

# On the $m$ -step solvable anabelian geometry of mixed-characteristic local fields

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## 1 Background

*Anabelian geometry* is a branch of arithmetic geometry that studies how the arithmetic information of a scheme  $X$  is embedded within its étale fundamental group. One could easily translate this description into the language of field arithmetic by setting

$$X = \text{Spec } k$$

for a field  $k$ : *Anabelian geometry of fields* studies how the field-theoretic/arithmetic invariants (including the field itself) of  $k$  can be “recovered” from its absolute Galois group  $G_k$  as a profinite group. The celebrated theorem of Neukirch-Uchida, one of the earliest validations of anabelian geometry’s central philosophy, demonstrates that number fields can be recovered by their absolute Galois groups [11, Corollary 2], i.e., for two number fields  $F_\circ$  and  $F_\bullet$ , it holds that

$$F_\circ \cong F_\bullet \iff G_{F_\circ} \cong G_{F_\bullet}.$$

This monumental result naturally led to investigations into the analogous statement in a local setting. That is, for two mixed-characteristic local fields  $K_\circ$  and  $K_\bullet$ , it holds that

$$K_\circ \cong K_\bullet \stackrel{?}{\iff} G_{K_\circ} \cong G_{K_\bullet}.$$

It is known that this statement does *not* hold in general (cf., e.g., [12] for a counterexample). However, by imposing stronger constraints on the side of the absolute Galois groups, Mochizuki [6] proved that

$$K_\circ \cong K_\bullet \iff G_{K_\circ} \cong_{\text{filt}} G_{K_\bullet},$$

where  $\cong_{\text{filt}}$  means that the two objects are isomorphic as *filtered* profinite groups. (Here, the absolute Galois groups and their quotients shall be regarded as filtered profinite groups by the *ramification groups in upper numbering*, cf. [9, Chap. IV, §3].)

On the other hand, the recent work of Saïdi-Tamagawa [7] has shown that number fields can be characterized by the “maximal 3-step solvable quotients” (cf. Definition 2.4) of their absolute Galois groups. That is, for number fields  $F_\circ$  and  $F_\bullet$ ,

$$F_\circ \cong F_\bullet \iff G_{F_\circ}^3 \cong G_{F_\bullet}^3,$$

where  $G^m$  denotes the maximal  $m$ -step solvable quotient of a profinite group  $G$ . An implication of this result is that these quotients, despite carrying less information than the full absolute Galois groups, still retain enough arithmetic information to determine the field structure.

## 2 Main results

In this paper, we mainly discuss the following result on mixed-characteristic local fields (analogous to the result of Saïdi-Tamagawa): For two mixed-characteristic local fields  $K_\circ$  and  $K_\bullet$ ,

$$K_\circ \cong K_\bullet \iff G_{K_\circ}^2 \cong_{\text{filt}} G_{K_\bullet}^2.$$

To do so, we begin by recalling Mochizuki's and Saïdi-Tamagawa's results in a *functorial* form.

**Theorem 2.1** (Mochizuki [6, Theorem 4.2]). *Let  $K_\circ$  and  $K_\bullet$  be mixed-characteristic local fields. Then, for an isomorphism*

$$\alpha: G_{K_\circ} \xrightarrow{\cong} G_{K_\bullet}$$

*of filtered profinite groups, there exists a unique isomorphism  $\theta: K_\circ^{\text{alg}} \rightarrow K_\bullet^{\text{alg}}$  such that*

$$\alpha(\sigma) = \theta \circ \sigma \circ \theta^{-1} \tag{1}$$

*for every  $\sigma \in G_{K_\circ}$ .* ◇

Theorem 2.1 implies that the isomorphism class of a given mixed-characteristic local field  $K$  can be determined *functorially* from the isomorphism class of its absolute Galois group  $G_K$  (as a filtered profinite group).

**Corollary 2.2.** *Let  $K_\circ$  and  $K_\bullet$  be mixed-characteristic local fields, and assume that there exists an isomorphism*

$$\alpha: G_{K_\circ} \xrightarrow{\cong} G_{K_\bullet}$$

*of filtered profinite groups. Then there exists an isomorphism  $f: K_\circ \xrightarrow{\cong} K_\bullet$ .* ◇

*Proof.* By Theorem 2.1, there exists an isomorphism  $\theta: K_\circ^{\text{alg}} \rightarrow K_\bullet^{\text{alg}}$  such that (1) holds for all  $\sigma \in G_{K_\circ}$ . Hence

$$\theta(K_\circ) = \theta((K_\circ^{\text{alg}})^{G_{K_\circ}}) = (K_\bullet^{\text{alg}})^{G_{K_\bullet}} = K_\bullet$$

and we obtain the isomorphism  $f = \theta|_{K_\circ}$ . □

We restate the result of Saïdi-Tamagawa.

**Theorem 2.3** (Saïdi-Tamagawa [7, Theorem 1]). *Let  $F_\circ$  and  $F_\bullet$  be number fields, and assume that there exists an isomorphism*

$$A_3: G_{F_\circ}^3 \xrightarrow{\cong} G_{F_\bullet}^3$$

*of profinite groups. Then there exists an isomorphism  $h: F_\circ \xrightarrow{\cong} F_\bullet$ .* ◇

In Theorem 2.3 (resp. Corollary 2.2), there is no clear description of how  $A_3$  (resp.  $\alpha$ ) and  $h$  (resp.  $f$ ) are related to each other; we often call this kind of results *weak bi-anabelian*. Contrastingly, *strong bi-anabelian* (or simply *bi-anabelian*) results are those that provide a functorial description of the relationship between *the morphisms of the étale fundamental groups (absolute Galois groups)* and *the morphisms of the geometric objects (fields)*. For instance, Theorem 2.1 is a strong bi-anabelian result. To proceed to the strong bi-anabelian version of Saïdi-Tamagawa's result, we define the *maximal  $m$ -step solvable extension* of a field.

**Definition 2.4.** For a profinite group  $G$ , we denote by  $\overline{[G, G]}$  the closed subgroup generated by the *commutators* of  $G$ , i.e., the elements of the form  $\sigma\tau\sigma^{-1}\tau^{-1}$ , where  $\sigma, \tau \in G$ .

(1) We inductively define the decreasing sequence

$$G = G^{[0]} \supseteq G^{[1]} \supseteq \dots \supseteq G^{[m]} \supseteq \dots$$

of closed normal subgroups of  $G$ , by  $G^{[m+1]} = \overline{[G^{[m]}, G^{[m]}]}$ . We say that a profinite group  $G$  is *m-step solvable* (resp. *abelian*, resp. *metabelian*) if  $G^{[m]}$  (resp.  $G^{[1]}$ , resp.  $G^{[2]}$ ) is trivial.

(2) We denote by  $G^m$  the quotient  $G/G^{[m]}$ , and call it the *maximal m-step solvable quotient* of  $G$ . We will often write  $G^{\text{ab}}$  instead of  $G^1$ , and call it the *maximal abelian quotient* or *abelianization* of  $G$ .

(3) Let  $k$  be a field. For an integer  $m \geq 0$ , we denote by  $k^m/k$  the subfield of  $k^{\text{sep}}$  fixed by  $G_k^{[m]}$ , and call it the *maximal m-step solvable extension* of  $k$ , so that  $G_k^m = \text{Gal}(k^m/k)$ . We will often write  $k^{\text{ab}}$  instead of  $k^1$ , and call it the *maximal abelian extension* of  $k$ .

◇

**Theorem 2.5** (Saïdi-Tamagawa [7, Theorem 2]). *Let  $m$  be an integer  $\geq 0$ ,  $F_\circ$  and  $F_\bullet$  number fields. Then, for an isomorphism*

$$A_{m+4}: G_{F_\circ}^{m+4} \rightarrow G_{F_\bullet}^{m+4}$$

*of profinite groups, there exists an isomorphism  $\Theta_{m+1}: F_\circ^{m+1} \rightarrow F_\bullet^{m+1}$  such that*

$$A_{m+1}(\sigma) = \Theta_{m+1} \circ \sigma \circ \Theta_{m+1}^{-1}$$

*for every  $\sigma \in G_{F_\circ}^{m+1}$ , where  $A_{m+1}: G_{F_\circ}^{m+1} \rightarrow G_{F_\bullet}^{m+1}$  is the isomorphism induced by  $A_{m+4}$ . Moreover,*

- *if  $m \geq 1$ , the isomorphism  $\Theta_{m+1}$  is uniquely determined by  $A_{m+4}$ ;*
- *if  $m = 0$ , the isomorphism  $\Theta_{m+1}|_{F_\circ}: F_\circ \rightarrow F_\bullet$  is uniquely determined by  $A_{m+4}$ .*

◇

Now, we are ready to proceed to the main results, which are:

- local analogues of Theorem 2.3 and Theorem 2.5;
- refinements of Theorem 2.1 for maximal  $m$ -step solvable quotients ( $m \geq 2$ ).

**Theorem 2.6** ([4, Theorem 1.2]). *Let  $K_\circ$  and  $K_\bullet$  be mixed-characteristic local fields, and assume that there exists an isomorphism*

$$\alpha_2: G_{K_\circ}^2 \xrightarrow{\cong} G_{K_\bullet}^2$$

*of filtered profinite groups. Then there exists an isomorphism  $f: K_\circ \xrightarrow{\cong} K_\bullet$ .*

◇

**Theorem 2.7** ([4, Theorem 1.3]). *Let  $m$  be an integer  $\geq 0$ ,  $K_\circ$  and  $K_\bullet$  mixed-characteristic local fields. Then, for an isomorphism*

$$\alpha_{m+3}: G_{K_\circ}^{m+3} \xrightarrow{\cong} G_{K_\bullet}^{m+3}$$

*of filtered profinite groups, there exists an isomorphism  $\theta_{m+1}: K_\circ^{m+1} \rightarrow K_\bullet^{m+1}$  such that*

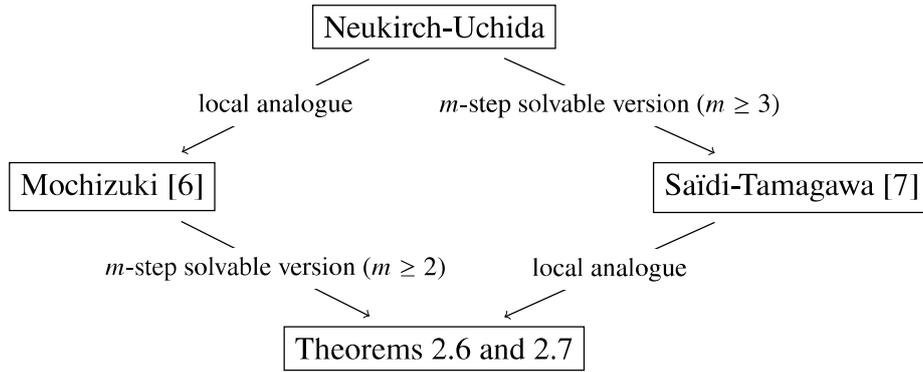
$$\alpha_{m+1}(\sigma) = \theta_{m+1} \circ \sigma \circ \theta_{m+1}^{-1}$$

*for every  $\sigma \in G_{K_\circ}^{m+1}$ , where  $\alpha_{m+1}: G_{K_\circ}^{m+1} \rightarrow G_{K_\bullet}^{m+1}$  is the isomorphism induced by  $\alpha_{m+3}$ . Moreover,*

(i) *if  $m \geq 1$ , the isomorphism  $\theta_{m+1}$  is uniquely determined by  $\alpha_{m+3}$ ;*

(ii) *if  $m = 0$ , the isomorphism  $\theta_{m+1}|_{K_\circ}: K_\circ \rightarrow K_\bullet$  is uniquely determined by  $\alpha_{m+3}$ .*

◇



**Definition 2.8.**

(1) A group  $G$  is said to be *center-free* if the center  $Z(G)$  of  $G$  is trivial, i.e.,

$$\{\sigma \in G \mid \sigma\tau = \tau\sigma \text{ for all } \tau \in G\} = \{1\}.$$

(2) A topological group  $G$  is said to be *slim* if the centralizer  $Z_G(H)$  of any open subgroup  $H$  of  $G$  is trivial in  $G$ , i.e.,

$$\{\sigma \in G \mid \sigma\tau = \tau\sigma \text{ for all } \tau \in H\} = \{1\}.$$

for all open subgroup  $H \subseteq G$ .

◇

**Proposition 2.9.** *Let  $K$  be a mixed-characteristic local field.*

(1) *The absolute Galois group  $G_K$  is slim.*

(2) *The maximal  $m$ -step solvable quotient  $G_K^m$  of  $G_K$  is center-free for all  $m \geq 2$ .*

◇

*Proof.* (1) cf. [3, Lemma 1.8]. (2) cf. [4, Proposition A.1], [7, Proposition 1.1 (ix)]. □

The “uniqueness” parts of Theorems 2.1 and 2.7 are consequences of the *slimness* of the absolute Galois group  $G_{K_\circ}$  and the *center-freeness* of the maximal  $m$ -step solvable quotients  $G_{K_\circ}^m$  ( $m \geq 2$ ), respectively.

*Uniqueness of the isomorphism  $\theta$  in Theorem 2.1.* Suppose that both isomorphisms

$$\theta_1, \theta_2: K_\circ^{\text{alg}} \rightarrow K_\bullet^{\text{alg}}$$

satisfy the condition (1). Then

$$\theta_2^{-1} \circ \theta_1 \in Z_G(G_{K_\circ}),$$

where  $G$  is the absolute Galois group of  $\mathbf{Q}_{p_{K_\circ}}$ . Since  $G$  is slim,  $\theta_2^{-1} \circ \theta_1 = \text{id}_{K_\circ^{\text{alg}}}$ .

*Uniqueness of the isomorphism  $\theta_{m+1}$  in Theorem 2.7.* Suppose that both isomorphisms

$$\theta_{m+1,1}, \theta_{m+1,2}: K_\circ^{m+1} \rightarrow K_\bullet^{m+1}$$

satisfy the condition. Then there exist isomorphisms

$$\theta_1, \theta_2: K_\circ^{\text{alg}} \rightarrow K_\bullet^{\text{alg}}$$

that extend  $\theta_{m+1,1}, \theta_{m+1,2}$ , respectively; we have

$$(\gamma :=) (\theta_2)^{-1} \circ \theta_1 \in \text{Gal}(K_\circ^{\text{alg}}/\mathbf{Q}_{p_{K_\circ}})$$

and

$$\gamma|_{K_\circ^{m+1}} \circ \sigma \circ \gamma^{-1}|_{K_\circ^{m+1}} = (\theta_{m+1,2})^{-1} \circ \theta_{m+1,1} \circ \sigma \circ (\theta_{m+1,1})^{-1} \circ \theta_{m+1,2} = \sigma, \quad (2)$$

for all  $\sigma \in G_{K_\circ}^{m+1}$ . Hence we see that, for all  $x \in K_\circ^\times$ ,

$$\gamma|_{K_\circ^{\text{ab}}} \circ \text{Art}_{K_\circ}(x) \circ \gamma^{-1}|_{K_\circ^{\text{ab}}} = \text{Art}_{K_\circ}(x),$$

and  $\gamma(x) = x$  by local class field theory; it follows that  $\gamma \in \text{Gal}(K_\circ^{\text{alg}}/K_\circ)$ . Furthermore,

$$\gamma|_{K_\circ^{m+1}} \in Z(G_{K_\circ}^{m+1})$$

since (2) holds for all  $\sigma \in G_{K_\circ}^{m+1}$ ; together with the center-freeness of  $Z(G_{K_\circ}^{m+1})$  for  $m \geq 1$ , we conclude that  $\gamma|_{K_\circ^{m+1}} = \text{id}_{K_\circ^{m+1}}$ .

For the rest of this paper, we give a sketch of the proof of Theorem 2.6 and the “existence” part of Theorem 2.7 via the following steps:

- (1) We first explain how Mochizuki utilized  $p$ -adic Hodge theory to prove Theorem 2.1. (§4.1)
- (2) We then explain how Mochizuki’s method can be applied to prove Theorems 2.6 and 2.7 with appropriate modifications. (§4.2)

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### 3 Notation

Given a mixed-characteristic local field  $K$ , we write:

- $\mathcal{O}_K$  for the *ring of integers* of  $K$ ;
- $\mathfrak{p}_K$  for the (unique) *maximal ideal* of  $\mathcal{O}_K$ ;
- $U_K$  for the *unit group*  $\mathcal{O}_K^\times$  of  $\mathcal{O}_K$ ;
- $\mathfrak{k}_K$  for the *residue field*  $\mathcal{O}_K/\mathfrak{p}_K$  of  $K$ ;
- $p_K$  for the *residue characteristic* of  $K$ , i.e., the characteristic of  $\mathfrak{k}_K$ ;
- $a_K$  for the largest integer  $\geq 0$  such that  $K$  contains a  $(p_K^{a_K})^{\text{th}}$  root of unity;
- $d_K$  for the *absolute degree*  $[K : \mathbf{Q}_{p_K}]$  of  $K$ ;
- $e_K$  for the *absolute ramification index* of  $K$ , so that  $p_K \mathcal{O}_K = \mathfrak{p}_K^{e_K}$ ;
- $f_K$  for the *absolute inertia degree*  $[\mathfrak{k}_K : \mathbf{F}_{p_K}]$ , so that  $|\mathfrak{k}_K| = p_K^{f_K}$ , and  $d_K = e_K f_K$ ;
- $\chi_K : G_K \rightarrow \mathbf{Z}_{p_K}^\times$  for the  $p_K$ -*adic cyclotomic character* of  $K$ ;
- $\text{Art}_K : K^\times \rightarrow G_K^{\text{ab}}$  for the *local reciprocity map* of  $K$ .

## 4 Sketch of the proof

### 4.1 Group-theoretic algorithms and $p$ -adic Hodge theory

Mochizuki's proof of Theorem 2.1 relies on the following key steps that characterize a mixed-characteristic local field by its absolute Galois group, equipped with the ramification filtration.

*Recovering the cyclotomic character.* The first step involves recovering the  $p_K$ -adic cyclotomic character

$$\chi_K : G_K \rightarrow \mathbf{Z}_{p_K}^\times$$

of  $K$  from the profinite group  $G_K$ , for a mixed-characteristic local field  $K$ . There are several known *group-theoretic algorithms* (cf. [3]) that recover important invariants of  $K$ , such as the residue characteristic  $p_K$ , the absolute ramification index  $e_K$ , the absolute inertia degree  $f_K$ , the absolute degree  $d_K$  and even the cyclotomic character  $\chi_K$ . This means that, for example, we have a “procedure”  $p$  (defined in language that only depends on the topological group structure) which takes a profinite group  $G$  as an input, and returns an integer  $p(G)$ , such that

$$p(G_K) = p_K$$

for any mixed-characteristic local field  $K$ . Likewise, we have a group-theoretic algorithm  $\chi$ , which takes a profinite group  $G$  and returns a character

$$\chi(G) : G \rightarrow \mathbf{Z}_{p(G)}^\times,$$

such that  $\chi(G_K) = \chi_K$  for any mixed-characteristic local field  $K$  [6, Proposition 1.1]. (Note that we have not used the ramification filtration yet.)

*Recovering the additive structure.* The second key step involves reconstructing the topological  $G_K$ -module  $K^{\text{alg},+}$  (the additive group of  $K^{\text{alg}}$ ) equipped with the  $p_K$ -adic topology (and hence its  $p_K$ -adic completion  $\mathcal{C}(K^{\text{alg}})^+ (= \mathbf{C}_{p_K}^+)$ ) from the *filtered* profinite group  $G_K$ . As in the previous step, this is achieved by establishing group-theoretic algorithms that recover the topological  $G_K$ -module  $K^{\text{alg},+}$  and  $\mathcal{C}(K^{\text{alg}})^+$  from  $G_K$ : We first establish a group-theoretic algorithm  $k^{\text{alg},+}$ , which takes a profinite group  $G$  and returns a  $G$ -module  $k^{\text{alg},+}(G)$  such that

$$k^{\text{alg},+}(G_K) \cong K^{\text{alg},+}$$

as  $G_K$ -modules. We then establish  $\mathcal{O}_{k^{\text{alg}}}^+$ , which takes a filtered profinite group  $G$  and returns a sub- $G$ -module  $\mathcal{O}_{k^{\text{alg}}}^+(G)$  of  $k^{\text{alg},+}(G)$ , such that  $\mathcal{O}_{k^{\text{alg}}}^+(G)$  is mapped onto the additive group of the integer ring of  $K^{\text{alg}}$ . (This is the reason why the ramification filtration is crucial in this step.) This yields another group-theoretic algorithm: Let  $\mathcal{C}_{k^{\text{alg}}}^+(G)$  be the  $p(G) \cdot \mathcal{O}_{k^{\text{alg}}}^+(G)$ -adic completion of  $k^{\text{alg},+}(G)$  for a filtered profinite group  $G$ . Then

$$\mathcal{C}_{k^{\text{alg}}}^+(G_K) \cong \mathcal{C}(K^{\text{alg}})^+$$

as topological  $G_K$ -modules [6, Proposition 2.2].

*Hodge-Tate representations.* Let  $G$  be a profinite group, and  $(\rho, V)$  an  $\ell$ -adic representation of  $G$  for a prime number  $\ell$ . For an  $\ell$ -adic character  $\chi: G \rightarrow \mathbf{Z}_\ell^\times$ , we shall write  $\mathbf{Z}_\ell(\chi)$  for the  $\mathbf{Z}_\ell$ -representation  $(\chi, \mathbf{Z}_\ell^+)$  of  $G$ , and  $V(\chi)$  for the  $\ell$ -adic representation  $V \otimes_{\mathbf{Q}_\ell} (\mathbf{Q}_\ell \otimes_{\mathbf{Z}_\ell} \mathbf{Z}_\ell(\chi))$  of  $G$ .

For a  $p_K$ -adic representation  $(\rho, V)$  of  $G_K$  and an integer  $i$ , we define the  $i^{\text{th}}$  *Hodge-Tate number*  $d_{\text{HT},K}^i(\rho, V)$  of  $(\rho, V)$  as the dimension of the  $K$ -vector space

$$\left( \mathbf{C}_{p_K} \otimes_{\mathbf{Q}_{p_K}} V(-i) \right)^{G_K},$$

where  $V(-i) = (\rho(-i), V(-i))$  denotes the  $(-i)^{\text{th}}$  Tate twist  $V(\chi_K^{-i})$  of  $V$ . It is known that

$$\sum_{i \in \mathbf{Z}} d_{\text{HT},K}^i(\rho, V) \leq \dim_{\mathbf{Q}_{p_K}}(V)$$

holds [2, §5.1], and we say that  $(\rho, V)$  is *Hodge-Tate* when the equality holds.

By the previous steps, it is clear that the Hodge-Tate numbers  $d_{\text{HT},K}^i(\rho, V)$  of a given  $p_K$ -adic representation  $(\rho, V)$  of  $G_K$  can be recovered group-theoretically from the *filtered* profinite group  $G_K$  and its action on  $V$ , i.e., if  $K_\circ$  and  $K_\bullet$  are mixed-characteristic local fields and  $G_{K_\circ} \rightarrow G_{K_\bullet}$  is an isomorphism of filtered profinite groups, then the  $i^{\text{th}}$  Hodge-Tate numbers of the representation

$$G_{K_\bullet} \rightarrow \text{Aut}_{\mathbf{Q}_{p_K}}(V)$$

and the representation defined by the composition

$$G_{K_\circ} \rightarrow G_{K_\bullet} \rightarrow \text{Aut}_{\mathbf{Q}_{p_K}}(V)$$

coincide for all  $i \in \mathbf{Z}$ .

*Uniformizing representations.* Let  $(\rho, V)$  be a  $p_K$ -adic representation of  $G_K$ , and  $E/K$  a finite extension such that  $E/\mathbf{Q}_{p_K}$  is Galois. Suppose that  $V$  is an  $E$ -vector space of dimension 1 and the  $G_K$ -action on  $V$  is  $E$ -linear, so that  $\rho: G_K \rightarrow \text{Aut}_{\mathbf{Q}_{p_K}}(V)$  factors through  $\rho: G_K^{\text{ab}} \rightarrow E^\times$ . We say that a representation  $(\rho, V)$  of this type is *uniformizing* if there exist an open subgroup  $I \subseteq U_K$  and a field homomorphism  $\iota: K \rightarrow E$  such that

$$(\rho \circ \text{Art}_K)|_I = \iota^\times|_I,$$

where  $\iota^\times: K^\times \rightarrow E^\times$  is the group homomorphism induced by  $\iota$ . For example,  $V_E = E^+$  can be regarded as a uniformizing representation by local class field theory: We define the  $G_K$ -action on  $V_E$  via the composition

$$\rho_E: G_K^{\text{ab}} \rightarrow G_E^{\text{ab}} (\cong G_E^{\text{ab}}(0) \oplus \text{Gal}(E^{\text{un}}/E)) \twoheadrightarrow G_E^{\text{ab}}(0) \rightarrow U_E$$

of continuous homomorphisms, where the first (resp. second, resp. third) arrow is the transfer map (resp. the natural surjection, resp. the isomorphism  $\text{Art}_E^{-1}$ ).

**Proposition 4.1.** *Let  $K$  be a mixed-characteristic local field, and let  $E/K$  be a finite extension such that  $E/\mathbf{Q}_{p_K}$  is Galois. Suppose that  $(\rho, V)$  is an  $E$ -linear representation of  $G_K$ , of  $E$ -dimension 1. Then  $(\rho, V)$  is a uniformizing representation if and only if*

$$d_{\text{HT},K}^i(\rho, V) = \begin{cases} [E : K]([K : \mathbf{Q}_{p_K}] - 1) & i = 0 \\ [E : K] & i = 1 \end{cases}.$$

◇

*Proof.* cf. [8, Chap. III, A5]. □

In other words, the uniformizing representations are characterized by their Hodge-Tate numbers; if

- $K_\circ$  and  $K_\bullet$  are mixed-characteristic local fields;
- $\alpha: G_{K_\circ} \rightarrow G_{K_\bullet}$  is an isomorphism of *filtered* profinite groups;
- $E/\mathbf{Q}_p$  is a finite Galois extension containing both  $K_\circ$  and  $K_\bullet$ , where  $p = p_{K_\circ} = p_{K_\bullet}$ ;
- $(\rho_\circ, V)$  is the (uniformizing) representation  $(\rho_E, V_E)$  of  $G_{K_\circ}$ ,

then  $(\rho_\bullet := \rho_\circ \circ \alpha_1^{-1}, V)$  is also uniformizing representation of  $G_{K_\bullet}$ , where  $\alpha_1: G_{K_\circ}^{\text{ab}} \rightarrow G_{K_\bullet}^{\text{ab}}$  is the isomorphism induced by  $\alpha$ . Hence there exist an open subgroup  $I_\circ$  (resp.  $I_\bullet$ ) of  $U_{K_\circ}$  (resp.  $U_{K_\bullet}$ ) and a field homomorphism  $\iota_\circ: K_\circ \rightarrow E$  (resp.  $\iota_\bullet: K_\bullet \rightarrow E$ ) such that  $\alpha_1(I_\circ) = I_\bullet$  and the diagram

$$\begin{array}{ccccccc} & & & K_\circ^\times & & & \\ & & & \nearrow & & \searrow & \\ & & & \subseteq & & \iota_\circ^\times & \\ I_\circ & \xrightarrow{\subseteq} & U_{K_\circ} & \xrightarrow{\text{Art}_{K_\circ}} & G_{K_\circ}^{\text{ab}} & \xrightarrow{\rho_\circ} & E^\times \\ & \downarrow \cong, \alpha_1|_{I_\circ} & \downarrow \cong, \alpha_1|_{U_{K_\circ}} & \downarrow \cong, \alpha_1 & & & \parallel \\ I_\bullet & \xrightarrow{\subseteq} & U_{K_\bullet} & \xrightarrow{\text{Art}_{K_\bullet}} & G_{K_\bullet}^{\text{ab}} & \xrightarrow{\rho_\bullet} & E^\times \\ & \downarrow \subseteq & & & & & \nearrow \\ & & & K_\bullet^\times & & & \end{array}$$

commutes. In particular, we have  $\iota_\bullet|_{I_\bullet} \circ \alpha_1|_{I_\bullet} = \iota_\bullet|_{I_\bullet}$ . Since the sub- $\mathbf{Q}_p$ -vector space generated by  $I_\circ$  (resp.  $I_\bullet$ ) in  $E$  equals  $K_\circ$  (resp.  $K_\bullet$ ) (cf. [6, Lemma 4.1]), it follows that we have the field isomorphism

$$f: K_\circ \xrightarrow{\cong, \iota_\circ} \iota_\circ(K_\circ) = \iota_\bullet(K_\bullet) \xrightarrow{\cong, \iota_\bullet^{-1}} K_\bullet.$$

Finally, we obtain the desired isomorphism  $\theta$  by passage to the limit.

## 4.2 The $m$ -step solvable anabelian geometry

The proofs of Theorems 2.6 and 2.7 follow similar steps but require additional machinery and refinement: We can observe from the previous subsection that it suffices to prove the Hodge-Tate numbers  $d_{\text{HT}, K}^i(\rho, V)$  of a given *abelian*  $p_K$ -adic representation  $(\rho, V)$  of  $G_K$  can be recovered group-theoretically from the *filtered* profinite group  $G_K^2$  (instead of  $G_K$ ) and its action on  $V$ , so we can repeat the restoration process of §4.2, starting from the isomorphism

$$\alpha_2: G_{K_\circ}^2 \rightarrow G_{K_\bullet}^2.$$

of filtered profinite groups. (From the exactly same discussion as the previous subsection, we obtain the isomorphism  $f: K_\circ \rightarrow K_\bullet$ , and by passage to the limit again, the desired isomorphism  $\theta_{m+1}$  (with the 2-step-loss of solvability) [4, §6]. These amount to the proofs of Theorem 2.6 and the ‘‘existence’’ part of Theorem 2.7, respectively.)

*Recovering the cyclotomic character from  $G_K^2$ .* The first step is to observe that:

- the residue characteristic  $p_K$ , the absolute ramification index  $e_K$ , the absolute inertia degree  $f_K$ , and the absolute degree  $d_K$  can be recovered group-theoretically from  $G_K^{\text{ab}}$  [4, Proposition 3.1];
- the cyclotomic character  $\chi_K$  can be recovered group-theoretically from  $G_K^2$  [4, Proposition 3.2].

For example, let  $G$  be a profinite group isomorphic to  $G_K^{\text{ab}}$  for some mixed-characteristic local field  $K$ . Then it follows from local class field theory that we have

$$G \cong (\mathbf{Z}/(p_K^{f_K} - 1)\mathbf{Z})^+ \oplus (\mathbf{Z}/p_K^{a_K}\mathbf{Z})^+ \oplus (\mathbf{Z}_{p_K}^+)^{\oplus d_K} \oplus \widehat{\mathbf{Z}}^+$$

as profinite groups. (Here,  $\widehat{\mathbf{Z}}$  is the profinite completion of the ring  $\mathbf{Z}$  of rational integers.) We denote by  $p(G)$  the uniquely determined prime number  $\ell$  such that

$$\log_\ell |G_{\text{tor}}/\ell \cdot G_{\text{tor}}| \geq 2,$$

where  $G_{\text{tor}}$  is the torsion subgroup of  $G$  and  $G_{\text{tor}} := G/G_{\text{tor}}$ . Then we have

$$p(G_K^{\text{ab}}) = p_K,$$

and see that the invariant  $p_K$  can be recovered group-theoretically from  $G_K^{\text{ab}}$ , instead of the full absolute Galois group  $G_K$ .

*Recovering the additive structure of  $K^m$ .* The second step involves reconstructing the topological  $G_K^{m+1}$ -module  $\mathcal{C}_{K^m}^+$ , the  $p_K$ -adic completion of  $K^{m,+}$ , from the *filtered* profinite group  $G_K^{m+1}$ .

Again, this is achieved by establishing a group-theoretic algorithm that recovers the topological  $G_K^{m+1}$ -module  $\mathcal{E}_{K^m}^+$  from  $G_K^{m+1}$  [4, Proposition 4.3].

*Hodge-Tate representations, revisited.* By the preceding steps, it is clear that the following invariant

$$\dim_K \left( \mathcal{E}_{K^{\text{ab}}} \otimes_{\mathbf{Q}_{p_K}} V(-i) \right)^{G_K}$$

of a given  $p_K$ -adic representation  $(\rho, V)$  of  $G_K$  can be recovered group-theoretically from the *filtered* profinite group  $G_K^2$  and its action on  $V$ , where  $\mathcal{E}_{K^{\text{ab}}}$  denotes the  $p_K$ -adic completion of  $K^{\text{ab}}$ . We complete the proof by showing that this invariant is equal to the  $i^{\text{th}}$  Hodge-Tate number  $d_{\text{HT},K}^i(\rho, V)$  if  $(\rho, V)$  is an *abelian* representation (and hence we can recover the Hodge-Tate numbers  $d_{\text{HT},K}^i(\rho, V)$  of a given abelian  $p_K$ -adic representation  $(\rho, V)$ ).

**Lemma 4.2** ([4, Lemma 5.1]). *Let  $(\rho, V)$  be a  $p_K$ -adic representation of  $G_K$ . Then we have*

$$\left( \mathbf{C}_{p_K} \otimes_{\mathbf{Q}_{p_K}} V \right)^{G_K} = \left( \mathcal{E}((K^{\text{alg}})^{\text{Ker}(\rho)}) \otimes_{\mathbf{Q}_{p_K}} V \right)^{G_K}. \quad (3)$$

◇

*Proof.* We choose a basis  $v_1, \dots, v_n$  of  $V$ . For each  $\sigma \in G_K$ , we shall write  $(a_{ij}(\sigma)) \in \mathbf{GL}_n(\mathbf{Q}_{p_K})$  for the matrix of the linear transformation  $\rho_\sigma := \rho(\sigma)$  with respect to the basis  $v_1, \dots, v_n$ , so that

$$(\rho_\sigma(v_1) \cdots \rho_\sigma(v_n)) = (v_1 \cdots v_n)(a_{ij}(\sigma)).$$

Suppose that  $c_1, \dots, c_n \in \mathbf{C}_{p_K}$ , and that  $c_1 \otimes v_1 + \cdots + c_n \otimes v_n$  belongs to the left hand side of (3). Then for all  $\sigma \in G_K$ ,

$$\begin{aligned} c_1 \otimes v_1 + \cdots + c_n \otimes v_n &= \sigma(c_1) \otimes \rho_\sigma(v_1) + \cdots + \sigma(c_n) \otimes \rho_\sigma(v_n) \\ &= \left( \sum_{j=1}^n \sigma(c_j) a_{1j}(\sigma) \right) \otimes v_1 + \cdots + \left( \sum_{j=1}^n \sigma(c_j) a_{nj}(\sigma) \right) \otimes v_n, \end{aligned}$$

and hence

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = (a_{ij}(\sigma)) \begin{pmatrix} \sigma(c_1) \\ \vdots \\ \sigma(c_n) \end{pmatrix}.$$

In particular, we have

$$c_1 \otimes v_1 + \cdots + c_n \otimes v_n \in \mathbf{C}_{p_K}^{\text{Ker}(\rho)} \otimes_{\mathbf{Q}_{p_K}} V = \mathcal{E}((K^{\text{alg}})^{\text{Ker}(\rho)}) \otimes_{\mathbf{Q}_{p_K}} V,$$

since it holds that  $\sigma(c_1) = c_1, \dots, \sigma(c_n) = c_n$  for all  $\sigma \in \text{Ker}(\rho)$ . (Note that, for any closed subgroup  $H$  of  $G_K$ ,

$$\mathbf{C}_{p_K}^H = \mathcal{E}((K^{\text{alg}})^H)$$

by the theorem of Ax-Sen-Tate—cf. [10], [1], [2, Chap. 3].) □

**Corollary 4.3.** *For an abelian  $p_K$ -adic representation  $(\rho, V)$  of  $G_K$ , we have*

$$d_{\text{HT},K}^i(\rho, V) = \dim_K \left( \mathcal{E}_{K^{\text{ab}}} \otimes_{\mathbf{Q}_{p_K}} V(-i) \right)^{G_K},$$

for all  $i \in \mathbf{Z}$ . ◇

*Proof.* Since  $\rho(-i)$  annihilates the  $G_K^{[1]}$ , the equality holds by the above lemma. □

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