as the following convex quadratic program:

$$\max_{x \in \mathcal{X}} \left\{ -x^\top Q_{sym} x + q(\tau)^\top x \right\}. \tag{3.13}$$

Applying the method of Lagrange multipliers, the unique solution to Problem (3.13) satisfies the following equation:

$$\nabla_x \mathcal{L}(x, \lambda) = 2Q_{sym}x - q(\tau) + \lambda \mathbf{1} = 0, \mathbf{1}^\top x - x_0 = 0. \tag{3.14}$$

Thus, we obtain the desired result. □

We then turn to the client's problem. By substituting the dealer's optimal strategy $x^*(\tau)$ into the client's problem, we can reduce it to a mathematical program as follows.

Theorem 2 *Problem 1 is equivalent to the following mathematical program:*

$$\begin{aligned} \min_{\tau \in \mathbb{R}^T} \quad & \frac{1}{2}\tau^\top Q' \tau + q'\tau \\ s.t. \quad & 1^\top \tau = 1, \\ & \tau \geq 0. \end{aligned}$$

where

$$Q' := x_0 A Q_{sym}^{-1} \left(I - \frac{\mathbf{1}\mathbf{1}^\top Q_{sym}^{-1}}{\mathbf{1}^\top Q_{sym}^{-1} \mathbf{1}} \right) (\eta A^\top + \gamma \sigma^2 C^\top C), q' := \frac{x_0}{\mathbf{1}^\top Q_{sym}^{-1} \mathbf{1}} \mathbf{1}^\top Q_{sym}^{-1} A^\top.$$

Proof. We first observe that the IR constraint is always satisfied with equality under the deterministic strategy $x = x_0 \tau$. Therefore, we can ignore the IR constraint below. Substituting the dealer's optimal strategy $x^*(\tau)$, obtained in Theorem 1, into Problem 1, we have

$$\begin{aligned} E[x_0 \langle \tau, p(x^*(\tau)) \rangle] &= E\left[\sum_{t=1}^T x_0 \tau_t \left(p_0 + \sigma \sum_{s=1}^t \epsilon_s + \eta \sum_{s=1}^t x_s^*(\tau) G(|t-s|) \right) \right] \\ &= x_0 p_0 + x_0 \eta \sum_{t=1}^T \sum_{s=1}^t \tau_t x_s^*(\tau) G(|t-s|) \\ &= x_0 p_0 + x_0 \eta \tau^\top A x^*(\tau), \end{aligned} \tag{3.15}$$

which implies that Problem 1 is equivalent to the minimization of $\tau^\top A x^*(\tau)$.

Furthermore, a straightforward calculation yields

$$
\begin{aligned}
\tau^\top Ax^*(\tau) &= \tau^\top A\frac{1}{2}Q_{sym}^{-1}\left(q(\tau)-\mathbf{1}\frac{\mathbf{1}^\top Q_{sym}^{-1}q(\tau)-2x_0}{\mathbf{1}^\top Q_{sym}^{-1}\mathbf{1}}\right)\\
&= \tau^\top A\frac{1}{2}Q_{sym}^{-1}\left(I-\frac{1}{\mathbf{1}^\top Q_{sym}^{-1}\mathbf{1}}\mathbf{1}\mathbf{1}^\top Q_{sym}^{-1}\right)q(\tau)+\frac{x_0}{\mathbf{1}^\top Q_{sym}^{-1}\mathbf{1}}\tau^\top AQ_{sym}^{-1}\mathbf{1}\\
&= \tau^\top A\frac{1}{2}Q_{sym}^{-1}\left(I-\frac{1}{\mathbf{1}^\top Q_{sym}^{-1}\mathbf{1}}\mathbf{1}\mathbf{1}^\top Q_{sym}^{-1}\right)x_0(\eta A^\top+\gamma\sigma^2C^\top C)\tau+\frac{x_0}{\mathbf{1}^\top Q_{sym}^{-1}\mathbf{1}}\tau^\top AQ_{sym}^{-1}\mathbf{1}\\
&= \tau^\top A\frac{1}{2}Q_{sym}^{-1}\left(I-\frac{1}{\mathbf{1}^\top Q_{sym}^{-1}\mathbf{1}}\mathbf{1}\mathbf{1}^\top Q_{sym}^{-1}\right)x_0(\eta A^\top+\gamma\sigma^2C^\top C)\tau+\frac{x_0}{\mathbf{1}^\top Q_{sym}^{-1}\mathbf{1}}\mathbf{1}^\top Q_{sym}^{-1}A^\top\tau\\
&= \frac{1}{2}\tau^\top Q'\tau+q'\tau \qquad (3.16)
\end{aligned}
$$

and this is our desired result. □

Whereas the transformed client's problem in Theorem 2 is a quadratic program with linear constraints (LCQP), the matrix Q' is not necessarily positive semidefinite, and thus the client's problem is non-convex in general. Consequently, a closed-form solution for the optimal contract is not available, making some numerical methods necessary. For instance, a standard quadratic programming solver can be applied to obtain a locally optimal solutions. Furthermore, a globally optimal solution may be attainable by employing existing global solvers (e.g., `BARON`, `Couenne` and `Gurobi`) that incorporates techniques such as branch-and-bound and convex relaxation. In the next section, we conduct numerical experiment with `scipy.optimize` and demonstrate that plausible solutions can be obtained.

4 Numerical experiments

In this section, we conduct numerical experiments to demonstrate the availability of numerical solutions and to investigate how the optimal contract and the dealer's optimal strategy depend on the model parameters. The computations are performed using the `scipy.optimize` module in Python. Although this module is basically designed for convex problems, our experiments indicate that it can be effectively applied to our non-convex problem. In our model, there are the following six parameters: the trading horizon $T \in \mathbb{N}$, the order size $x_0 > 0$, the volatility parameter $\sigma > 0$, the dealer's risk aversion $\gamma > 0$, the resilience parameter $\kappa > 0$ and the impact parameter $\eta > 0$. However, it is clear that the risk aversion γ has a similar effect to the volatility σ and the impact parameter η acts in opposition to the resilience κ. We thus focus on only the parameters σ and κ, fixing $T = 20, x_0 = 1.0, \sigma = 0.5$ and $\gamma = \kappa = \eta = 1.0$ except for the parameter being varied.

4.1 Effect of the price volatility

At first the effect of the price volatility is investigated. Figure 1 and Figure 2 respectively show the dealer's optimal strategies and the optimal contracts for different volatility parameters, where the volatility σ is varied from 0 to a extremely large value. From Figure 1, we can observe

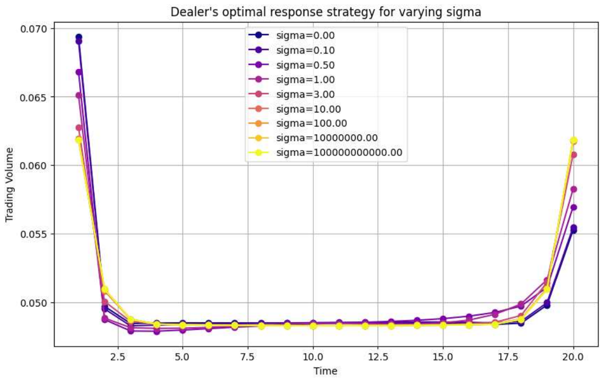

Figure 1: The dealer's optimal strategies for different volatility parameters

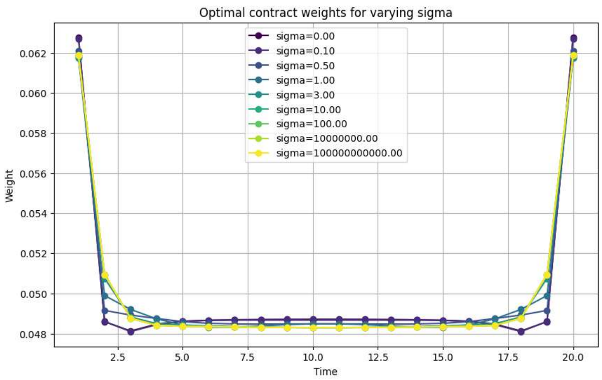

Figure 2: The optimal contracts for different volatility parameters

that as the volatility σ varies, the dealer's optimal strategy changes from inverse J-shaped to U-shaped. In [4], it is shown that the optimal strategy converges to the TWAP strategy as $\sigma \to \infty$ and thus our result indicates that the existence of resilience preserves the U-shape of the optimal strategy. A similar pattern appears in the optimal contracts shown in Figure 2, where the optimal contract converges to the U-shaped contract as σ grows. This U-shape can be attributed to the existence of price resilience as well because the optimality of the TWAP contract when $\kappa = 0$ and $\sigma \to \infty$ is proved in [4].

4.2 Effect of the resilience

We next examine how optimal contracts changes when the price resilience parameter varies. Similar to the previous experiments, Figures 3 and 4 show the dealer's optimal strategies and the optimal contracts for different resilience parameters, respectively. The resilience parameter κ is varied from 0 to a large value. As shown in Figure 3, increasing κ causes the optimal

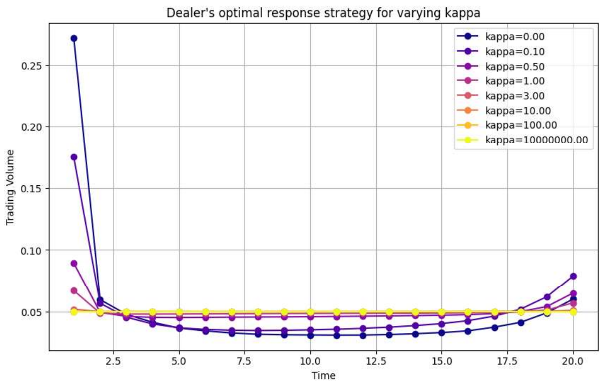

Figure 3: The dealer's optimal strategies for different resilience parameters

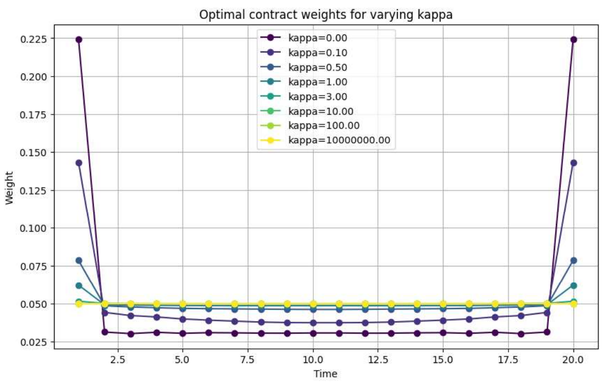

Figure 4: The optimal contracts for different resilience parameters

strategy to converge from an inverse J-shaped strategy to the TWAP strategy. This result is consistent with [4], in which the TWAP execution is proved to be optimal when the risk is only price volatility, i.e., $\eta = 0$. We can also see from Figure 4 that the optimal contract evolves from extreme U-shaped contract to the TWAP contract as κ increases. Since the limit $\kappa \to \infty$

corresponds to the case of no price impact, this result also aligns with [4], which establishes the optimality of the TWAP contract in the absence of price impact. Finally, Figure 3 and Figure 4 both illustrate that an increase in κ enhances the components around the middle of the trading horizon.

4.3 Interpretation for the results

The findings from the experiments above allows us to conclude that even solvers designed for non-convex problems are capable of producing plausible numerical solutions to our problem. In addition, the results suggest that price resilience mitigates the effect of price impact, thereby diminishing the dealer's incentive to manipulate prices through front-running. In this sense, incorporating price resilience can lead to a more realistic assessment of market manipulation risk and may refine the contract design proposed by [4].

5 Conclusion

In this paper, we have studied the optimal contracting problem for off-exchange block trading under the transient price impact model. By extending the framework in [4], we derived the dealer's optimal strategy in a closed form and transformed the client's problem to a non-convex quadratic program. Through numerical experiments, we demonstrated that even a standard solver for convex quadratic programs can provide plausible solutions to the client's problem and we investigated how the dealer's optimal strategy and the optimal contract are affected by the model parameters. In particular, our results show that price resilience serves to mitigate the influence of price impact and maintain the U-shaped structure of the optimal contract and strategy. This finding suggests that incorporating price resilience effects may improve the risk assessment of dealer's market manipulation and refine the design of block trade contracts. We leave several directions for future research. One is to provide a clear interpretation for why the U-shaped form is preserved in the presence of price resilience. Another is to generalize the class of dealers' strategies from deterministic to stochastic ones and investigate the difference between them. Finally, we need to introduce a intraday volume process to the model to analyze practically important contracts dependent on volumes, such as guaranteed VWAP contract.

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