ON THE (φ, Γ)-MODULES CORRESPONDING TO CRYSTALLINE REPRESENTATIONS, SEMI-STABLE REPRESENTATIONS AND DE RHAM REPRESENTATIONS

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ABSTRACT. This is a survey of the author's papers [Wat24, Watc, Watb] and it is an expanded version of the author's talk in the conference "Algebraic Number Theory and Related Topics 2024" at Kyoto University. We explain how we can determine the (φ, Γ)-modules corresponding to crystalline, semi-stable and de Rham representations.

CONTENTS

Notation and Convention. Throughout this paper, all rings are assumed commmutative and unital. We fix a prime number p and a p-adic field K, i.e., a complete discrete valuation field of characteristic 0 whose residue field k is a perfect field of characteristic p. We put $K_0 := W(k)[1/p]$. We fix a uniformizer $\pi \in K$ and an algebraic closure $\overline{K}$ of K and let $\mathbb{C}_p$ denote its completion. Let v_p denote the valuation of $\mathbb{C}_p$ normalized as $v_p(p) = 1$. For any subfield L in $\mathbb{C}_p/K_0$, $\mathcal{O}_L$ denotes the ring of integers, $\mathfrak{m}_L$ the maximal ideal and G_L the absolute Galois group. We fix a compatible sequence of

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p-power roots of unity $\{\zeta_{p^n}\}_n$ in $\overline{K}$ and we put $K_\infty := \bigcup_{n=1}^\infty K(\zeta_{p^n})$. $\widehat{K_\infty}$ denotes the completion of K_∞. We set $\Gamma_K := \mathrm{Gal}\,(K_\infty/K)$ and $H_K := \mathrm{Gal}\,(\overline{K}/K_\infty)$. We define $\mathcal{O}^\flat_{\mathbb{C}_p} := \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_p}/p$. Let $\mathbb{C}^\flat_p$ denote its field of fractions and $\mathfrak{m}^\flat_{\mathbb{C}_p}$ the maximal ideal. By abuse of notation, the valuation on $\mathbb{C}^\flat_p$ induced from v_p is also denoted by v_p. We similarly define $\mathcal{O}^\flat_{\widehat{K_\infty}}$, $\widehat{K_\infty}^\flat$ and $\mathfrak{m}^\flat_{\widehat{K_\infty}}$. We set $\epsilon := (1, \zeta_p, \zeta_{p^2}, \dots) \in \mathcal{O}^\flat_{\widehat{K_\infty}}$, $\mu := [\epsilon] - 1 \in W(\mathcal{O}^\flat_{\widehat{K_\infty}})$ and $\xi_r := \mu/\varphi^{-r}(\mu) \in W(\mathcal{O}^\flat_{\widehat{K_\infty}})$ for any $r \in \mathbb{Z}_{\geq 1}$. We put $\xi := \xi_1$. For any ring A and A-modules M and N, we write $M \widehat{\otimes}_A N$ for the p-adic completion of $M \otimes_A N$.

1. BACKGROUND

In number theory, it is important to study p-adic Galois representations.

Definition 1.1.

(1) A *p-adic Galois representation of G_K* is a finite dimensional $\mathbb{Q}_p$-vector space equipped with a continuous and linear action of G_K. We write $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ for the category of p-adic Galois representations of G_K.

(2) A *free $\mathbb{Z}_p$-representation of G_K* is a finite free $\mathbb{Z}_p$-module equipped with a continuous and linear action of G_K. We write $\mathrm{Rep}_{\mathbb{Z}_p}(G_K)$ for the category of free $\mathbb{Z}_p$-representations of G_K.

Example 1.2. Let E be an elliptic curve over K. For any $n \in \mathbb{Z}_{\geq 1}$, we define $E[n] := \{x \in E(\overline{K}) \mid nx = 0\}$. Then, we can construct an inverse system

$$E[p] \xleftarrow{\times p} E[p^2] \xleftarrow{\times p} E[p^3] \xleftarrow{\times p} \cdots .$$

$E[p^n]$ is a $\mathbb{Z}/p^n\mathbb{Z}$-module equipped with a G_K-action. Then, $T_p E := \varprojlim_n E[p^n]$ is a free $\mathbb{Z}_p$-representation of G_K of rank 2. It is called a *Tate module associated to E*.

Example 1.3. Let X be a smooth and proper scheme over K. Then, a p-adic étale cohomology $H^m(X_{\overline{K}}, \mathbb{Q}_p)$ is a p-adic Galois representation of G_K.

Every p-adic Galois representation comes from a (not unique) free $\mathbb{Z}_p$-representation and free $\mathbb{Z}_p$-representations of G_K have connections with mod p representations of G_K, i.e., finite dimensional $\mathbb{F}_p$-vector spaces equipped with continuous and linear actions of G_K. In this sense, it is more important to study free $\mathbb{Z}_p$-representations, so we focus on them from now on. In general, free $\mathbb{Z}_p$-representations are difficult. In order to handle them, Jean-Marc Fontaine developed a theory of (φ, Γ)-module. Fontaine constructed subrings $\mathbb{A}_K \subseteq W(\widehat{K_\infty}^\flat)$ and $\mathbb{A} \subseteq W(\mathbb{C}^\flat_p)$ which are stable under the Galois actions and the Frobenii. When K is absolutely unramified, i.e., $K = K_0$, $\mathbb{A}_K := W(k)((\mu))^\wedge_p$.

Remark 1.4. In [Fon90], Fontaine writes $\mathbb{A}_K = \mathcal{O}_{\mathcal{E}}$ and $\mathbb{A} = \mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}}$.

Definition 1.5 ([Fon90, 3.3.2.]). A *(φ, Γ)-module over $\mathbb{A}_K$* is a finite free $\mathbb{A}_K$-module M equipped with a continuous and semi-linear Γ_K-action and a Γ_K-equivariant $\mathbb{A}_K$-linear isomorphism $M \otimes_{\mathbb{A}_K, \varphi} \mathbb{A}_K \cong M$. We write $\mathrm{Mod}^{\text{ét}}_{\varphi, \Gamma}(\mathbb{A}_K)$ for the category of (φ, Γ)-modules over $\mathbb{A}_K$.

Remark 1.6. Usually, it is called an *étale (φ, Γ)-module over $\mathbb{A}_K$*.

An important point of the theory of (φ, Γ)-modules is that the Γ_K-action is simple. Since Γ_K is a Galois group of p-cyclotomic extension of K, we have an injection $\Gamma_K \hookrightarrow \mathbb{Z}_p^\times$ via the cyclotomic character. Thus, a continuous and semi-linear Γ_K-action is determined

by an action of a topological generator of Γ_K, which is much simpler than a G_K-action. In summary, (φ, Γ)-modules over $\mathbb{A}_K$ are simpler than free $\mathbb{Z}_p$-representations of G_K. However, Fontaine proved an equivalence of categories between them.

Theorem 1.7 ([Fon90, Théorème 3.4.3.]). *There exists an equivalence of categories between the category of free $\mathbb{Z}_p$-representations of G_K and the category of (φ, Γ)-modules over $\mathbb{A}_K$ via the functors $M \mapsto (M \otimes_{\mathbb{A}_K} \mathbb{A})^{\varphi=1}$ for any (φ, Γ)-module M over $\mathbb{A}_K$ and $T \mapsto (T \otimes_{\mathbb{Z}_p} \mathbb{A})^{H_K}$ for any free $\mathbb{Z}_p$-representation T of G_K.*

2. Problem

Fontaine defined some important classes of p-adic Galois representations or free $\mathbb{Z}_p$-representations in [Fon82, Fon94]. They are called *crystalline representations, semi-stable representations* and *de Rham representations*. Crystalline representations are semi-stable and semi-stable representations are de Rham.

Example 2.1. Let X be a proper and smooth scheme over K. Then, the p-adic étale cohomology $H^m(X_{\overline{K}}, \mathbb{Q}_p)$ is always de Rham. If X has a semi-stable reduction, it is semi-stable. If X has a good reduction, it is crystalline. These are highly non-trivial results by T. Tsuji ([Tsu99]), G. Faltings ([Fal89, Fal02]), or W. Nizioł ([Niz98, Niz08]).

Example 2.1 shows that these classifications reflect geometry. We consider the following question.

Question 2.2. Which (φ, Γ)-modules correspond to crystalline representations, semi-stable representations and de Rham representations?

This question has been considered for a long time. When K is absolutely unramified, the question in the crystalline case was studied by Fontaine ([Fon90]), N. Wach ([Wac96, Wac97]) and P. Colmez ([Col99]). By using these results, L. Berger finally constructed a theory of Wach modules ([Ber04]). For any absolutely unramified p-adic field K, we define $\mathbb{A}_K^+ := W(k)[[\mu]]$.

Definition 2.3 ([Ber04, DÉFINITION III.4.1.]). Assume K is absolutely unramified. A *Wach module* is a finite free $\mathbb{A}_K^+$-module N equipped with a continuous and semi-linear Γ_K-action that is trivial on N/μ and a Γ_K-equivariant $\mathbb{A}_K^+[1/\varphi(\xi)]$-linear isomorphism $N \otimes_{\mathbb{A}_K^+, \varphi} \mathbb{A}_K^+[1/\varphi(\xi)] \cong N[1/\varphi(\xi)]$.

Theorem 2.4 ([Ber04, PROPOSITION III.4.2.]). *Assume K is absolutely unramified. Then, the category of Wach modules is equivalent to the category of free crystalline $\mathbb{Z}_p$-representations of G_K via the functor $N \mapsto (N \otimes_{\mathbb{A}_K^+} \mathbb{A})^{\varphi=1}$ for any Wach module N.*

Recall that when K is absolutely unramified, the coefficient ring of (φ, Γ)-module is $\mathbb{A}_K := W(k)((\mu))_p^\wedge$. So $\mathbb{A}_K^+$ is a subring of $\mathbb{A}_K$ and in some sense an integral ring.

Proposition 2.5 (cf. proof of [Ber04, PROPOSITION III.4.2.]). *Assume K is absolutely unramified. Then, for any Wach module N, $N \otimes_{\mathbb{A}_K^+} \mathbb{A}_K$ is the (φ, Γ)-module over $\mathbb{A}_K$ corresponding to the free crystalline $\mathbb{Z}_p$-representation of G_K.*

Proposition 2.5 says that a Wach module is an $\mathbb{A}_K^+$-lattice of the (φ, Γ)-module over $\mathbb{A}_K$ satisfying some conditions. So, Berger determined the (φ, Γ)-modules over $\mathbb{A}_K$ corresponding to free crystalline $\mathbb{Z}_p$-representations of G_K by the existence of special lattices.

The theory of Wach modules are very beautiful and useful. It has applications to, for example, Iwasawa theory and p-adic Langlands correspondence. However, it can be used

only when K is absolutely unramified. So, we would like to remove the assumption and obtain similar results for semi-stable representations and de Rham representations. Why do we assume K is absolutely unramified in the theory of Wach modules?

Problem 2.6. We do not know a good definition of "$\mathbb{A}_K^+$" for general p-adic field K.

$\mathbb{A}_K^+$ should satisfy some conditions. For example, it should have a Γ_K-action, a Frobenius map and a "good" relationship with $\mathbb{A}_K$, $W(\mathcal{O}_{\widehat{K_\infty}}^\flat)$ and $\mathbb{E}_K^+$. However, we do not know how to construct such a ring. Regarding this problem, an important fact is the following pointed out by Wach.

Proposition 2.7 ([Wac96, THÉORÈME, REMARQUES on page 381], [Wata]). *Let k_∞ denote the residue field of K_∞. For general p-adic field K, we have an equality*

$$\mathbb{A}_K \cap W(\mathcal{O}_{\widehat{K_\infty}}^\flat) = W(k_\infty)[[\mu]] \quad in \ W(\widehat{K_\infty}^\flat).$$

When K is absolutely unramified, we have an equality $\mathbb{A}_K^+ = \mathbb{A}_K \cap W(\mathcal{O}_{\widehat{K_\infty}}^\flat)$. So taking the intersection seems to be a good candidate of the definition of "$\mathbb{A}_K^+$" for general p-adic field K. However, Proposition 2.7 implies that it does not work. The point is that the right hand side only depends on the residue field of K_∞ and it does not have any information of the ramification of K. In order to conquer this problem, we use (φ, Γ)-modules over $\widetilde{\mathbb{A}}_K := W(\widehat{K_\infty}^\flat)$. We have a natural inclusion $\mathbb{A}_K \subseteq \widetilde{\mathbb{A}}_K$ and $\widetilde{\mathbb{A}}_K$ coincides with the p-adic completion of the perfection of $\mathbb{A}_K$.

Remark 2.8. One difference between $\mathbb{A}_K$ and $\widetilde{\mathbb{A}}_K$ is that the Frobenius map on $\mathbb{A}_K$ is not bijective, but the Frobenius map on $\widetilde{\mathbb{A}}_K$ is an isomorphism.

The category of (φ, Γ)-modules over $\widetilde{\mathbb{A}}_K$ is also equivalent to the category of free $\mathbb{Z}_p$-representations of G_K. Let us write explicitly.

Definition 2.9. A (φ, Γ)-*module over* $\widetilde{\mathbb{A}}_K$ is a finite free $\widetilde{\mathbb{A}}_K$-module M equipped with a continuous and semi-linear Γ_K-action and a Γ_K-equivariant $\widetilde{\mathbb{A}}_K$-linear isomorphism $M \otimes_{\widetilde{\mathbb{A}}_K, \varphi} \widetilde{\mathbb{A}}_K \cong M$. We write $\mathrm{Mod}_{\varphi, \Gamma}^{\text{ét}}(\widetilde{\mathbb{A}}_K)$ for the category of (φ, Γ)-modules over $\widetilde{\mathbb{A}}_K$.

Theorem 2.10 ([Fon84, 2.], [Fon90, Théorème 3.4.3.]). *There exists an equivalence of categories between the category of free $\mathbb{Z}_p$-representations of G_K and the category of (φ, Γ)-modules over $\widetilde{\mathbb{A}}_K$ via the functors*

$$\mathrm{Mod}_{\varphi, \Gamma}^{\text{ét}}(\widetilde{\mathbb{A}}_K) \xleftrightarrow{\ \sim\ } \mathrm{Rep}_{\mathbb{Z}_p}(G_K)$$

$$M \longmapsto \left(M \otimes_{\widetilde{\mathbb{A}}_K} W(\mathbb{C}_p^\flat)\right)^{\varphi=1}$$

$$\left(T \otimes_{\mathbb{Z}_p} W(\mathbb{C}_p^\flat)\right)^{H_K} \longleftarrow\!\shortmid\ T.$$

Remark 2.11. As indicated in the citation, (φ, Γ)-modules over $\widetilde{\mathbb{A}}_K$ appeared in 1984 and Fontaine first considered them as far as the author knows.

In this case, we can use $\widetilde{\mathbb{A}}_K^+ := W(\mathcal{O}_{\widehat{K_\infty}}^\flat)$ instead of $\mathbb{A}_K^+$ for general p-adic field K.

Remark 2.12. As G. Yamashita pointed out in the conference, Berger studied the relationship between (φ, Γ)-modules and crystalline representations, semi-stable representations and de Rham representations in terms of p-adic differential equations in [Ber02].

3. Main theorem for crystalline representations

Definition 3.1.

(1) We define an ideal $\mathfrak{q}_\infty$ of $\widetilde{\mathbb{A}}_K^+$ by
$$\mathfrak{q}_\infty := \{[a]x \mid a \in \mathfrak{m}_{\widehat{K_\infty}}^\flat, x \in \widetilde{\mathbb{A}}_K^+\}.$$

(2) ([Du19, Lemma 4.11.]) For any p-torsion free $W(k)$-module M equipped with a semi-linear homomorphism $\varphi \colon M \to M[1/p]$ with respect to the Frobenius endomorphism of $W(k)$, we define a φ-*finite part* of M by
$$M_{\varphi\text{-fin}} := \{x \in M \mid \dim_{K_0}\langle \varphi^n(x) \colon n \in \mathbb{Z}_{\geq 0}\rangle_{K_0} < \infty\}.$$

Remark 3.2. We do not know whether the natural morphism $W(k) \to (\widetilde{\mathbb{A}}_K^+/\mathfrak{q}_\infty)^{\Gamma_K}$ is isomorphic or not. However, we can show that $W(k) \to (\widetilde{\mathbb{A}}_K^+/\mathfrak{q}_\infty)^{\Gamma_K}_{\varphi\text{-fin}}$ is an isomorphism ([Du19, Lemma 4.11.], [Wat24, Proposition 2.17]). This is the motivation of φ-finite part.

Remark 3.3. φ-finite part is a Frobenius analog of K-finite in the sense of [Sen81, page 98], see also [BHH+22, Remark 2.6.5.].

Definition 3.4 ([Wat24, Definition 2.18.]). A *crystalline* (φ, Γ)-*module over* $\widetilde{\mathbb{A}}_K^+$ is a finite free $\widetilde{\mathbb{A}}_K^+$-module N equipped with a continuous and semi-linear Γ_K-action and a Γ_K-equivariant $\widetilde{\mathbb{A}}_K^+[1/\varphi(\xi)]$-linear isomorphism $\Phi_N \colon N \otimes_{\widetilde{\mathbb{A}}_K^+, \varphi} \widetilde{\mathbb{A}}_K^+[1/\varphi(\xi)] \cong N[1/\varphi(\xi)]$ such that the following conditions hold.

(1) (Triviality modulo μ) $(N/\mu)^{\Gamma_K}$ is a finite projective $(\widetilde{\mathbb{A}}_K^+/\mu)^{\Gamma_K}$-module and we have a canonical isomorphism
$$(N/\mu)^{\Gamma_K} \otimes_{(\widetilde{\mathbb{A}}_K^+/\mu)^{\Gamma_K}} \widetilde{\mathbb{A}}_K^+/\mu \xrightarrow{\cong} N/\mu.$$

(2) $(N/\mathfrak{q}_\infty)^{\Gamma_K}_{\varphi\text{-fin}}$ is a finite free $W(k)$-module and we have a canonical isomorphism
$$(N/\mathfrak{q}_\infty)^{\Gamma_K}_{\varphi\text{-fin}} \otimes_{W(k)} \widetilde{\mathbb{A}}_K^+/\mathfrak{q}_\infty \xrightarrow{\cong} N/\mathfrak{q}_\infty.$$

We write $\mathrm{Mod}^{\mathrm{fh,crys}}_{\varphi,\Gamma}(\widetilde{\mathbb{A}}_K^+)$ for the category of crystalline (φ, Γ)-modules over $\widetilde{\mathbb{A}}_K^+$.

Remark 3.5. The condition (1) is called *triviality modulo* μ in [MT20]. If $(N/\mu)^{\Gamma_K}$ is a finite free $(\widetilde{\mathbb{A}}_K^+/\mu)^{\Gamma_K}$-module, the isomorphism implies that N/μ has a basis consisting of elements fixed by the Γ_K-actions. So, the condition (1) means that the Γ_K-action on N/μ is "trivial" in some sense. Thus, the condition (1) is an analog of Wach modules.

The following theorem is our main theorem on crystalline representations.

Theorem 3.6 ([Wat24, Theorem 2.43.]). *There exists an equivalence of categories between the category of crystalline* (φ, Γ)-*modules over* $\widetilde{\mathbb{A}}_K^+$ *and the category of free crystalline* $\mathbb{Z}_p$-*representations of* G_K *denoted by* $\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Z}_p}(G_K)$ *via the functor*
$$T_{\mathrm{crys}} \colon \mathrm{Mod}^{\mathrm{fh,crys}}_{\varphi,\Gamma}(\widetilde{\mathbb{A}}_K^+) \xrightarrow{\sim} \mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Z}_p}(G_K), \qquad N \mapsto T_{\mathrm{crys}}(N) := \left(N \otimes_{\widetilde{\mathbb{A}}_K^+} W(\mathbb{C}_p^\flat)\right)^{\varphi=1}.$$

Proposition 3.7 ([Wat24, Corollary 1.26.]). *Let* N *be a crystalline* (φ, Γ)-*module over* $\widetilde{\mathbb{A}}_K^+$. *Then* $N \otimes_{\widetilde{\mathbb{A}}_K^+} \widetilde{\mathbb{A}}_K$ *is the* (φ, Γ)-*module over* $\widetilde{\mathbb{A}}_K$ *associated to* $T_{\mathrm{crys}}(N)$.

This proposition implies that similarly to Wach modules, a crystalline (φ, Γ)-module over $\widetilde{\mathbb{A}}_K^+$ is an $\widetilde{\mathbb{A}}_K^+$-lattice of the (φ, Γ)-module over $\widetilde{\mathbb{A}}_K$ satisfying some conditions.

Let us explain an outline of the proof of Theorem 3.6. Roughly speaking,

- the condition (1) (Triviality modulo μ) implies the functor T_{crys} is full and faithful,
- the condition (2) implies the free $\mathbb{Z}_p$-representation $T_{\mathrm{crys}}(N)$ is crystalline, and
- we show the essentially surjectiveness of T_{crys} by using a theory of prisms.

First, we see that $T_{\mathrm{crys}}(N)$ is a free $\mathbb{Z}_p$-representation of G_K by Proposition 3.7 and Theorem 2.10. We tried to show that $T_{\mathrm{crys}}(N)$ is crystalline from the condition (1) since it is natural from the view point of the theory of Wach modules, but we failed. This is because our coefficient ring $\widetilde{\mathbb{A}}_K^+$ is bigger than the coefficient ring $\mathbb{A}_K^+ := W(k)[[\mu]]$ of Wach modules. Especially, $\mathbb{A}_K^+/\mu = W(k)$ and Γ_K acts on it trivially, but $\widetilde{\mathbb{A}}_K^+/\mu \neq W(k)$ and Γ_K acts on it non-trivially. This is why we add the condition (2). Let us explain how to show $T_{\mathrm{crys}}(N)$ is crystalline. The first key lemma is the following due to B. Bhatt, M. Morrow and P. Scholze. We put $A_{\mathrm{inf}} := W(\mathcal{O}_{\mathbb{C}_p}^\flat)$.

Lemma 3.8 ([BMS18, Lemma 4.26.]). *For any crystalline (φ, Γ)-module N over $\widetilde{\mathbb{A}}_K^+$, we have an isomorphism*

$$N \otimes_{\widetilde{\mathbb{A}}_K^+} A_{\mathrm{inf}}[1/\mu] \cong T_{\mathrm{crys}}(N) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}[1/\mu]$$

compatible with the Frobenii and the G_K-actions and functorial in N.

Lemma 3.8 implies $N \otimes_{\widetilde{\mathbb{A}}_K^+} B_{\mathrm{crys}} \cong T_{\mathrm{crys}}(N) \otimes_{\mathbb{Z}_p} B_{\mathrm{crys}}$, so it is enough to show that $(N \otimes_{\widetilde{\mathbb{A}}_K^+} B_{\mathrm{crys}})^{G_K}$ is enough large. We prove it by constructing an injection

$$(N/\mathfrak{q}_\infty)^{\Gamma_K}_{\varphi\text{-fin}}[1/p] \hookrightarrow (N \otimes_{\widetilde{\mathbb{A}}_K^+} B_{\mathrm{crys}}^+)^{G_K}.$$

This argument is due to [Du19, Lemma 3.27., Lemma 4.14.]. In this way, we prove the well-definedness of the functor T_{crys}.

As a crystalline (φ, Γ)-module over $\widetilde{\mathbb{A}}_K^+$ is a submodule of the (φ, Γ)-module over $\widetilde{\mathbb{A}}_K$, we see that T_{crys} is faithful. The fullness can be shown by the same argument as in [Tsu20, Proposition 76]. The essential proposition is Lemma 3.8 and the following due to Tsuji:

Proposition 3.9 ([Tsu20, Proposition 76], [Wat24, Proposition 2.4]). *Let N_1 and N_2 be finite free $\widetilde{\mathbb{A}}_K^+$-modules endowed with semi-linear Γ_K-actions trivial modulo μ. Then, any Γ_K-equivariant $\widetilde{\mathbb{A}}_K^+[1/\mu]$-linear map $f \colon N_1[1/\mu] \to N_2[1/\mu]$ satisfies $f(N_1) \subseteq N_2$.*

It remains to show the essentially surjectiveness, i.e., we need to construct crystalline (φ, Γ)-modules from free crystalline $\mathbb{Z}_p$-representations. Mathematically, it is the most difficult point. However, we can do relatively easily by virtue of a theory of prisms. The essential theorem is the following due to Bhatt and Scholze:

Theorem 3.10 ([BS23, Theorem 5.6]). *There exists an equivalence of categories between the category of prismatic F-crystals in $\mathcal{O}_{\mathbb{\Delta}}$-modules on $\mathcal{O}_{K,\mathbb{\Delta}}$ denoted by $\mathrm{Vect}^\varphi(\mathcal{O}_{K,\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}})$ and the category of free crystalline $\mathbb{Z}_p$-representations of G_K via the functor*

$$\mathrm{Vect}^\varphi(\mathcal{O}_{K,\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}}) \to \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{crys}}(G_K), \quad \mathcal{N} \mapsto \left(\mathcal{N}\left(\widetilde{\mathbb{A}}_K^+, \varphi(\xi) \right) \otimes_{\widetilde{\mathbb{A}}_K^+} W(\mathbb{C}_p^\flat) \right)^{\varphi=1}.$$

Hence, in order to show the essentially surjectiveness, it suffices to construct crystalline (φ, Γ)-modules over $\widetilde{\mathbb{A}}_K^+$ from prismatic F-crystals in $\mathcal{O}_{\mathbb{\Delta}}$-modules on $\mathcal{O}_{K,\mathbb{\Delta}}$. We show that $\mathcal{N}(\widetilde{\mathbb{A}}_K^+, \varphi(\xi))$ is a crystalline (φ, Γ)-module over $\widetilde{\mathbb{A}}_K^+$ for any prismatic F-crystal $\mathcal{N}$. The first condition (Triviality modulo μ) can be shown from the diagram

$$(\widetilde{\mathbb{A}}_K^+, \varphi(\xi)) \twoheadrightarrow (\widetilde{\mathbb{A}}_K^+/\mu, p) \hookleftarrow ((\widetilde{\mathbb{A}}_K^+/\mu)^{\Gamma_K}, p)$$

in the prismatic site $\mathcal{O}_{K,\mathbb{\Delta}}$. We deduce the second condition from the diagram

$$(\widetilde{\mathbb{A}}_K^+, \varphi(\xi)) \twoheadrightarrow (\widetilde{\mathbb{A}}_K^+/[a], p) \leftarrow (W(k), p)$$

in $\mathcal{O}_{K,\mathbb{\Delta}}$ where a is an element in $\mathcal{O}_{\widehat{\overline{K_\infty}}}^\flat$ such that $v_p(a) \leq p v_p(\pi)$. We remark that $\widetilde{\mathbb{A}}_K^+/\mathfrak{q}_\infty$ cannot be a prism since it is not p-adically complete. The following commutative diagram summarizes the relationship between the categories.

$$
\begin{array}{ccc}
\mathrm{Mod}_{\varphi,\Gamma}^{\text{ét}}(\widetilde{\mathbb{A}}_K) & \xrightarrow[\cong]{\text{[Fon90]}} & \mathrm{Rep}_{\mathbb{Z}_p}(G_K) \\[2mm]
{\scriptstyle \otimes_{\widetilde{\mathbb{A}}_K^+}\widetilde{\mathbb{A}}_K}\Big\uparrow & \circlearrowleft & \Big\uparrow \\[2mm]
\mathrm{Mod}_{\varphi,\Gamma}^{\text{fh,crys}}(\widetilde{\mathbb{A}}_K^+) & \xrightarrow[\cong]{\text{Theorem 3.6}} & \mathrm{Rep}_{\mathbb{Z}_p}^{\text{crys}}(G_K)
\end{array}
$$

$$
\begin{array}{c}
\nwarrow{\scriptstyle \cong} \qquad \circlearrowleft \qquad {\scriptstyle \cong}\nearrow \\
{\scriptstyle \text{evaluate at } (\widetilde{\mathbb{A}}_K^+, \varphi(\xi))} \qquad\qquad {\scriptstyle \text{[BS23, Theorem 5.6]}} \\
\mathrm{Vect}^\varphi(\mathcal{O}_{K,\mathbb{\Delta}}, \mathcal{O}_\mathbb{\Delta})
\end{array}
$$

4. Main theorem for semi-stable representations

Definition 4.1 ([Watc]). A *semi-stable (φ, Γ)-module over $\widetilde{\mathbb{A}}_K^+$* is a finite free $\widetilde{\mathbb{A}}_K^+$-module N equipped with a continuous and semi-linear Γ_K-action and a Γ_K-equivariant $\widetilde{\mathbb{A}}_K^+[1/\varphi(\xi)]$-linear isomorphism $\Phi_N\colon N \otimes_{\widetilde{\mathbb{A}}_K^+, \varphi} \widetilde{\mathbb{A}}_K^+[1/\varphi(\xi)] \cong N[1/\varphi(\xi)]$ such that the following condition holds.

> (Triviality modulo $< \mu$) For all integer $r \geq 1$, $(N/\xi_r)^{\Gamma_K}$ is a finite projective $(\widetilde{\mathbb{A}}_K^+/\xi_r)^{\Gamma_K}$-module and we have a canonical isomorphism
> $$(N/\xi_r)^{\Gamma_K} \otimes_{(\widetilde{\mathbb{A}}_K^+/\xi_r)^{\Gamma_K}} \widetilde{\mathbb{A}}_K^+/\xi_r \xrightarrow{\cong} N/\xi_r$$

We write $\mathrm{Mod}_{\varphi,\Gamma}^{\text{fh,st}}(\widetilde{\mathbb{A}}_K^+)$ for the category of semi-stable (φ, Γ)-modules over $\widetilde{\mathbb{A}}_K^+$.

Remark 4.2. The condition is called a *triviality modulo $< \mu$* in [MT20]. Triviality modulo μ implies triviality modulo $< \mu$, so crystalline (φ, Γ)-modules are semi-stable (φ, Γ)-modules. Triviality modulo $< \mu$ is slightly weaker than triviality modulo μ. This can be seen from the equality of ideals $\mu\widetilde{\mathbb{A}}_K^+ = \bigcap_{r \geq 1} \xi_r\widetilde{\mathbb{A}}_K^+$ ([BMS18, Lemme 3.23.]). Also, Morrow and Tsuji show that these two conditions are equivalent in the geometric context ([MT20, Theorem 1.14.]).

The following theorem is our main theorem on semi-stable representations.

Theorem 4.3 ([Watc]). *There exists an equivalence of categories between the category of semi-stable (φ, Γ)-modules over $\widetilde{\mathbb{A}}_K^+$ and the category of free semi-stable $\mathbb{Z}_p$-representations of G_K denoted by $\mathrm{Rep}_{\mathbb{Z}_p}^{\text{st}}(G_K)$ via the functor*

$$T_{\text{st}}\colon \mathrm{Mod}_{\varphi,\Gamma}^{\text{fh,st}}(\widetilde{\mathbb{A}}_K^+) \xrightarrow{\sim} \mathrm{Rep}_{\mathbb{Z}_p}^{\text{st}}(G_K), \qquad N \mapsto T_{\text{st}}(N) := \left(N \otimes_{\widetilde{\mathbb{A}}_K^+} W(\mathbb{C}_p^\flat)\right)^{\varphi=1}.$$

We have the same relationship with (φ, Γ)-modules over $\widetilde{\mathbb{A}}_K$ as in the crystalline case.

Proposition 4.4 ([Watc]). *Let N be a semi-stable (φ, Γ)-module over $\widetilde{\mathbb{A}}_K^+$. Then $N \otimes_{\widetilde{\mathbb{A}}_K^+} \widetilde{\mathbb{A}}_K$ is the (φ, Γ)-module over $\widetilde{\mathbb{A}}_K$ associated to $T_{\text{st}}(N)$.*

Let us explain an outline of the proof of Theorem 4.3. $T_{\mathrm{st}}(N)$ is a free $\mathbb{Z}_p$-representation of G_K by Theorem 2.10 and Proposition 4.4. In order to show that $T_{\mathrm{st}}(N)$ is semi-stable, we use p-adic monodromy theorem proved by Berger.

Theorem 4.5 ([Ber02, Corollaire 5.22.]). *Let V be a de Rham representation of G_K. Then, there exists a finite Galois extension L/K such that $V|_{G_L}$ is a semi-stable representation of G_L.*

We first show that triviality modulo ξ (when $r = 1$) implies that $T_{\mathrm{st}}(N)$ is a de Rham representation of G_K by constructing an injection

$$(N/\xi)^{\Gamma_K}[1/p] \hookrightarrow (N \otimes_{\widetilde{\mathbb{A}}_K^+} B_{\mathrm{dR}}^+)^{G_K}.$$

This argument is similar to [Wac96, LEMME 1.]. Note that similar lemma to Lemma 3.8 holds in the semi-stable case. Then, there exists a finite Galois extension L/K such that $T_{\mathrm{st}}(N)|_{G_L}$ is a semi-stable representation of G_L by Theorem 4.5. In order to show that it is a semi-stable representation of G_K, we introduce a new filtration on B_{crys} defined by Tsuji. We first recall that for any $r \in \mathbb{Z}_{\geq 0}$, Fontaine defined in [Fon94, 5.1.1.]

$$I^{[r]}A_{\mathrm{crys}} := \{x \in A_{\mathrm{crys}} \mid \varphi^n(x) \in \mathrm{Fil}^r A_{\mathrm{crys}} \text{ for all } n \geq 0\}.$$

We also put $I^{[r]}A_{\mathrm{crys}} := A_{\mathrm{crys}}$ for any $r \in \mathbb{Z}_{<0}$. We set $t := \log[\epsilon] := \sum_{i=1}^{\infty} (-1)^{i+1}\mu^i/i$. Then, we define a filtration on B_{crys} due to Tsuji as follows:

Definition 4.6 (T. Tsuji). For any $r \in \mathbb{Z}$, we define

$$I^{[r]}B_{\mathrm{crys}} := \bigcup_{n \in \mathbb{Z}} t^{-n} I^{[n+r]}A_{\mathrm{crys}}.$$

$I^{[0]}B_{\mathrm{crys}}$ is a subring of B_{crys} stable under the G_K-action and the Frobenius map which contains B_{crys}^+. We have $I^{[1]}B_{\mathrm{crys}} = tI^{[0]}B_{\mathrm{crys}}$ and a canonical isomorphism $A_{\mathrm{inf}}/\mu[1/p] \xrightarrow{\cong} I^{[0]}B_{\mathrm{crys}}/I^{[1]}B_{\mathrm{crys}}$ (cf. [Tsu99, Proposition A3.20]). Then, we consider the following commutative diagram:

$$
\begin{array}{ccccc}
D_{\mathrm{st},L}(V) \otimes_{L_0} B_{\mathrm{st}} & \xrightarrow{\cong} & V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}} & \xrightarrow{\cong} & N \otimes_{\widetilde{\mathbb{A}}_K^+} B_{\mathrm{st}} \\
\cup| & & \cup| & & \cup| \\
D_{\mathrm{st},L}(V) \otimes_{L_0} B_{\mathrm{crys}} \xrightarrow[\cong]{\exp(\mathbf{N}\otimes\mathfrak{u})} (D_{\mathrm{st},L}(V) \otimes_{L_0} B_{\mathrm{st}})^{\mathbf{N}=0} & \xrightarrow{\cong} & V \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}} & \xrightarrow{\cong} & N \otimes_{\widetilde{\mathbb{A}}_K^+} B_{\mathrm{crys}} \\
\cup| & & \cup| & & \cup| \\
D_{\mathrm{st},L}(V) \otimes_{L_0} I^{[0]}B_{\mathrm{crys}} \xrightarrow[\cong]{\exp(\mathbf{N}\otimes\mathfrak{u})} (D_{\mathrm{st},L}(V) \otimes_{L_0} I^{[0]}B_{\mathrm{crys}}[\mathfrak{u}])^{\mathbf{N}=0} & & & & N \otimes_{\widetilde{\mathbb{A}}_K^+} I^{[0]}B_{\mathrm{crys}}
\end{array}
$$

where $V := T_{\mathrm{st}}(N)[1/p]$ and $\mathbf{N}$ is the monodromy operator. Note that the second line is obtained by taking the $\mathbf{N} = 0$ part of the first line. We show that $(D_{\mathrm{st},L}(V) \otimes_{L_0} I^{[0]}B_{\mathrm{crys}}[\mathfrak{u}])^{\mathbf{N}=0}$ and $N \otimes_{\widetilde{\mathbb{A}}_K^+} I^{[0]}B_{\mathrm{crys}}$ coincide via the isomorphisms in the second line by using a similar lemma to Proposition 3.9. Thus, we get an isomorphism $D_{\mathrm{st},L}(V) \otimes_{L_0} I^{[0]}B_{\mathrm{crys}} \cong N \otimes_{\widetilde{\mathbb{A}}_K^+} I^{[0]}B_{\mathrm{crys}}$. By taking a reduction modulo $I^{[1]}B_{\mathrm{crys}}$ and more and then taking a G_K-fixed part, we get an isomorphism $D_{\mathrm{st},K}(V) \cong (N/W(\mathfrak{m}_{\widehat{K_\infty}}^\flat)[1/p])^{\Gamma_K}$, which implies V is a semi-stable representation of G_K. In this way, we see that T_{st} is well-defined.

The functor T_{st} is faithful by the same argument in the crystalline case and fullness can also be proved similarly. So, it remains to show the essentially surjectiveness. We use a theory of log prisms. The essential theorem is the following due to H. Du and T. Liu:

Theorem 4.7 ([DL23, Theorem 5.0.18.]). *There exists an equivalence of categories between the category of log prismatic F-crystals in $\mathcal{O}_{\mathbbl{\Delta}}$-modules on $(\mathcal{O}_K, \pi^{\mathbb{N}})_{\mathbbl{\Delta}_{\log}}$ denoted by $\mathrm{Vect}^{\varphi}((\mathcal{O}_K, \pi^{\mathbb{N}})_{\mathbbl{\Delta}_{\log}}, \mathcal{O}_{\mathbbl{\Delta}})$ and the category of free semi-stable $\mathbb{Z}_p$-representations of G_K via the functor*

$$\mathrm{Vect}^{\varphi}((\mathcal{O}_K, \pi^{\mathbb{N}})_{\mathbbl{\Delta}_{\log}}, \mathcal{O}_{\mathbbl{\Delta}}) \to \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Z}_p}(G_K), \quad \mathcal{N} \mapsto \left(\mathcal{N}\left((\widetilde{\mathbb{A}}_K^+, \varphi(\xi), [a]^{\mathbb{N}})^a \right) \otimes_{\widetilde{\mathbb{A}}_K^+} W(\mathbb{C}_p^{\flat}) \right)^{\varphi=1},$$

where $a \in \mathcal{O}^{\flat}_{\widehat{\overline{K_{\infty}}}}$ such that $v_p(a) = p v_p(\pi)$ and $(\widetilde{\mathbb{A}}_K^+, \varphi(\xi), [a]^{\mathbb{N}})^a$ is the log prism associated to a prelog prism $(\widetilde{\mathbb{A}}_K^+, \varphi(\xi), [a]^{\mathbb{N}})$.

We show that for any log prismatic F-crystal $\mathcal{N}$, $\mathcal{N}\left((\widetilde{\mathbb{A}}_K^+, \varphi(\xi), [a]^{\mathbb{N}})^a \right)$ is a semi-stable (φ, Γ)-module over $\widetilde{\mathbb{A}}_K^+$. Let us explain how we deduce the condition (Triviality modulo $< \mu$). We first prove that it is equivalent to the condition that $(N \otimes_{\widetilde{\mathbb{A}}_K^+, \theta_{\infty}} W(\mathcal{O}_{\widehat{\overline{K_{\infty}}}}))^{\Gamma_K}$ is a finite projective $W(\mathcal{O}_K)$-module and the canonical map

$$(N \otimes_{\widetilde{\mathbb{A}}_K^+, \theta_{\infty}} W(\mathcal{O}_{\widehat{\overline{K_{\infty}}}}))^{\Gamma_K} \otimes_{W(\mathcal{O}_K)} W(\mathcal{O}_{\widehat{\overline{K_{\infty}}}}) \to N \otimes_{\widetilde{\mathbb{A}}_K^+, \theta_{\infty}} W(\mathcal{O}_{\widehat{\overline{K_{\infty}}}})$$

is an isomorphism. This is due to [MT20, Proposition 1.11.]. By [BMS18, Lemma 3.12.], we have an isomorphism $\varprojlim_r \widetilde{\mathbb{A}}_K^+ / \xi_r \xrightarrow[\cong]{\theta_{\infty}} W(\mathcal{O}_{\widehat{\overline{K_{\infty}}}})$. Consider a map

$$\varphi \colon \varprojlim_r \widetilde{\mathbb{A}}_K^+ / \xi_r \to \varprojlim_r \widetilde{\mathbb{A}}_K^+ / \xi_r, \quad (\overline{a_1}, \overline{a_2}, \overline{a_3}, \dots) \mapsto (\overline{\varphi(a_2)}, \overline{\varphi(a_3)}, \overline{\varphi(a_4)}, \dots).$$

It is a Frobenius lift and gives us a bounded prism $(W(\mathcal{O}_{\widehat{\overline{K_{\infty}}}}), p)$. We define a prelog structure $[\pi^p]^{\mathbb{N}}$ on $W(\mathcal{O}_{\widehat{\overline{K_{\infty}}}})$ and we obtain an object $(W(\mathcal{O}_{\widehat{\overline{K_{\infty}}}}), p, [\pi^p]^{\mathbb{N}})^a$ in the log prismatic site $(\mathcal{O}_K, \pi^{\mathbb{N}})_{\mathbbl{\Delta}_{\log}}$. Then, the following diagram shows that the Γ_K-action on $\mathcal{N}\left((\widetilde{\mathbb{A}}_K^+, \varphi(\xi), [a]^{\mathbb{N}})^a \right)$ satisfies triviality modulo $< \mu$.

$$(\widetilde{\mathbb{A}}_K^+, \varphi(\xi), [a]^{\mathbb{N}})^a \xrightarrow{\theta_{\infty}} (W(\mathcal{O}_{\widehat{\overline{K_{\infty}}}}), p, [\pi^p]^{\mathbb{N}})^a \leftarrow (W(\mathcal{O}_K), p, [\pi^p]^{\mathbb{N}})^a$$

The following commutative diagram summarizes the relationship between the categories.

$$
\begin{array}{ccc}
\mathrm{Mod}^{\text{ét}}_{\varphi, \Gamma}(\widetilde{\mathbb{A}}_K) & \xrightarrow[\cong]{[\text{Fon90}]} & \mathrm{Rep}_{\mathbb{Z}_p}(G_K) \\
\otimes_{\widetilde{\mathbb{A}}_K^+} \widetilde{\mathbb{A}}_K \Big\uparrow & \circlearrowleft & \Big\uparrow \\
\mathrm{Mod}^{\text{fh,st}}_{\varphi, \Gamma}(\widetilde{\mathbb{A}}_K^+) & \xrightarrow[\cong]{\text{Theorem 4.3}} & \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Z}_p}(G_K)
\end{array}
$$

evaluate at $(\widetilde{\mathbb{A}}_K^+, \varphi(\xi), [a]^{\mathbb{N}})^a$ $\cong$ $\circlearrowleft$ $\cong$ [DL23, Theorem 5.0.18.]

$$\mathrm{Vect}^{\varphi}((\mathcal{O}_K, \pi^{\mathbb{N}})_{\mathbbl{\Delta}_{\log}}, \mathcal{O}_{\mathbbl{\Delta}})$$

5. MAIN THEOREM FOR DE RHAM REPRESENTATIONS

Definition 5.1 ([Watb]). A *de Rham (φ, Γ)-module over $\widetilde{\mathbb{A}}_K^+$* is a p-adically complete and separated and torsion free $\widetilde{\mathbb{A}}_K^+$-module N equipped with a continuous and semi-linear Γ_K-action with respect to the (p, ξ)-adic topology and a Γ_K-equivariant $\widetilde{\mathbb{A}}_K^+[1/\varphi(\xi)]$-linear isomorphism $\Phi_N \colon N \otimes_{\widetilde{\mathbb{A}}_K^+, \varphi} \widetilde{\mathbb{A}}_K^+[1/\varphi(\xi)] \cong N[1/\varphi(\xi)]$ satisfying the following conditions:

(1) There exists $a, b \in \mathbb{Z}$ with $a \le b$ such that $\varphi(\xi)^{-a} N \subseteq \varphi_N(N) \subseteq \varphi(\xi)^{-b} N$, where $\varphi_N \colon N \to N[1/\varphi(\xi)]$ is the semi-linear map induced from Φ_N.

(2) N/ξ is an almost finite projective $\widetilde{\mathbb{A}}_K^+/\xi$-module.

(3) We have a canonical isomorphism
$$(N/\xi[1/p])^{\Gamma_K} \otimes_K \widehat{K_\infty} \xrightarrow{\cong} N/\xi[1/p].$$

(4) $\mathrm{Hom}_{\widetilde{\mathbb{A}}_K^+}\big([\mathfrak{m}^\flat_{\widehat{K_\infty}}]\widetilde{\mathbb{A}}_K^+, N\widehat{\otimes}_{\widetilde{\mathbb{A}}_K^+} A_{\mathrm{inf}}\big)$ is a finite free A_{inf}-module.

(5) We have a canonical isomorphism
$$N \xrightarrow{\cong} \mathrm{Hom}_{\widetilde{\mathbb{A}}_K^+}\big([\mathfrak{m}^\flat_{\widehat{K_\infty}}]\widetilde{\mathbb{A}}_K^+, N\big); \quad x \mapsto (m \mapsto mx).$$

We write $\mathrm{Mod}_{\varphi,\Gamma}^{\mathrm{fh,dR}}(\widetilde{\mathbb{A}}_K^+)$ for the category of de Rham (φ, Γ)-modules over $\widetilde{\mathbb{A}}_K^+$.

Remark 5.2. The conditions $(1), (2), (4)$ and (5) are satisfied if N is a finite free $\widetilde{\mathbb{A}}_K^+$-module and triviality modulo $< \mu$ implies the condition (3). Thus, semi-stable (φ, Γ)-modules are de Rham (φ, Γ)-modules. Note that de Rham (φ, Γ)-modules are not free nor finitely generated in general, see Example 5.6.

The following theorem is our main theorem on de Rham representations.

Theorem 5.3 ([Watb]). *There exists an equivalence of categories between the category of de Rham (φ, Γ)-modules over $\widetilde{\mathbb{A}}_K^+$ and the category of free de Rham $\mathbb{Z}_p$-representations of G_K via the functor*
$$T_{\mathrm{dR}} \colon \mathrm{Mod}_{\varphi,\Gamma}^{\mathrm{fh,dR}}(\widetilde{\mathbb{A}}_K^+) \xrightarrow{\sim} \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{dR}}(G_K), \qquad N \mapsto T_{\mathrm{dR}}(N) := \Big(N\widehat{\otimes}_{\widetilde{\mathbb{A}}_K^+} W(\mathbb{C}_p^\flat)\Big)^{\varphi=1}.$$

Remark 5.4. By using Theorem 5.3, we can show that for any de Rham (φ, Γ)-module N, N is (p, ξ)-adically complete and separated, N/p^n (resp. N/ξ^n) is an almost finite projective $\widetilde{\mathbb{A}}_K^+/p^n$-module (resp. $\widetilde{\mathbb{A}}_K^+/\xi^n$-module) and torsion free in the sense that for any non-zero divisor $a \in \widetilde{\mathbb{A}}_K^+/p^n$ (resp. $a \in \widetilde{\mathbb{A}}_K^+/\xi^n$), the multiplication map $N/p^n \xrightarrow{\times a} N/p^n$ (resp. $N/\xi^n \xrightarrow{\times a} N/\xi^n$) is injective.

Remark 5.5. Since the underlying module is not finite free in general, we take the p-adic completion in the definition of the functor.

Example 5.6. Let L/K be a finite Galois extension such that $L \cap K_\infty = K$. Consider a regular representation of G_K defined by $G_K \twoheadrightarrow \mathrm{Gal}\,(L/K) \curvearrowright \mathbb{Z}_p[\mathrm{Gal}\,(L/K)]$. It is a de Rham representation of G_K since the G_K-action factors through a finite quotient $\mathrm{Gal}\,(L/K)$. Then, the corresponding (φ, Γ)-module over $\widetilde{\mathbb{A}}_K$ is $\widetilde{\mathbb{A}}_L$ and the corresponding de Rham (φ, Γ)-module over $\widetilde{\mathbb{A}}_K^+$ is $\widetilde{\mathbb{A}}_L^+$. It is known that $\widetilde{\mathbb{A}}_K^+$-module $\widetilde{\mathbb{A}}_L^+$ is not necessarily finite nor free. However, $\widetilde{\mathbb{A}}_L^+/\xi \cong \mathcal{O}_{\widehat{L_\infty}}$ is an almost finite projective $\widetilde{\mathbb{A}}_K^+/\xi \cong \mathcal{O}_{\widehat{K_\infty}}$-module.

Similarly to Theorem 5.3, we take the p-adic completion to obtain the (φ, Γ)-module over $\widetilde{\mathbb{A}}_K$ from a de Rham (φ, Γ)-module over $\widetilde{\mathbb{A}}_K^+$.

Proposition 5.7 ([Watb]). *Let N be a de Rham (φ, Γ)-module over $\widetilde{\mathbb{A}}_K^+$. Then $N\widehat{\otimes}_{\widetilde{\mathbb{A}}_K^+} \widetilde{\mathbb{A}}_K$ is the (φ, Γ)-module over $\widetilde{\mathbb{A}}_K$ associated to $T_{\mathrm{dR}}(N)$.*

Let us explain an outline of the proof of Theorem 5.3. We use arithmetic Breuil-Kisin-Fargues modules introduced by Du as an intermediate object.

Definition 5.8 ([Du19, Definition 4.4.]). An *arithmetic Breuil-Kisin-Fargues module* is a finite free A_{inf}-module $\mathfrak{N}$ equipped with a continuous and semi-linear G_K-action and a G_K-equivariant $A_{\mathrm{inf}}[1/\varphi(\xi)]$-linear isomorphism $\mathfrak{N} \otimes_{A_{\mathrm{inf}},\varphi} A_{\mathrm{inf}}[1/\varphi(\xi)] \cong \mathfrak{N}[1/\varphi(\xi)]$ such that the canonical map $(\mathfrak{N}/\xi[1/p])^{G_K} \otimes_K \mathbb{C}_p \to \mathfrak{N}/\xi[1/p]$ is an isomorphism. We write $\mathbf{BKF}^a(G_K)$ for the category of arithmetic Breuil-Kisin-Fargues modules.

Du showed that $\mathbf{BKF}^a(G_K)$ is equivalent to $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{dR}}(G_K)$ in [Du19, Theorem 4.17.]. We simultaneously give another proof of it. Consider a diagram of categories

$$
\begin{array}{ccc}
 & & \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{dR}}(G_K) \\
 & \overset{N}{\swarrow} & \uparrow T \\
\mathrm{Mod}_{\varphi,\Gamma}^{\mathrm{fh,dR}}(\widetilde{\mathbb{A}}_K^+) & \xrightarrow[F]{} & \mathbf{BKF}^a(G_K)
\end{array}
$$

where each functor is defined as follows:

$$N\colon \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{dR}}(G_K) \to \mathrm{Mod}_{\varphi,\Gamma}^{\mathrm{fh,dR}}(\widetilde{\mathbb{A}}_K^+), \qquad T \mapsto (N_L)^{H_{L/K}}$$

$$F\colon \mathrm{Mod}_{\varphi,\Gamma}^{\mathrm{fh,dR}}(\widetilde{\mathbb{A}}_K^+) \to \mathbf{BKF}^a(G_K), \qquad N \mapsto \mathrm{Hom}_{\widetilde{\mathbb{A}}_K^+}\!\left([\mathfrak{m}_{\widehat{K_\infty}}^\flat]\widetilde{\mathbb{A}}_K^+, N\widehat{\otimes}_{\widetilde{\mathbb{A}}_K^+} A_{\mathrm{inf}}\right)$$

$$T\colon \mathbf{BKF}^a(G_K) \to \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{dR}}(G_K), \qquad \mathfrak{N} \mapsto (\mathfrak{N} \otimes_{A_{\mathrm{inf}}} W(\mathbb{C}_p^\flat))^{\varphi=1}.$$

Here, L/K is a finite Galois extension such that $T|_{G_L}$ is a semi-stable representation of G_L, N_L is the semi-stable (φ,Γ)-module over $\widetilde{\mathbb{A}}_L^+$ corresponding to $T|_{G_L}$, and $H_{L/K} := \mathrm{Gal}\,(L_\infty/K_\infty)$. Note that $T_{\mathrm{dR}} = T \circ F$. By [BMS18, Remark 4.29.], T is full and faithful. For any de Rham (φ,Γ)-module N, we show that the canonical map $N \to F(N)^{H_K}$ is an isomorphism, which implies F is full and faithful. We also prove that $T \circ F \circ N \cong \mathrm{id}$. Then, it formally follows that T, F and N are equivalent.

6. Comparison to other theories

Comparing with the theory of Wach modules ([Ber04]), our results can be applied to any p-adic field. We also have results for semi-stable representations and de Rham representations. However, the condition on crystalline (φ,Γ)-modules is more complicated and the coefficient ring is bigger. Indeed, $\mathbb{A}_K^+$ is Noether, but $\widetilde{\mathbb{A}}_K^+$ is not Noether.

Comparing with the theory of Breuil-Kisin modules ([Kis06]), we construct not only a full and faithful functor but also an equivalence of categories. Also, our results lie in the framework of the theory of (φ,Γ)-modules, which are equivalent to free $\mathbb{Z}_p$-representations. Thus, we can investigate the positions of crystalline, semi-stable and de Rham representations in the whole p-adic Galois representations. Another strong point is that they have a good compatibility with a restriction of G_K-actions of p-adic Galois representations. More precisely, let T be a free crystalline $\mathbb{Z}_p$-representation of G_K, for example, and N be the corresponding crystalline (φ,Γ)-module over $\widetilde{\mathbb{A}}_K^+$. For any finite extension L/K, the crystalline (φ,Γ)-module corresponding to $T|_{G_L}$ is given by $N \otimes_{\widetilde{\mathbb{A}}_K^+} \widetilde{\mathbb{A}}_L^+$. However, Breuil-Kisin modules do not have such a compatibility since their coefficient rings depend on the choice of a compatible sequence of p-power roots of uniformizers.

The theory of arithmetic Breuil-Kisin-Fargues modules ([Du19]) is similar to our results. However, it uses an action of G_K, which is much more complicated than an action of Γ_K.

Comparing with the theory of $(\varphi,\widehat{G})$-modules ([Liu10]), we use smaller field K_∞. The former uses the field generated by all p^n-th root of a uniformizer over K_∞. However, such a big field should not be necessary taking the theory of (φ,Γ)-modules into consideration.

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