Refined arithmetic topology and arithmetic Chern-Simons theory

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1. Arithmetic topology and 3-dimensional foliated dynamical systems

A fundamental question of arithmetic topology is to find “a good set of knots” in a 3-manifold, which plays a role similar to the set of all (finite) primes of a number ring (cf. [Mo1; Chapter 7]). Candidates of such a set of knots were proposed by Niibo-Ueki [NU] and Mihara [Mi] in purely topological framework. We propose such a good set of knots in the dynamical framework. Our construction is based on the following

Theorem 1.1 (T. Soma [So]). *Any oriented connected closed* 3*-manifold contains a fibered and hyperbolic link.*

Here is the construction. Let M be a given oritented, connected, closed 3-manifold. By Theorem 1.1, there is a fibered and hyperbolic link L in M. Let Σ be a Seifert surface of L and let X be the complement of L in M. Then there is a pseudo-Anosov homeomorphism $\varphi : \Sigma \overset{\approx}{\to} \Sigma$ such that X is homeomorphic to the mapping torus $\Sigma \times_\varphi \mathbb{R}$, the quotient of the product space $\Sigma \times \mathbb{R}$ by the relation $(x, s) \sim (\varphi(x), s + 1)$ $(x \in \Sigma, s \in \mathbb{R})$. Define the dynamical system $\phi : \mathbb{R} \times X \to X$ by $\phi^t([x, s]) := [x, s + t]$ and let $\mathcal{P}$ be the union of the set of closed orbits of ϕ and components of L. Since φ is pseudo-Anosov, $\mathcal{P}$ is a countable infinite set. The following Theorem 1.2 asserts that $\mathcal{P}$ satisfies a property similar to Chebotarev density theorem and so $\mathcal{P}$ looks a good set of knots which may play a role analogous to the set of primes.

Theorem 1.2. *Notations being as above, let $\ell(\gamma)$ be the period* (*length*) *of $\in \mathcal{P}$. Then, for any a finite Galois cover $f : N \to M$ ramified over a finite link L in $\mathcal{P}$ with natural homomorphism $\rho : \pi_1(M \setminus L) \to G := \mathrm{Gal}(N/M)$ and for any conjugacy class $C \subset G$, we have*

$$\lim_{x\to\infty} \frac{1}{x} \#\{\gamma \in \mathcal{P} \mid \rho([\gamma]) \in C, \ell(\gamma) \le x\} = \frac{\#C}{\#G}.$$

This is proved as in a theorem of McMullen [Mc]. We call $(M, \mathcal{P})$ a *number ring like* 3-manifold.

Next, we will see that our number ring provides a standard example of Deninger's foliated dynamical systems in 3-dimension. Recall that a 3-dimensional *foliated dynamical system* is a triple $(M, \mathcal{F}, \phi)$, where

- M is an oriented, connected, closed, smooth 3-manifold,
- $\mathcal{F}$ is a smooth 1-codimensional foliation on M,
- ϕ is a smooth dynamical system on M,

which satisfies the following properties:

- There is a finite number of disjoint compact leaves $C_1, \dots, C_r$ such that $\phi^t(C_i) = C_i$ $(\forall t, i)$ for any i and $t \in \mathbb{R}$.
- For ant $t \in \mathbb{R}$, $\phi^t : M_0 \xrightarrow{\approx} M_0$ is a diffeomorphism, where $M_0 := M \backslash \bigsqcup_{i=1}^r C_i$.
- The flow ϕ is transverse to any leaf in M_0 and ϕ^t sends a leaf to leaf.

We note that the set of non-compact leaves $\{C_1, \dots, C_r\}$ plays a role similar to the set of infinite prime of a number field. The above requirements for a foliated dynamical system seem to be rather strict at a glance, but, the following proposition tells us that any 3-manifold admits the structure of a foliated dynamical system.

Proposition 1.3. *A number ring like* 3*-manifold gives rise to a foliated dynamical system.*

In fact, the construction of a number ring like 3-manifold $(M, \mathcal{P})$ works in the smooth category. So it suffices to define the smooth foliation structure $\mathcal{F}$ on $(M, \mathcal{P})$. For this, let us take (turbulized) fiberes of $\pi : X \to S^1 : [x, s] \mapsto s \bmod \mathbb{Z}$ as leaves outside the tubular neighborhood $V(L)$ of L, and take the Reeb foliation inside $V(L)$, and finally add $\partial V(L)$ as the compact non-transverse leaves. The flow inside $V(L)$ is the natural one transverse to the Reeb foliation.

2. Deninger's program and Connes' adèle class spaces

The motivation that Deninger has introduced the notion of a foliated dynamical system comes from his cohomological approach to number-theoretic zeta functions. Suggested by Grothendieck's formula for the zeta function of a variety over a finite field ([Gr]), he made a vast program for a new arithmetic site and arithmetic cohomology describing L-functions of arithmetic schemes ([D1]). We recall it focusing on the case of a number ring.

Let k be a number field with the ring $\mathcal{O}_k$ of integers and let $\overline{\mathrm{Spec}(\mathcal{O}_k)} := \mathrm{Spec}(\mathcal{O}_k) \sqcup \{\text{infinite primes}\}$. Then Deninger expected that one could associate a 3-dimensional arithmetic foliated space "$\overline{\mathrm{Spec}(\mathcal{O}_k)}$" equipped with smooth dynamical system ϕ^t, where there is a 1 to 1 correspondence between finite primes and closed orbits (knots), and ∞-dimensional arithmetic leafwise cohomology $H^n(\text{“}\overline{\mathrm{Spec}(\mathcal{O}_k)}\text{”})$ on which the infinitesimal generator $\Theta := \lim_{t\to 0}(\phi^{t^*} - 1)/t$ acts such that the complete zeta function of k has the following regularized determinant expression:

$$\begin{aligned}\hat{\zeta}_k(s) &:= \prod_{\mathfrak{p}\in\mathrm{Max}(\mathcal{O}_k)} (1 - \mathrm{N}\mathfrak{p}^{-s})^{-1} \cdot \Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2} \\ &= \prod_{n=0}^{2} \mathrm{det}_{\infty}(\frac{1}{2\pi}(s - \Theta_*)|H^n(\text{“}\overline{\mathrm{Spec}(\mathcal{O}_k)}\text{”})^{(-1)^{i+1}}.\end{aligned}$$

As an evidence of his program, Deninger showed an interpretation of the explicit formula in analytic number theory in his cohomological framework [ibid]. Recently, he constructed a would-be arithmetic foliated phase space “X” with flow for an arithmetic scheme X ([D4]).

On the other hand, there is known a different approach to search for the phase space “$\overline{\mathrm{Spec}(\mathbb{Z})}$”, due to Connes, from non-commutative geometry. It was motivated by his study of the zeros of the Riemann zeta function ([C]). Connes's phase space is the noncommutative quotient of adèles, $\mathbb{Q}^\times\backslash\mathbb{A}/\hat{\mathbb{Z}}^\times$, so that the space of functions on it is dual to the Bost-Connes system ([BC]), which provides the functional analytic tool in the study of the Riemann zeta function. It is equipped with a foliation and the action of $\mathbb{R} \simeq \mathrm{Gal}(\mathbb{R}_+^{\max}/\mathbb{B})$, where $\mathbb{B}$ is the Boolean semiring and $\mathbb{R}_+^{\max}$ is the tropical semiring. Here a prime p is seen as the closed orbit $\mathbb{R}/(\log p)\mathbb{Z}$ in the adèle class space. Recently, Connes and Concani made a discovery that the analogy between knots and primes admits a geometric realization within the adèle class spaces ([CC]).

Although both Deninger's phase space and Connes' one carry the structures of foliation and dynamical system, their approaches seems deeply different. Deninger's approach can be generalized for higher dimensional arithmetic schemes and their motivic L-functions, while Connes' approach generalizes readily to automorphic L-functions but not to motivic L-functions. A relation between them has been unknown for a long time*.

3. Regularized determinant expression of the zeta function of a 3-dimensional Riemannian foliated dynamical system

Searching for his conjectural arithmetic cohomology theory, Deninger has noticed the intimate analogies between $\overline{\mathrm{Spec}(\mathcal{O}_k)}$ and 3-dimensional Riemannian foliated dynamical system $\mathfrak{S} = (M, g, \mathcal{F}, \phi^t)$ and conjectured the regularized determinant formula for the zeta function of $\mathfrak{S}$, using the ∞-dimensional

*After the conference, I found a relation between Deninger's foliated dynamical systems for Spec($\mathbb{Z}$) and Connes' adèle class spaces. See [Mo2].

leafwise cohomology ([D2], [D3]). One of our results is a proof of this conjecture, which supports the existence of an ∞-dimensional cohomological description for the Dedekind zeta function $\zeta_k(s)$.

Let $\mathfrak{S} = (M, g, \mathcal{F}, \phi^t)$ be a 3-dimensional Riemannian foliated dynamical system, namely,
- M is an oriented, connected, closed, Riemannian 3-manifold with metric g,
- $\mathcal{F}$ is a 1-codimensional foliation by closed surfaces,
- $\phi : \mathbb{R} \times M \to M$ is a smooth dynamical system

such that
- ϕ is transverse to leaves and send leaves to leaves,
- g is bundle-like, i.e., $T(M) = T(\mathcal{F}) \perp T_0(M)$,

where $T(\mathcal{F})$ = the tangent bundle to the foliation and $T_0(M)$ is the line bundle over M generated by the vector field of ϕ.
First, we note the following classification theorem.

Theorem 3.1 ([ÁKL], [KMNT]). *Let $\mathfrak{S} = (M, g, \mathcal{F}, \phi^t)$ be a 3-dimensional Riemannian foliated dynamical system. Then one of the followings holds.*
(i) *M is a fiber bundle over S^1 (i.e., mapping torus of a diffeomorphism of a closed surface) and $\mathcal{F}$ is a bundle foliation.*
(ii) *$\mathcal{F}$ is a minimal $\mathbb{R}$-Lie foliation so that any leaf is dense.*

The leafwise cohomology group is defined by the cohomology of the leafwise de Rham complex $(\Gamma(M, \wedge^n T^*(\mathcal{F})), d^n_{\mathcal{F}})$:

$$\bar{H}^n_{\mathcal{F}}(M) := \mathrm{Ker}(d^n_{\mathcal{F}})/\overline{\mathrm{Im}(d^{n-1}_{\mathcal{F}})},$$

which is ∞-dimensional in general, and is acted by the infinitesimal operator defined by

$$\Theta(h) := \lim_{t\to 0} \frac{\phi^{t*}(h) - h}{t} \quad (h \in H^n_{\mathcal{F}}(M)).$$

Let $\mathcal{P}$ the set of closed orbits of the flow ϕ and $\ell(\gamma)$ denotes the period (length) of $\gamma \in \mathcal{P}$. The index of γ for $k \in \mathbb{N}$ is defined by

$$\varepsilon_\gamma(k) := \mathrm{sgn}\det(\mathrm{id} - T_x(\phi^{k\ell(\gamma)}) \mid T_x(\mathcal{F})), \ \ \varepsilon_\gamma := \varepsilon_\gamma(1).$$

We assume that $a := \limsup_{x\to\infty} \frac{1}{x}\#\{\gamma \in \mathcal{P} \mid \ell(\gamma) < x\} < \infty$ exists and that $\varepsilon_\gamma(k) = \varepsilon_\gamma \in \{\pm 1\}$ for all $\gamma \in \mathcal{P}$ and k.
The zeta function of $\mathfrak{S}$ is then defined by

$$\zeta_{\mathfrak{S}}(s) := \prod_{\gamma\in\mathcal{P}} (1 - e^{-s\ell(\gamma)})^{-\varepsilon_\gamma} \ \ (\mathrm{Re}(s) > a).$$

For the case that $\mathfrak{S}$ is of type (i) in Theorem 3.1, we have the following

Theorem 3.2 ([ÁKM]). *Let $\mathfrak{S} = (M, g, \mathcal{F}, \phi^t)$ be a 3-dimensional Riemannian foliated dynamical system of type (i) so that M is a mapping torus of a diffeomorphism f of a closed surface Σ of genus ≥ 1. Assume that f is Anosov when S is a torus and f is an Axiom A diffeomorphism when the genus of S ≥ 2. Then we have*
(1) $\mathrm{Sp}(\Theta|\bar{H}^n_{\mathcal{F}}(M)_{\mathbb{C}}) = \{\log\alpha + 2\pi\sqrt{-1}k|\alpha \in \mathrm{Sp}(H^n(f)), k \in \mathbb{Z}\}$.
(2) $\zeta_{\mathfrak{S}}(s) = \prod_{n=0}^{2} \det_\infty(\mathrm{id} - \Theta \mid \bar{H}^n_{\mathcal{F}}(M)_{\mathbb{C}})^{(-1)^{n+1}}$, *where* $\det_\infty$ *stands for the regularized determinant.*

The case of type (i) has the similar nature to the case of an algebraic curve over a finite field. The proof follows essentially from the classical Lefschetz trace formula for the diffeomorphism f of Σ.
The case that $\mathfrak{S}$ is of type (ii) in Theorem 3.1 is much harder and much more interesting. We have the following

Theorem 3.3 ([ÁKM]). *Let $\mathfrak{S} = (M, g, \mathcal{F}, \phi)$ be a 3-dimensional Riemannian foliated dynamical system of type (ii).*
Assume that ϕ is leafwise homotopic to a foliated flow ψ, which is isometric, i.e., $g(T_x(\psi^t)(v), T_x(\psi^t)(w)) = g(v, w)$ for any $x \in M$ and $v, w \in T_x(\mathcal{F})$ and $t \in \mathbb{R}$. Then we have
(1) $\Theta = 0$ *on* $\bar{H}^0_{\mathcal{F}}(M) = \bar{H}^2_{\mathcal{F}}(M)$ *and* $\mathrm{Sp}(\Theta|\bar{H}^1_{\mathcal{F}}(M)_{\mathbb{C}}) \subset \mathbb{R}\sqrt{-1}$.
(2) $\zeta_{\mathfrak{S}}(s) = \prod_{n=0}^{2} \det_\infty(\mathrm{id} - \Theta \mid \bar{H}^n_{\mathcal{F}}(M)_{\mathbb{C}})^{(-1)^{n+1}}$.

A key ingredient for the proof is the following distributional trace formula due to Álvarez López and Kordyukov:

Theorem 3.4 ([ÁK]). *In the space of distributions $C^\infty_c(\mathbb{R}_+)'$, one has the equality*

$$\sum_{n=0}^{2}(-1)^n \mathrm{Tr}(\phi^* \mid \bar{H}^n_{\mathcal{F}}(M)) = \sum_{\gamma} \ell(\gamma) \sum_{k \in \mathbb{Z}\setminus 0} \varepsilon_\gamma(n)\delta_{k\ell(\gamma)}.$$

Here the l.h.s. is $2 - \sum_{\rho \in \mathrm{Sp}^1(\Theta)} e^{\rho t}$, *where* $e^{\rho t}(F) = \int_0^\infty e^{\rho t} F(t) dt$.

This formula may seen as a dynamical analog of the explicit formula in analytic number theory. Another key object is the dynamical spectral ξ-function

$\xi_n(z,s)$ of $\mathfrak{S}$ ($n=0,1,2$) defined by

$$\xi_n(z,s) := \sum_{\alpha\in \mathrm{Sp}^n(\Theta)} (s-\alpha)^{-z} \quad (\mathrm{Re}(s), \mathrm{Re}(z) > 1).$$

Using the distributional trace formula, we can show the following Proposition 3.4, from which Theorem 3.3 is deduced.

Proposition 3.5. *For* $\mathrm{Re}(s) > c$ *(*c *is an explicit constant) and* $\mathrm{Re}(z) > 5$*,*

$$\sum_{n=0}^{2}(-1)^n \xi_n(z,s) = -\frac{1}{2\pi\sqrt{-1}}\int_{\mathcal{C}} t^{-z}\frac{\zeta'_{\mathfrak{S}}}{\zeta_{\mathfrak{S}}}(s-t)dt,$$

where $\mathcal{C}$ *be the contour consisting of the lower edge of the cut from* $-\infty$*, the circle* $x=\delta e^{\sqrt{-1}\varphi}$ *with* $-\pi\le\varphi\le\pi$ *and the upper edge of the cut from* $-\delta$ *to* $-\infty$*.*

4. Arithmetic Chern-Simons theory from the viewpoint of étale metaplectic covers

In recent years, M. Kim and his collaborators employed arithmetic topology to construct an arithmetic analogue of 3-dimensional Chern-Simons theory with finite or p-adic gauge group ([Ki], [CKKPY]). We give an alternative to arithmetic Chern-Simons theory whose gauge group is any reductive group scheme, using the theory of étale metaplectic covers due to Zhao ([Z]).

Let G be a reductive group scheme. We assume here for simplicity that G is semi-simple, simply-connected group scheme. Let BG denote the classifying stack. Our phase space of "gauge fields" is the groupoid $\mathcal{F}_X^G$ of G-torsors over a scheme X. Fix $n\in\mathbb{Z}\ge 2$.

Recall that the level set for Dijkgraaf-Witten theory for 3-manifolds and a finite gauge group G is $H^3(BG,\mathbb{R}/\mathbb{Z}) = H^4(BG,\mathbb{Z})$ ([DW], [Go]). So Kim takes $H^3(G,\mathbb{Z}/n\mathbb{Z}(1)$ as the level set for his arithmetic Chern-Simons theory for number rings. But we take $H^4(BG;\mathbb{Z}/n\mathbb{Z}(2))$ as the level set, since it is where the universal Chern class $c_2(EG)$ is living and it fits with the following "motivic" analogy:

Geometry	**Arithmetic**
$\mathrm{U}(1)=\mathbb{R}/\mathbb{Z}$	$\mathbb{G}_m=\mathbb{Z}(1)[1]$
$H^4(BG;\mathbb{Z})$ $= H^3(BG;\mathrm{U}(1))$	$H^4(BG;\mathbb{Z}/n\mathbb{Z}(1))$ $= H^3(BG;\mathbb{Z}/n\mathbb{Z}(1)[1])$
$H^4(BG;\mathbb{Z}(2)_D)=\mathbb{Z}c_2(EG)$	$H^4(BG;\mathbb{Z}/n\mathbb{Z}(2))=\mathbb{Z}/n\mathbb{Z}c_2(EG)$

For a scheme X with étale cohomological dimension d and $H^d(X, \mathbb{Z}/n\mathbb{Z}(1)) = \mathbb{Z}/n\mathbb{Z}$, the key ingredient for our construction is the transgression map

$$\mathrm{trans}^d : H^4(BG, \mathbb{Z}/n\mathbb{Z}(2)) \longrightarrow H^{4-d}(\mathcal{F}_X^G, \mathbb{Z}/n\mathbb{Z}(1)).$$

Here $H^4(BG, \mathbb{Z}/n\mathbb{Z}(2))$ classifies rigidified sections $BG \to B^4\mathbb{Z}/n\mathbb{Z}(2)$, called universal étale metaplectic covers of G (cf. [Z]). For arithmetic examples, we have the following description:

- $d = 3$, $X = \mathrm{Spec}(\mathcal{O}_k)$ with $\mu_n := \mathbb{Z}/n\mathbb{Z}(1)$ being contained in k. Then we have $H^3(\mathrm{Spec}(\mathcal{O}_k), \mathbb{Z}/n\mathbb{Z}(1)) \simeq \mathbb{Z}/n\mathbb{Z}$ and $\mathcal{F}_X^G = G(k)\backslash G(\mathbb{A}_k)/G(\widehat{\mathcal{O}})$. So we have
$$H^1(\mathcal{F}_X^G, \mathbb{Z}/n\mathbb{Z}(1)) = \{\mu_n\text{-torsors over } G(k)\backslash G(\mathbb{A}_k)/G(\widehat{\mathcal{O}})\}.$$
Hence $\mathrm{trans}^3(c)$ is an μ_n-torsor $G(k)\backslash \widetilde{G(\mathbb{A}_k)}/G(\widehat{\mathcal{O}})$ over $G(k)\backslash G(\mathbb{A}_k)/G(\widehat{\mathcal{O}})$.
- $d = 2$, $X = \mathrm{Spec}(k_\mathfrak{p})$ with $\mu_n \subset k_\mathfrak{p}$. Then we have $H^2(\mathrm{Spec}(k_\mathfrak{p}); \mathbb{Z}/n\mathbb{Z}(1)) \simeq \mathbb{Z}/n\mathbb{Z}$ and $\mathcal{F}_X^G$ is the groupoid $\mathrm{pt}/G(k_\mathfrak{p})$ (Note $H^1(\mathrm{Spec}(k_\mathfrak{p}); G) = 0$). So we have
$$\begin{aligned} H^2(\mathcal{F}_X^G, \mathbb{Z}/n\mathbb{Z}(1)) &= H^2(G(k_\mathfrak{p}), \mu_n) \\ &= \{\text{central extensions of } G(k_\mathfrak{p}) \text{ by } \mu_n\}/\text{isom.} \end{aligned}$$

Hence $\mathrm{trans}^2(c)$ is a metaplectic cover $1 \to \mu_n \to \widetilde{G(k_\mathfrak{p})} \to G(k_\mathfrak{p}) \to 1$.
For example, when $G = \mathrm{SL}(2)$, $\mathrm{trans}^2(c_2(E\mathrm{SL}(2)))$ coincides with the class of Kubota's 2-cocycle ([Ku]) defined by

$$\begin{array}{ccc} \mathrm{SL}_2(k_\mathfrak{p}) \times \mathrm{SL}_2(k_\mathfrak{p}) & \to & \mu_n \\ (\sigma, \tau) & \mapsto & \{\alpha(\sigma), \alpha(\tau)\}_\mathfrak{p}\{-\alpha(\sigma)^{-1}\alpha(\tau), \alpha(\sigma\tau)\}_\mathfrak{p} \end{array},$$

where $\alpha(\sigma) := \begin{cases} c & \cdots c \neq 0, \\ d & \cdots c = 0 \end{cases}$ for $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(k_\mathfrak{p})$ and $\{\cdot\,,\,\cdot\}_\mathfrak{p}$ is n-th the Hilbert symbol in $k_\mathfrak{p}$.

Now let us see the connection with Kim's arithmetic Chern-Simons functional for a number ring $X = \mathrm{Spec}(\mathcal{O}_k)$ with $\mu_n \subset k$, where the level set is $H^3(BG, \mathbb{Z}/n\mathbb{Z})$. For this, we note the isomorphisms obtained by working in higher category

$$H^{4-i}(BG; \mathbb{Z}/n\mathbb{Z}(2)) \simeq \pi_i(\Gamma(BG, B^4\mathbb{Z}/n\mathbb{Z}(2))).$$

By this isomorphism, a class $c \in H^3(BG; \mathbb{Z}/n\mathbb{Z}(2))$ corresponds to an element of $\pi_1(\Gamma(BG, B^4\mathbb{Z}/n\mathbb{Z}(2)))$, which gives an automorphism of the base point, namely, the trivial metaplectic cover in $\Gamma(BG, B^4\mathbb{Z}/n\mathbb{Z}(2))$. By the transgression with $H^4(BG, \mathbb{Z}/n\mathbb{Z}(2)$ being the level set , it gives an automorphism of the trivial $\mathbb{Z}/n\mathbb{Z}(1)$-torsor over $\mathcal{F}_X^G$, hence the functional:

$$\mathcal{F}_X^G \longrightarrow \mathbb{Z}/n\mathbb{Z}(1).$$

This is the image of c under the transgression with $H^3(BG, \mathbb{Z}/n\mathbb{Z}(2))$ being the level set

$$H^3(BG, \mathbb{Z}/n\mathbb{Z}(2)) \longrightarrow H^0(\mathcal{F}_X^G, \mathbb{Z}/n\mathbb{Z}(1))$$

and may be seen as Kim's functional for the case that the phase space of gauge fields is $\mathcal{F}_X^G$.

Actually, Kim's gauge fields are characters of Galois representations. Let $X_S := \mathrm{Spec}(\mathcal{O}_k) \setminus S$, where S contains primes over p. His phase space with gauge group being a p-adic Lie group is $\mathcal{M}_{X_S}^G := \mathrm{Hom}_{\mathrm{cont}}(\pi_1(X_S), G(\mathbb{Z}_p))/\mathrm{conj.}$. If we fix a prime number ℓ, then elements of $H^4(BG; \mathbb{Z}_\ell(2))$ transgress to $H^3(BG(\mathbb{Z}_p), \mathbb{Z}_\ell(2))$, which defines the $\mathbb{Z}_\ell(1)$-valued functional on $\mathcal{M}_{X_S}^G$:

$$\mathcal{M}_{X_S}^G \longrightarrow \mathbb{Z}_\ell(1).$$

Question: For $\ell = p$, would this functional be the p-adic L-function which Kim looked for in [Ki] ?

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