On the Supersingular Locus of the $\mathrm{GU}(2, n-2)$ Shimura Variety

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1 Introduction

Shimura varieties have been used, with great success, towards applications in number theory. There are many such applications based on the study of integral models and their reductions. It is known that in some cases, the supersingular (or basic) locus of the reduction of a Shimura variety admits a simple description. For example, Vollaard-Wedhorn [27] described the supersingular locus of the Shimura variety of $\mathrm{GU}(1, n-1)$ at an inert prime as a union of (classical) Deligne-Lusztig varieties. Also in the $\mathrm{GU}(2, 2)$-case, Howard-Pappas [14] proved the existence of a similar description. After [27] and [14], Görtz, He and Nie classified the cases where the supersingular locus is naturally a union of Deligne-Lusztig varieties, called the *fully Hodge-Newton decomposable* cases (cf. [7], [9]). As a result, the Shimura variety of $\mathrm{GU}(2, n-2)$ is not fully Hodge-Newton decomposable if $n \geq 5$. The studies by Görtz, He and Nie are based on the fact that the study of the perfection of the supersingular locus can be reduced to a study of an affine Deligne-Lusztig variety via the Rapoport-Zink uniformization. Such simple descriptions have been applied towards the Kudla-Rapoport program [16], Zhang's Arithmetic Fundamental Lemma [29] and the Tate conjecture for certain Shimura varieties [25], [13].

Recently, new simple descriptions have been discovered in some cases which are not fully Hodge-Newton decomposable (cf. [26], [21], [23]). In the $\mathrm{GU}(2, n-2)$-case, Fox, Howard and Imai studied irreducible components of the supersingular locus [5],[4]. The core of their method is the Chen-Zhu conjecture, which is a theorem on irreducible components of affine Deligne-Lusztig varieties. After these results, the author found an explicit stratification of the affine Deligne-Lusztig variety associated to the Shimura variety of $\mathrm{GU}(2, n-2)$ in terms of Deligne-Lusztig varieties [22]. This stratification gives another description of the irreducible components by taking the closure of the top-dimensional strata. In this paper, we will give a survey of [22] (see [5, §11] for the Rapoport-Zink uniformization in this case).

Let F be a non-archimedean local field with finite residue field $\mathbb{F}_q$ of prime characteristic p, and let L be the completion of the maximal unramified extension of F. We write $\mathcal{O}$ for the valuation ring of L. To simplify the exposition, we assume that F has mixed characteristic in the introduction. Let G be the unramified general unitary group of degree n over F.

Let μ be the cocharacter of G corresponding to $z \to (\mathrm{diag}(1, \dots, 1, z^{-1}, z^{-1}), z^{-1})$ under an isomorphism $G_L \cong \mathrm{GL}_n \times \mathbb{G}_m$. Let $X_\mu(b)$ denote the affine Deligne-Lusztig variety attached to μ and $b = (\mathrm{diag}(1, \dots, 1), \varpi^{-1})$, where ϖ denotes a uniformizer of F. Including $X_\mu(b)$, all varieties in the introduction are perfect schemes. For example, Deligne-Lusztig varieties actually mean the perfection of them.

In the fully Hodge-Newton decomposable cases, the decomposition into Deligne-Lusztig varieties is a refinement of the *Ekedahl-Oort stratification* of affine Deligne-Lusztig varieties (see §2.2). This stratification itself exists in general even outside the fully Hodge-Newton decomposable cases. It is the local analogue of the stratification defined in the global context of Shimura varieties in [12]. An Ekedahl-Oort stratum in $X_\mu(b)$ actually corresponds to the intersection of a global Ekedahl-Oort stratum with the supersingular locus (cf. [10, §2.5]).

Set $\varpi^\mu = (\mathrm{diag}(1, \dots, 1, \varpi^{-1}, \varpi^{-1}), \varpi^{-1})$. Then the Ekedahl-Oort stratification of $X_\mu(b)$ (and the corresponding global one) is parametrized by

$$^S\mathrm{Adm}(\mu) = \{w_{k,l} \coloneqq \varpi^\mu (n-1 \ \cdots \ k+1 \ k)(n \ \cdots \ l+1 \ l) \mid 1 \leq k < l \leq n\},$$

which is a subset of the Iwahori-Weyl group. Let us denote by $\pi(X_{w_{k,l}}(b))$ the Ekedahl-Oort stratum corresponding to $w_{k,l}$. See §2.2 for the precise definition. Let $^S\mathrm{Adm}(\mu)_{\neq\emptyset} \coloneqq \{w_{k,l} \in {}^S\mathrm{Adm}(\mu) \mid \pi(X_{w_{k,l}}(b)) \neq \emptyset\}$. Let $^S\mathrm{Adm}(\mu)_{\mathrm{DL}} \subseteq {}^S\mathrm{Adm}(\mu)_{\neq\emptyset}$ denote a certain subset consisting of all $w_{k,l}$ such that $\pi(X_{w_{k,l}}(b))$ is naturally a union of Deligne-Lusztig varieties (see Proposition 2.2). The equality holds if and only if $n \leq 4$. It is also known by Görtz-He-Nie that the global Ekedahl-Oort stratum corresponding to $w_{k,l} \in {}^S\mathrm{Adm}(\mu)$ is contained in the supersingular locus if and only if $w_{k,l} \in {}^S\mathrm{Adm}(\mu)_{\mathrm{DL}}$ (cf. [9, Proposition 5.6], [28, Proposition 4.2]). Let $^S\mathrm{Adm}(\mu)_{\neq\mathrm{DL}}$ denote $^S\mathrm{Adm}(\mu)_{\neq\emptyset} \setminus {}^S\mathrm{Adm}(\mu)_{\mathrm{DL}}$. This set parametrizes the global Ekedahl-Oort strata which intersect but are not contained in the supersingular locus (cf. [9, Lemma 7.6]). The following theorem gives a complete description of $^S\mathrm{Adm}(\mu)_{\neq\emptyset}$. This was only known in fully Hodge-Newton decomposable cases $n \leq 4$ and the case $n = 5$ studied in [1, Theorem 6.7]. See Example 3.4 for the cases where $n = 13, 14$.

Theorem A (Theorem 3.3)**.** We have

$$^S\mathrm{Adm}(\mu)_{\mathrm{DL}} = \{w_{k,l} \mid k = 1 \text{ or } l \leq \tfrac{n+2}{2}\}.$$

Moreover, $w_{k,l} \in {}^S\mathrm{Adm}(\mu)_{\neq\mathrm{DL}}$ if and only if $3 \leq k < \frac{n+2}{2} < l \leq n-1$ and one of the following conditions is satisfied:

(i) k is odd and $k + l \leq n + 2$.

(ii) $l \equiv n - 1 \pmod 2$ and $k + l \geq n + 3$.

The following theorem is our main theorem, whose description coincides with the conventional ones in fully Hodge-Newton cases $n \leq 4$.

Theorem B (Theorem 3.6)**.** Let $w_{k,l} \in {}^S\mathrm{Adm}(\mu)_{\neq\emptyset}$. Then there exists a standard parahoric subgroup $P_{k,l}$, a Deligne-Lusztig variety $X_{k,l}$ and an irreducible component $Y_{k,l}$ of $\pi(X_{w_{k,l}}(b))$ such that $\pi(X_{w_{k,l}}(b)) = \sqcup_{j \in G(F)/G(F) \cap P_{k,l}} jY_{k,l}$ and $Y_{k,l}$ is an iterated fibration over $X_{k,l}$. In particular, the variety $X_\mu(b)$ is naturally a disjoint union of iterated fibrations over Deligne-Lusztig varieties.

An iterated fibration is the composite of some Zariski-locally trivial $\mathbb{A}^1$-bundles (cf. §2.3). The variety $X_\mu(b)$ and each Ekedahl-Oort stratum admit an action of $G(F)$. The fibration in Theorem B is $G(F)$-equivariant. In Theorem 3.6, both $P_{k,l}$ and $X_{k,l}$ are described explicitly. For example, if $w_{k,l} \in {}^S\mathrm{Adm}(\mu)_{\neq \mathrm{DL}}$, then $P_{k,l} = G(\mathcal{O})$ and $X_{k,l}$ is a Deligne-Lusztig variety in a partial flag variety for $G(\overline{\mathbb{F}}_q)$ determined by $w_{k,l}$ (see Remark 3.10).

2 Preliminaries

We will usually drop the adjective "perfect" for notational convenience.

2.1 Notation

Let F be a non-archimedean local field with finite residue field $\mathbb{F}_q$ of prime characteristic p, and let L be the completion of the maximal unramified extension of F. Let σ denote the Frobenius automorphism of L/F. Further, we write $\mathcal{O}$ (resp. $\mathcal{O}_F$) for the valuation ring of L (resp. F). Finally, we denote by ϖ a uniformizer of F and by v_L the valuation of L such that $v_L(\varpi) = 1$.

Let G be an unramified connected reductive group over $\mathcal{O}_F$. Let $B \subset G$ be a Borel subgroup and $T \subset B$ a maximal torus in B, both defined over $\mathcal{O}_F$. For a cocharacter $\mu \in X_*(T)$, let ϖ^μ be the image of $\varpi \in \mathbb{G}_m(F)$ under the homomorphism $\mu\colon \mathbb{G}_m \to T$.

Let $\Phi = \Phi(G, T)$ denote the set of roots of T in G. We denote by Φ_+ (resp. Φ_-) the set of positive (resp. negative) roots distinguished by B. Let Δ be the set of simple roots and $\Delta^\vee$ be the corresponding set of simple coroots. Let $X_*(T)$ be the set of cocharacters, and let $X_*(T)_+$ be the set of dominant cocharacters.

The Iwahori-Weyl group $\tilde{W} = \tilde{W}_G$ is defined as the quotient $N_{G(L)}T(L)/T(\mathcal{O})$. This can be identified with the semi-direct product $W_0 \ltimes X_*(T)$, where W_0 is the finite Weyl group of G. We denote the projection $\tilde{W} \to W_0$ by p. We have a length function $\ell\colon \tilde{W} \to \mathbb{Z}_{\geq 0}$ given as $\ell(u\varpi^\lambda) = \sum_{\alpha \in \Phi_+, u\alpha \in \Phi_-} |\langle \alpha, \lambda \rangle + 1| + \sum_{\alpha \in \Phi_+, u\alpha \in \Phi_+} |\langle \alpha, \lambda \rangle|$, where $u \in W_0$ and $\lambda \in X_*(T)$.

Let $S \subset W_0$ denote the subset of simple reflections, and let $\tilde{S} \subset \tilde{W}$ denote the subset of simple affine reflections. We often identify Δ and S. The affine Weyl group W_a is the subgroup of $\tilde{W}$ generated by $\tilde{S}$. Then we can write the Iwahori-Weyl group as a semi-direct product $\tilde{W} = W_a \rtimes \Omega$, where $\Omega \subset \tilde{W}$ is the subgroup of length 0 elements. Moreover, $(W_a, \tilde{S})$ is a Coxeter system. We denote by $\leq$ the Bruhat order on $\tilde{W}$. For any $J \subseteq \tilde{S}$, let ${}^J\tilde{W}$ be the set of minimal length elements for the cosets in $W_J\backslash\tilde{W}$, where W_J denotes the subgroup of $\tilde{W}$ generated by J.

For $w \in W_a$, we denote by $\mathrm{supp}(w) \subseteq \tilde{S}$ the set of simple affine reflections occurring in every (equivalently, some) reduced expression of w. Note that $\tau \in \Omega$ acts on $\tilde{S}$ by conjugation. Let $\mathrm{supp}_\sigma(w\tau)$ be the smallest $\tau\sigma$-stable subset of $\tilde{S}$ which contains $\mathrm{supp}(w)$.

For $w, w' \in \tilde{W}$ and $s \in \tilde{S}$, we write $w \xrightarrow{s} w'$ if $w' = sw\sigma(s)$ and $\ell(w') \leq \ell(w)$. We write $w \to w'$ if there is a sequence $w = w_0, w_1, \ldots, w_k = w'$ of elements in $\tilde{W}$ such that for any i, $w_{i-1} \xrightarrow{s_i} w_i$ for some $s_i \in \tilde{S}$. If $\ell(w) = \ell(w')$, we write $w \approx w'$. Note that in many other papers, $\to$ and $\approx$ are denoted as $\to_\sigma$ and $\approx_\sigma$ respectively.

For $\alpha \in \Phi$, let $U_\alpha \subseteq G$ denote the corresponding root subgroup. We set

$$I = T(\mathcal{O}) \prod_{\alpha \in \Phi_+} U_\alpha(\varpi \mathcal{O}) \prod_{\beta \in \Phi_-} U_\beta(\mathcal{O}) \subseteq G(L),$$

which is called the standard Iwahori subgroup associated to $T \subset B \subset G$. For $J \subset \tilde{S}$ with W_J finite, let $P_J \supset I$ be the standard parahoric subgroup associated to J. We denote by π_J the projection $G(L)/I \to G(L)/P_J$. Set $K = P_S = G(\mathcal{O})$ and $\pi = \pi_S$.

In the case $G = \mathrm{GL}_n$, we will use the following description. Let T be the torus of diagonal matrices, and we choose the subgroup of upper triangular matrices B as Borel subgroup. Let χ_{ij} be the character $T \to \mathbb{G}_m$ defined by $\mathrm{diag}(t_1, t_2, \dots, t_n) \mapsto t_i {t_j}^{-1}$. Then we have $\Phi = \{\chi_{ij} \mid i \neq j\}$, $\Phi_+ = \{\chi_{ij} \mid i < j\}$, $\Phi_- = \{\chi_{ij} \mid i > j\}$ and $\Delta = \{\chi_{i,i+1} \mid 1 \leq i < n\}$. Through the isomorphism $X_*(T) \cong \mathbb{Z}^n$, $X_*(T)_+$ can be identified with the set $\{(m_1, \cdots, m_n) \in \mathbb{Z}^n \mid m_1 \geq \cdots \geq m_n\}$. Let us write $s_1 = (1\ 2), s_2 = (2\ 3), \dots, s_{n-1} = (n-1\ n)$. Set $s_0 = \varpi^{\chi_{1,n}^\vee}(1\ n)$, where $\chi_{1,n}$ is the unique highest root. Then $S = \{s_1, s_2, \dots, s_{n-1}\}$ and $\tilde{S} = S \cup \{s_0\}$. The Iwahori subgroup $I \subset K$ is the inverse image of B^{op} under the projection $G(\mathcal{O}) \to G(\overline{\mathbb{F}}_q)$ sending ϖ to 0, where B^{op} is the subgroup of lower triangular matrices. Similarly, if $J \subset S$, then P_J is the inverse image of the standard parabolic subgroup (which contains B^{op}) corresponding to J. Finally, $\varpi^{(1,0^{(n-1)})} s_1 s_2 \cdots s_{n-1}$ is a generator of $\Omega \cong \mathbb{Z}$.

2.2 Affine Deligne-Lusztig Varieties

For $w \in \tilde{W}$ and $b \in G(L)$, the affine Deligne-Lusztig variety $X_w(b)$ in the affine flag variety $G(L)/I$ is defined as

$$X_w(b) = \{xI \in G(L)/I \mid x^{-1} b \sigma(x) \in IwI\}.$$

For $\mu \in X_*(T)_+$ and $b \in G(L)$, the affine Deligne-Lusztig variety $X_\mu(b)$ in the affine Grassmannian $G(L)/K$ is defined as

$$X_\mu(b) = \{xK \in G(L)/K \mid x^{-1} b \sigma(x) \in K \varpi^\mu K\}.$$

In the equal characteristic case, affine Deligne-Lusztig varieties are schemes, locally of finite type over $\overline{\mathbb{F}}_q$. In the mixed characteristic case, affine Deligne-Lusztig varieties are perfect schemes, locally perfectly of finite type over $\overline{\mathbb{F}}_q$. See [18], [30], [2] and [11, Lemma 1.1]. Left multiplication by $g^{-1} \in G(L)$ induces an isomorphism between affine Deligne-Lusztig varieties corresponding to b and $g^{-1} b \sigma(g)$. Thus the isomorphism class of the affine Deligne-Lusztig variety only depends on the σ-conjugacy class of b. Also, the affine Deligne-Lusztig varieties carry a natural action (by left multiplication) by the σ-centralizer of b

$$\mathbb{J}_b = \{g \in G(L) \mid g^{-1} b \sigma(g) = b\}.$$

Note that $\mathbb{J}_b \cong \mathbb{J}_{g^{-1} b \sigma(g)}$ by sending j to $g^{-1} j g$.

Remark 2.1. In [7, §3.4], $\pi_J(X_w(b))$ was denoted by $X_{J,w}(b)$.

The admissible subset of $\tilde{W}$ associated to μ is defined as

$$\mathrm{Adm}(\mu) = \{w \in \tilde{W} \mid w \leq \varpi^{u\mu} \text{ for some } u \in W_0\}.$$

Set ${}^S\mathrm{Adm}(\mu) = \mathrm{Adm}(\mu) \cap {}^S\tilde{W}$. Assume that μ is minuscule. Then, by [7, Theorem 3.2.1] (see also [10, §2.5]), we have $X_\mu(b) = \bigsqcup_{w \in {}^S\mathrm{Adm}(\mu)} \pi(X_w(b))$. This is called the *Ekedahl-Oort stratification.* Note that

$$\pi(X_w(b)) = \{gK \in G(L)/K \mid g^{-1}b\sigma(g) \in K \cdot_\sigma IwI\},$$

where $\cdot_\sigma$ denotes the σ-twisted conjugation action of $G(L)$.

Set $S(w,\sigma) = \max\{S' \subseteq S \mid \mathrm{Ad}(w)\sigma(S') = S'\}$. Then $P_{S(w,\sigma)} w P_{\sigma(S(w,\sigma))} = P_{S(w,\sigma)} \cdot_\sigma IwI$ for $w \in {}^S\tilde{W}$ by [7, Theorem 3.2.1]. Moreover, by [7, Theorem 4.1.2 (1)], the projection $G(L)/P_{S(w,\sigma)} \to G(L)/K$ induces an isomorphism $\pi_{S(w,\sigma)}(X_w(b)) \cong \pi(X_w(b))$.

It follows from [9, Proposition 5.7] that if $W_{\mathrm{supp}_\sigma(w)}$ is finite, then $W_{\mathrm{supp}_\sigma(w) \cup S(w,\sigma)}$ is also finite. The following proposition is a combination of [7, Proposition 2.2.1 & §4.1].

Proposition 2.2. Let $\tau \in \Omega$. Let $w \in W_a\tau$ such that $W_{\mathrm{supp}_\sigma(w)}$ is finite. Then

$$X_w(\tau) = \bigsqcup_{j \in \mathbb{J}_\tau / \mathbb{J}_\tau \cap P_{\mathrm{supp}_\sigma(w)}} jY(w),$$

where $Y(w) = \{gI \in P_{\mathrm{supp}_\sigma(w)}/I \mid g^{-1}\tau\sigma(g) \in IwI\}$ is a Deligne-Lusztig variety in the flag variety $P_{\mathrm{supp}_\sigma(w)}/I$.

Let $w \in {}^S\tilde{W} \cap W_a\tau$ such that $W_{\mathrm{supp}_\sigma(w)}$ is finite. Then

$$\pi(X_w(\tau)) \cong \pi_{S(w,\sigma)}(X_w(\tau)) = \bigsqcup_{j \in \mathbb{J}_\tau / \mathbb{J}_\tau \cap P_{\mathrm{supp}_\sigma(w) \cup S(w,\sigma)}} j\pi_{S(w,\sigma)}(Y(w)),$$

where $\pi_{S(w,\sigma)}(Y(w)) = \{gP_{S(w,\sigma)} \in P_{\mathrm{supp}_\sigma(w) \cup S(w,\sigma)}/P_{S(w,\sigma)} \mid g^{-1}\tau\sigma(g) \in P_{S(w,\sigma)} \cdot_\sigma IwI\}$ is a Deligne-Lusztig variety in the partial flag variety $P_{\mathrm{supp}_\sigma(w) \cup S(w,\sigma)}/P_{S(w,\sigma)}$.

Remark 2.3. Each $Y(w)$ or $\pi_{S(w,\sigma)}(Y(w))$ is irreducible and of dimension $\ell(w)$ (cf. [3]).

2.3 Deligne-Lusztig Reduction Method

The following Deligne-Lusztig reduction method was established in [6, Corollary 2.5.3] ($\mathbb{A}^1$ and $\mathbb{G}_m$ actually mean $\mathbb{A}^{1,\mathrm{pfn}}$ and $\mathbb{G}_m^{\mathrm{pfn}}$ respectively in the mixed characteristic case).

Proposition 2.4. Let $w \in \tilde{W}$ and let $s \in \tilde{S}$ be a simple affine reflection. Then the following two statements hold for any $b \in G(L)$.

(i) If $\ell(sw\sigma(s)) = \ell(w)$, then there exists a $\mathbb{J}_b$-equivariant universal homeomorphism $X_w(b) \to X_{sw\sigma(s)}(b)$.

(ii) If $\ell(sw\sigma(s)) = \ell(w) - 2$, then there exists a decomposition $X_w(b) = X_1 \sqcup X_2$ such that

- X_1 is open and there exists a $\mathbb{J}_b$-equivariant morphism $X_1 \to X_{sw}(b)$, which is the composition of a Zariski-locally trivial $\mathbb{G}_m$-bundle and a universal homeomorphism.
- X_2 is closed and there exists a $\mathbb{J}_b$-equivariant morphism $X_2 \to X_{sw\sigma(s)}(b)$, which is the composition of a Zariski-locally trivial $\mathbb{A}^1$-bundle and a universal homeomorphism.

Remark 2.5. We actually need a slight generalization of Proposition 2.4 to prove the main theorem. See [22, Proposition 2.6].

We say that a scheme X is an *iterated fibration* of rank a over a scheme Y (whose fibers are all $\mathbb{A}^1$) if there exist morphisms $X = Y_0 \to Y_1 \to \cdots \to Y_a = Y$ such that Y_i is a Zariski-locally trivial $\mathbb{A}^1$-bundle over Y_{i+1} for any $0 \leq i < a$.

2.4 Length Positive Elements

We denote by δ^+ the indicator function of the set of positive roots. Note that any element $w \in \tilde{W}$ can be written in a unique way as $w = x\varpi^\mu y$ with μ dominant, $x, y \in W_0$ such that $\varpi^\mu y \in {}^S\tilde{W}$. We have $p(w) = xy$ and $\ell(w) = \ell(x) + \langle \mu, 2\rho \rangle - \ell(y)$. We define the set of *length positive* elements by $\mathrm{LP}(w) \coloneqq \{v \in W_0 \mid \langle v\alpha, y^{-1}\mu \rangle + \delta^+(v\alpha) - \delta^+(xyv\alpha) \geq 0 \text{ for all } \alpha \in \Phi_+\}$. Then we always have $y^{-1} \in \mathrm{LP}(w)$. Indeed, y is uniquely determined by the condition that $\langle \alpha, \mu \rangle \geq \delta^+(-y^{-1}\alpha)$ for all $\alpha \in \Phi_+$. Since $\delta^+(\alpha) + \delta^+(-\alpha) = 1$, we have

$$\langle y^{-1}\alpha, y^{-1}\mu \rangle + \delta^+(y^{-1}\alpha) - \delta^+(x\alpha) = \langle \alpha, \mu \rangle - \delta^+(-y^{-1}\alpha) + \delta^+(-x\alpha) \geq 0.$$

Thanks to Kottwitz [15], a σ-conjugacy class $[b]$ of $b \in G(L)$ is uniquely determined by two invariants: the Kottwitz point $\kappa(b) \in \pi_1(G)/((1-\sigma)\pi_1(G))$ and the Newton point $\nu_b \in X_*(T)_{\mathbb{Q},+}$. Clearly, $X_w(b) = \emptyset$ if $\kappa(b) \neq \kappa(\dot{w})$. We say that $b \in G(L)$ is basic if ν_b is central. The following theorem is a refinement of the non-emptiness criterion in [8], which is conjectured by Lim in [17] and proved by Schremmer in [19, Proposition 5].

Theorem 2.6. Assume that the Dynkin diagram of G is σ-connected, i.e., σ acts transitively on the set of irreducible components of Φ. Let $b \in G(L)$ be a basic element with $\kappa(b) = \kappa(\dot{w})$. Then $X_w(b) = \emptyset$ if and only if the following two conditions are satisfied:

(i) $|W_{\mathrm{supp}_\sigma(w)}|$ is infinite.

(ii) There exists $v \in \mathrm{LP}(w)$ such that $\mathrm{supp}_\sigma(\sigma^{-1}(v)^{-1}p(w)v) \subsetneq S$.

3 Main results

We assume that $n \geq 2$. Let F_2 be the quadratic unramified extension of F. Let $\mathcal{O}_{F_2}$ denote the ring of integers of F_2. We put $\Lambda = \mathcal{O}_{F_2}^n$ equipped with the hermitian form

$$\Lambda \times \Lambda \to \mathcal{O}_{F_2}, \quad ((a_i)_{1 \leq i \leq n}, (b_i)_{1 \leq i \leq n}) \mapsto \sum_{i=1}^{n} \sigma(a_i) b_{n+1-i}.$$

From now on, we set $G = \mathrm{GU}(\Lambda)$. By taking the first factor of the isomorphism

$$\mathcal{O}_{F_2} \otimes_{\mathcal{O}_F} \mathcal{O}_{F_2} \cong \mathcal{O}_{F_2} \times \mathcal{O}_{F_2}, \quad a \otimes b \mapsto (ab, a\sigma(b)),$$

we obtain an isomorphism $G_{\mathcal{O}_{F_2}} \cong \mathrm{GL}_n \times \mathbb{G}_m$. Let $T \subset B \subset G$ be the maximal torus and the Borel subgroup determined by the diagonal torus and the upper triangular subgroup of GL_n under this isomorphism. Then $X_*(T)$ can be identified with $\mathbb{Z}^n \times \mathbb{Z}$ and $\tilde{W} = \tilde{W}_G \cong \tilde{W}_{\mathrm{GL}_n} \times \mathbb{Z}$. Under this identification, we have $\sigma(((m_i)_{1\leq i\leq n}, 0)) = ((-m_{n+1-i})_{1\leq i\leq n}, 0) \in X_*(T)$ and $\sigma((s_i, 0)) = (s_{n-i}, 0) \in \tilde{W}$ by setting $s_n = s_0$.

In the sequel, we set $\mu = ((0^{(n-2)}, -1, -1), -1)$ and $b = \varpi^{((0^{(n)}), -1)}$. Then b is a central basic element with $\kappa(b) = \kappa(\varpi^\mu)$ (cf. [24, §5.2]). By abuse of notation, we write s_i for $(s_i, 0) \in \tilde{W}$. For integers $1 \leq k, l \leq n$, we set

$$s_{[k,l]} = \begin{cases} s_k s_{k-1} \cdots s_l & \text{if } k \geq l \\ 1 & \text{otherwise} \end{cases}$$

and $w_{k,l} = \varpi^\mu s_{[n-2,k]} s_{[n-1,l]}$. Then ${}^S\mathrm{Adm}(\mu) = \{w_{k,l} \mid 1 \leq k < l \leq n\}$. In particular, $\tau := w_{1,2}$ is the length 0 element corresponding to μ. It is easy to check that $\ell(w_{k,l}) = k + l - 3$.

Set $\tau_1 := \varpi^{((1,0^{(n-1)}),0)} s_1 s_2 \cdots s_{n-1} \in \Omega$. Then $\tau_1^{-1} b \sigma(\tau_1) = \tau$. We define ${}^S\mathrm{Adm}(\mu)_{\neq\emptyset} := \{w \in {}^S\mathrm{Adm}(\mu) \mid X_w(b) \neq \emptyset\}$ and ${}^S\mathrm{Adm}(\mu)_{\mathrm{DL}} := \{w \in {}^S\mathrm{Adm}(\mu) \mid \mathrm{supp}_\sigma(w) \neq \tilde{S}\}$. By Theorem 2.6, we have ${}^S\mathrm{Adm}(\mu)_{\mathrm{DL}} \subseteq {}^S\mathrm{Adm}(\mu)_{\neq\emptyset}$. We denote ${}^S\mathrm{Adm}(\mu)_{\neq\emptyset} \setminus {}^S\mathrm{Adm}(\mu)_{\mathrm{DL}}$ by ${}^S\mathrm{Adm}(\mu)_{\neq\mathrm{DL}}$.

3.1 Non-Empty Ekedahl-Oort Strata

If n is odd, then σ-orbits on $\tilde{S}$ are

$$\{s_0\}, \{s_1, s_{n-1}\}, \{s_2, s_{n-2}\}, \ldots, \{s_{\frac{n-1}{2}}, s_{\frac{n+1}{2}}\}.$$

Since $\tau s_i \tau^{-1} = s_{i-2}$, $\tau\sigma$-orbits on $\tilde{S}$ are

$$\{s_{n-1}\}, \{s_0, s_{n-2}\}, \{s_1, s_{n-3}\}, \ldots, \{s_{\frac{n-3}{2}}, s_{\frac{n-1}{2}}\}.$$

If n is even, then σ-orbits on $\tilde{S}$ are

$$\{s_0\}, \{s_1, s_{n-1}\}, \{s_2, s_{n-2}\}, \ldots, \{s_{\frac{n}{2}-1}, s_{\frac{n}{2}+1}\}, \{s_{\frac{n}{2}}\}.$$

Since $\tau s_i \tau^{-1} = s_{i-2}$, $\tau\sigma$-orbits on $\tilde{S}$ are

$$\{s_{n-1}\}, \{s_0, s_{n-2}\}, \{s_1, s_{n-3}\}, \ldots, \{s_{\frac{n}{2}-2}, s_{\frac{n}{2}}\}, \{s_{\frac{n}{2}-1}\}.$$

Set $t_i = s_i s_{n-i} (= s_{n-i} s_i$ unless $\frac{n-1}{2} \leq i \leq \frac{n+1}{2})$. We simply write $\equiv$ for $\equiv \pmod 2$.

Lemma 3.1. Assume that $1 \leq k < l \leq n$ satisfies one of the following conditions:

(i) $\frac{n+2}{2} \leq k < l \leq n$.

(ii) $l \equiv n$, $\frac{n+3}{2} \leq l$ and $n - l + 2 \leq k \leq \frac{n+1}{2}$.

(iii) k is even, $k \leq \frac{n-1}{2}$ and $\frac{n+3}{2} \leq l \leq n - k + 2$.

Then $X_{w_{k,l}}(b) = \emptyset$.

Proof. This can be proved by using Theorem 2.6 (ii). See [22, Lemma 3.1] for details. □

The following lemma is proved by an explicit computation (cf. [22, Lemma 3.3]).

Lemma 3.2. Assume that $3 \leq k \leq \frac{n+1}{2}$ and $\frac{n+3}{2} \leq l \leq n-1$.

(i) If k is odd and $k + l \leq n + 2$, then there exists $s \in \tilde{S}$ and $w' \in \tilde{W}$ such that $w_{k,l} \approx w'$, $sw'\sigma(s) \approx w_{k-2,l}$ and $sw' \approx w_{k-1,l}$.

(ii) If $l \equiv n - 1$ and $k + l \geq n + 4$, then there exists $s \in \tilde{S}$ and $w' \in \tilde{W}$ such that $w_{k,l} \approx w'$, $sw'\sigma(s) \approx w_{k,l-2}$ and $sw' \approx w_{k,l-1}$.

(iii) If $l \equiv n - 1$ and $k + l = n + 3$, then there exists $s \in \tilde{S}$ and $w' \in \tilde{W}$ such that $w_{k,l} \approx w'$, $sw'\sigma(s) \approx w_{k-1,l-1}$ and $sw' \approx w_{k,l-1}$.

For $w_{k,l} \in {}^S\mathrm{Adm}(\mu)_{\neq \mathrm{DL}}$, let $w'_{k,l}$ denote $w_{k-2,l}$ (resp. $w_{k,l-2}$, resp. $w_{k-1,l-1}$) if $k+l \leq n+2$ (resp. $k+l \geq n+4$, resp. $k+l = n+3$). Combining the above two lemmas with Proposition 2.4 and Theorem 2.6, we obtain the following theorem.

Theorem 3.3. We have

$$ {}^S\mathrm{Adm}(\mu)_{\mathrm{DL}} = \{w_{k,l} \mid k = 1 \text{ or } l \leq \tfrac{n+2}{2}\}. $$

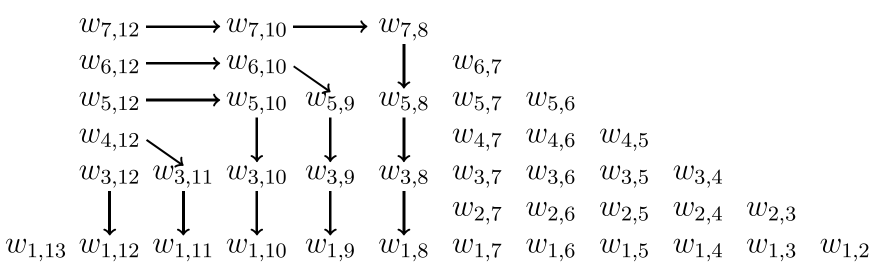
Moreover, $w_{k,l} \in {}^S\mathrm{Adm}(\mu)_{\neq \mathrm{DL}}$ if and only if $3 \leq k < \frac{n+2}{2} < l \leq n-1$ and one of the following conditions is satisfied:

(i) k is odd and $k + l \leq n + 2$.

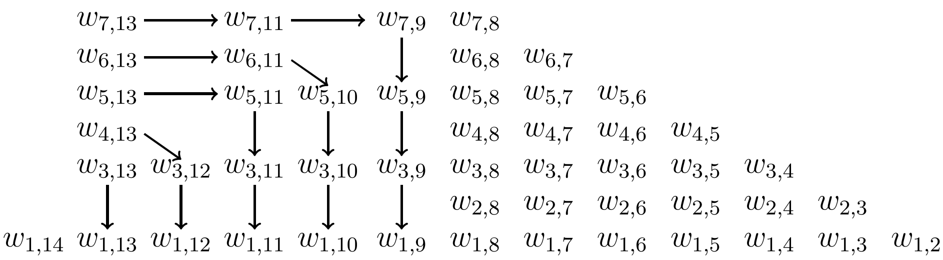
(ii) $l \equiv n - 1$ and $k + l \geq n + 3$.

If this is the case, $X_{w_{k,l}}(\tau)$ is $\mathbb{J}_\tau$-equivariant universally homeomorphic to a Zariski-locally trivial $\mathbb{A}^1$-bundle over $X_{w'_{k,l}}(\tau)$. In particular, if $w_{k,l} \in {}^S\mathrm{Adm}(\mu)_{\neq \mathrm{DL}}$ satisfies (i) (resp. (ii)), then $X_{w_{k,l}}(\tau)$ is an iterated fibration of rank $\frac{k-1}{2}$ (resp. $k + \frac{l-n-3}{2}$) over $X_{w_{1,l}}(\tau)$ (resp. $X_{w_{1,n-k+2}}(\tau)$).

Example 3.4. In the case $n = 13$, the non-empty Ekedahl-Oort strata are the following.



In the case $n = 14$, the non-empty Ekedahl-Oort strata are the following.



3.2 Geometric Structure of Non-Empty Ekedahl-Oort Strata

For $w_{k,l} \in {}^S\mathrm{Adm}(\mu)_{\mathrm{DL}}$, we have

$$\mathrm{supp}_\sigma(w_{k,l}) = \begin{cases} \{s_i, s_{n-i-2} \mid 0 \leq i \leq l-3\} \sqcup \{s_{n-1}\} & (k \geq 2) \\ \{s_i, s_{n-i-2} \mid 0 \leq i \leq l-3\} & (k = 1 \ \& \ l \leq \frac{n+2}{2}) \\ \tilde{S} \setminus \{s_{n-1}\} & (k = 1 \ \& \ l \geq \frac{n+3}{2}). \end{cases}$$

The following lemma can also be proved by an explicit computation (cf. [22, Lemma 4.1]).

Lemma 3.5. Let $w_{k,l} \in {}^S\mathrm{Adm}(\mu)_{\mathrm{DL}}$. If $l = k+1$, then

$$S(w_{k,k+1}, \sigma) = \begin{cases} \{s_i \mid k \leq i \leq n-k-2\} \sqcup \\ \{s_1, s_3, \ldots, s_{k-2}, s_{n-1}, s_{n-3}, \ldots, s_{n-k}\} & (\text{odd } k) \\ \{s_i \mid k \leq i \leq n-k-2\} & (\text{even } k). \end{cases}$$

If $l - k \geq 2$, then

$$S(w_{k,l}, \sigma) = \begin{cases} \{s_i \mid l-1 \leq i \leq n-l-1\} & (l \leq \frac{n+2}{2}) \\ \{s_i \mid n-l+2 \leq i \leq l-3\} & (k = 1 \ \& \ l \geq \frac{n+3}{2}). \end{cases}$$

In particular,

$$\mathrm{supp}_\sigma(w_{k,l}) \cup S(w_{k,l}, \sigma) = \begin{cases} \tilde{S} \setminus \{s_{l-2}, s_{n-l}\} & (k \geq 2 \text{ or } l = 2) \\ \tilde{S} \setminus \{s_{l-2}, s_{n-l}, s_{n-1}\} & (k = 1 \ \& \ 3 \leq l \leq \frac{n+2}{2}) \\ \tilde{S} \setminus \{s_{n-1}\} & (k = 1 \ \& \ l \geq \frac{n+3}{2}). \end{cases}$$

We set $\mathrm{supp}_\sigma(w)_1 := \tau_1 \mathrm{supp}_\sigma(w) \tau_1^{-1} = \{s_{i+1} \mid s_i \in \mathrm{supp}_\sigma(w)\}$ and $S(w,\sigma)_1 := \tau_1 S(w,\sigma) \tau_1^{-1} = \{s_{i+1} \mid s_i \in S(w,\sigma)\}$. Recall that for $w_{k,l} \in {}^S\mathrm{Adm}(\mu)_{\mathrm{DL}}$, $Y(w_{k,l})$ is a Deligne-Lusztig variety in $P_{\mathrm{supp}_\sigma(w_{k,l})}/I$ as in Proposition 2.2. Its image $\pi(Y(w_{k,l}))$ is isomorphic to a Deligne-Lusztig variety $\pi_{S(w_{k,l},\sigma)}(Y(w_{k,l}))$ in $P_{\mathrm{supp}_\sigma(w_{k,l}) \cup S(w_{k,l},\sigma)}/P_{S(w_{k,l},\sigma)}$. If $w_{k,l} \in {}^S\mathrm{Adm}(\mu)_{\neq \mathrm{DL}}$ and $k + l \leq n+2$ (resp. $k + l \geq n+3$), let $Y(w_{k,l}) \subset X_{w_{k,l}}(\tau)$ denote the inverse image of

$Y(w_{1,n-k+2})$ (resp. $Y(w_{1,l})$) under the iterated fibration in Theorem 3.3. Then $Y(w_{k,l})$ is an irreducible component of $X_{w_{k,l}}(\tau)$ (cf. Remark 2.3). By Proposition 2.2, we have

$$\pi(X_{w_{k,l}}(b)) \cong \pi_{S(w_{k,l},\sigma)}(X_{w_{k,l}}(b)) = \bigsqcup_{j\in G(F)/G(F)\cap P_{\mathrm{supp}_\sigma(w_{k,l})_1\cup S(w_{k,l},\sigma)_1}} j\tau_1\pi_{S(w_{k,l},\sigma)}(Y(w_{k,l})).$$

Here $\mathrm{supp}_\sigma(w_{k,l})$ and $S(w_{k,l},\sigma)$ are as in Lemma 3.5. We can prove the following theorem similarly as Theorem 3.3 using [22, Proposition 2.6] and Lemma 3.5 instead of Proposition 2.4. See [22, Lemma 4.2 & Theorem 4.3].

Theorem 3.6. Let $w_{k,l} \in {}^S\mathrm{Adm}(\mu)_{\neq \mathrm{DL}}$. Then $S(w_{k,l},\sigma) = S(w'_{k,l},\sigma)$ and $\pi(X_{w_{k,l}}(b))$ is $G(F)$-equivariant universally homeomorphic to a Zariski-locally trivial $\mathbb{A}^1$-bundle over $\pi(X_{w'_{k,l}}(b))$. In particular, if $k+l \leq n+2$ (resp. $k+l \geq n+3$), then $S(w_{k,l},\sigma) = S(w_{1,l},\sigma)$ (resp. $S(w_{1,n-k+2},\sigma)$),

$$\pi(X_{w_{k,l}}(b)) \cong \pi_{S(w_{k,l},\sigma)}(X_{w_{k,l}}(b)) = \bigsqcup_{j\in G(F)/G(\mathcal{O}_F)} j\tau_1\pi_{S(w_{k,l},\sigma)}(Y(w_{k,l}))$$

and $\tau_1\pi_{S(w_{k,l},\sigma)}(Y(w_{k,l}))$ is an irreducible component, which is $G(\mathcal{O}_F)$-equivariant universally homeomorphic to an iterated fibration of rank $\frac{k-1}{2}$ (resp. $k+\frac{l-n-3}{2}$) over $\tau_1\pi_{S(w_{k,l},\sigma)}(Y(w_{1,l}))$ (resp. $\tau_1\pi_{S(w_{k,l},\sigma)}(Y(w_{1,n-k+2}))$).

Remark 3.7. The perfection of a universal homeomorphism is an isomorphism.

Remark 3.8. If $w_{k,l} \in {}^S\mathrm{Adm}(\mu)_{\mathrm{DL}}$, then we set $P_{k,l} = P_{\mathrm{supp}_\sigma(w_{k,l})_1\cup S(w_{k,l},\sigma)_1}$ and $X_{k,l} = Y_{k,l} = \tau_1\pi(Y(w_{k,l})) \cong \tau_1\pi_{S(w_{k,l},\sigma)}(Y(w_{k,l}))$. If $w_{k,l} \in {}^S\mathrm{Adm}(\mu)_{\neq\mathrm{DL}}$ and $k+l\leq n+2$ (resp. $k+l \geq n+3$), then we set $P_{k,l} = G(\mathcal{O})$, $X_{k,l} = \tau_1\pi(Y(w_{1,l})) \cong \tau_1\pi_{S(w_{k,l},\sigma)}(Y(w_{1,l}))$ (resp. $X_{k,l} = \tau_1\pi(Y(w_{1,n-k+2})) \cong \tau_1\pi_{S(w_{k,l},\sigma)}(Y(w_{1,n-k+2}))$) and $Y_{k,l} = \tau_1\pi(Y(w_{k,l})) \cong \tau_1\pi_{S(w_{k,l},\sigma)}(Y(w_{k,l}))$. This notation justifies Theorem B.

Remark 3.9. The strata associated to $w_{k,l} \in {}^S\mathrm{Adm}(\mu)_{\mathrm{DL}}$ satisfy a nice closure relation, which is similar to that in the fully Hodge-Newton decomposable cases. See [22, Corollary 4.8].

Remark 3.10. Since we chose b to be $\varpi^{((0^{(n)}),-1)}$, the following description is possible. Let $w_{k,l} \in {}^S\mathrm{Adm}(\mu)_{\neq\emptyset}$. There is an isomorphism $G(L)/P_{S(w_{k,l},\sigma)} \xrightarrow{\sim} G(L)/P_{S(w_{k,l},\sigma)_1}$ given by $gP_{S(w_{k,l},\sigma)} \mapsto gP_{S(w_{k,l},\sigma)}\tau_1^{-1} = g\tau_1^{-1}P_{S(w_{k,l},\sigma)_1}$. Then $\tau_1 P_{\mathrm{supp}_\sigma(w_{k,l})\cup S(w_{k,l},\sigma)}/P_{S(w_{k,l},\sigma)}$ maps to $P_{\mathrm{supp}_\sigma(w_{k,l})_1\cup S(w_{k,l},\sigma)_1}/P_{S(w_{k,l},\sigma)_1}$ under this isomorphism. So if $w_{k,l} \in {}^S\mathrm{Adm}(\mu)_{\mathrm{DL}}$, then $\tau_1\pi_{S(w_{k,l})}(Y(w_{k,l}))$ maps to

$$\pi_{S(w_{k,l})_1}(Y(w^0_{k,l})) = \{gP_{S(w_{k,l},\sigma)_1} \mid g^{-1}\sigma(g) \in P_{S(w_{k,l},\sigma)_1} \cdot_\sigma Iw^0_{k,l}I\},$$

where $w^0_{k,l} = b^{-1}\tau_1 w_{k,l}\sigma(\tau_1)^{-1} \in W_a$. Note that this is a $G(F)\cap P_{\mathrm{supp}_\sigma(w_{k,l})_1\cup S(w_{k,l},\sigma)_1}$-equivariant isomorphism. If, moreover, $\mathrm{supp}_\sigma(w_{k,l})_1 \cup S(w_{k,l},\sigma)_1 = S$, then we have $w^0_{k,l} \in W_0$ and $\pi_{S(w_{k,l})_1}(Y(v_{k,l}))$ can be identified with a Deligne-Lusztig variety associated to $w^0_{k,l}$ in the partial flag variety of type $S(w_{k,l},\sigma)$ for $G(\mathcal{O}/\varpi) = G(\overline{\mathbb{F}}_q)$. Under this identification, the action of $G(\mathcal{O}_F)$ factors through $G(\mathcal{O}_F/\varpi) = G(\mathbb{F}_q)$, which coincides with the usual action on Deligne-Lusztig varieties for $G(\overline{\mathbb{F}}_q)$. Thus if $w_{k,l} \in {}^S\mathrm{Adm}(\mu)_{\neq \mathrm{DL}}$, then $\tau_1\pi_{S(w_{k,l},\sigma)}(Y(w_{k,l}))$ is $G(\mathcal{O}_F)$-equivariant universally homeomorphic to an iterated fibration over such a Deligne-Lusztig variety.

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