Beyond the minute admissible sets

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1 Introduction

The purpose of this announcement article is to summarize some results of an upcoming joint paper with Eva Viehmann, and how they relate to an upcoming joint paper with Sian Nie and Qingchao Yu.

The study of affine Deligne–Lusztig varieties arises from the reduction of Shimura varieties and their equal characteristic counterparts. They are affine analogues of Deligne–Lusztig varieties introduced to study representations of finite groups of Lie type [DL76]. We refer to [He18] for an overview.

Due to the preliminary nature of the results discussed here, I would like to urge the reader to take them with care and wait for more refined versions of the papers to appear online before relying on them.

2 Preliminaries

Let F be a non–archimedian local field with ring of integers $\mathcal{O}_F$ and uniformizer $\varepsilon \in \mathcal{O}_F$. We denote by $\breve F$ the completion of its maximal unramified extension and by $\mathcal{O}_{\breve F}$ the corresponding ring of integers.

Let G be a reductive group defined over $\mathcal{O}_F$ with maximal torus T and Borel $B \supseteq T$. For the purposes of this announcement, we assume the torus T to be split over F. Let $I \subseteq G(\mathcal{O}_{\breve F})$ be the Iwahori subgroup which is defined as the preimage of the opposite Borel of B under the reduction map $G(\mathcal{O}_{\breve F}) \to G(\mathcal{O}_{\breve F}/(\varepsilon))$. Define the Iwahori–Weyl group by $\widetilde W = N_G(T)(\breve F)/T(\mathcal{O}_{\breve F})$, so that we get the Iwahori–Bruhat decomposition

$$G(\breve F) = \bigsqcup_{x\in\widetilde W} IxI.$$

For $x \in \widetilde W$ and $b \in G(\breve F)$, the *(single) affine Deligne–Lusztig variety (ADLV)* $X_x(b)$ is the reduced locally closed subscheme of the affine flag variety with geometric points given by

$$X_x(b) = \{gI \in G(\breve F)/I \mid g^{-1}b\sigma(g) \in IxI\}.$$

Here, $\sigma \in \mathrm{Gal}(\breve F/F)$ is the Frobenius. It naturally acts on $G(\breve F)$ and stabilizes T, B, I.

Let $K = G(\mathcal{O}_{\breve F})$, which is a hyperspecial subgroup, and consider a cocharacter $\mu \in X_*(T)$. Then we get the following affine Deligne–Lusztig varieties in the affine Grassmannian:

$$X_\mu(b) = \{gK \in G(\breve F)/K \mid g^{-1}b\sigma(g) \in K\mu(\varepsilon)K\},$$
$$X_{\leqslant\mu}(b) = \{gK \in G(\breve F)/K \mid g^{-1}b\sigma(g) \in \overline{K\mu(\varepsilon)K}\}.$$

By [HR17], we obtain a combinatorially defined set $\mathrm{Adm}(\mu)^K \subseteq \widetilde W$ such that the map

$$\bigsqcup_{x\in\mathrm{Adm}(\mu)^K} X_x(b) \xrightarrow{gI\mapsto gK} X_{\leqslant\mu}(b)$$

is surjective with finite fibres (the pieces are known as Ekedahl–Oort strata). While some main geometric invariants of $X_\mu(b)$ are well–understood (cf. [ZZ20; Nie22]), it remains open to understand all pieces $X_x(b)$ where $x \in \mathrm{Adm}(\mu)^K$.

The geometry of $X_\mu(b)$ and $X_x(b)$ only depends on the σ–conjugacy class

$$[b] = \{g^{-1}b\sigma(g) \mid g \in G(\breve F)\}.$$

We denote the set of σ–conjugacy classes by $B(G)$ and the set of σ–conjugacy classes intersecting $\overline{K\mu(\varepsilon)K}$ by $B(G,\mu)$. Both sets have very accessible combinatorial descriptions [Kot85; He16].

3 Depth

While some special cases of ADLV have been very well understood in the past, the bigger picture remains opaque. In order to measure the expected "difficulty" of understanding the geometry of an ADLV $X_\mu(b)$, the following quantity is suggested in an upcoming joint article with Eva Viehmann.

Definition 1. Let $\mu \in X_*(T)$ be a dominant cocharacter. We set

$$\mathrm{depth}(G,\mu) := \max_\omega \langle \mu, \omega\rangle.$$

Here, ω runs through the fundamental weights inside $\mathbb{Q}\Phi$, where Φ is the set of roots of (G,T).

Theorem 2 (Görtz–He–Nie)**.** *The following are equivalent*

- *Every element in $B(G,\mu)$ satisfies some special properties, so that we say (G,μ) is fully Hodge–Newton decomposable.*

- *We have* $\dim X_\mu(b) = 0$ *for all non-basic* $[b] \in B(G,\mu)$.
- *For all* $x \in \mathrm{Adm}(\mu)^K$, *there exists a unique* $[b] \in B(G)$ *such that* $X_x(b) \neq \varnothing$.
- *The pair* (G,μ) *satisfies the* minute condition $\mathrm{depth}(G,\mu) \leqslant 1$.
- *(The authors give an explicit classification of all cases).*

So the next most accessible cases should have depth between 1 and 2. These will also satisfy the so–called *weak fully Hodge–Newton decomposability* property of [CT22]. A full classification of cases of depth between 1 and 2 is possible, and it largely overlaps with the list from [Shi24].

Regarding the geometry, some key results are summarized below.

Theorem 3 (S–Viehmann, to appear)**.** *Let* G *be absolutely quasi–simple and* $1 < \mathrm{depth}(G,\mu) < 2$. *Let* $x \in \mathrm{Adm}(\mu)^K$.

(a) There are elements $[b_{x,\min}], [b_{x,\max}] \in B(G,\mu)$ *such that for every* $[b] \in B(G)$, *we have*

$$X_x(b) \neq \varnothing \iff [b_{x,\min}] \leqslant [b] \leqslant [b_{x,\max}].$$

(b) For all $b \in G(\breve{F})$ *with* $X_x(b) \neq \varnothing$, *the ADLV* $X_x(b)$ *is equidimensional, with an explicit dimension formula. The group*

$$J_b(F) = \{g \in G(\breve{F}) \mid g^{-1} b\sigma(g) = b\}$$

acts transitively on the set of irreducible components of $X_x(b)$ *by left multiplication.*

4 Geometric Coxeter Type

In an upcoming joint paper with Sian Nie and Qingchao Yu, we introduce and study a class of elements inside $\widetilde{W}$ which we call of *geometric Coxeter type*. This is motivated by a procedure to analyse the geometry of ADLV $X_x(b)$ in an iterative fashion, developed by Görtz–He [GH10] following a similar method of Deligne–Lusztig [DL76] in the classical case, relying on combinatorial properties of $\widetilde{W}$ developed by He–Nie [HN14]. We identify certain element $x \in \widetilde{W}$ where this reduction method has very desirable properties, known as elements of geometric Coxeter type. This generalizes classes of elements studied previously by He–Nie–Yu [HNY22], as well as a joint paper with Shimada and Yu [SSY23]. Elements of geometric Coxeter type in particular enjoy all the properties listed in Theorem 3. In fact, we can get something even better.

Theorem 4 (Nie–S–Yu, to appear)**.** *Assume that* $F = \mathbb{F}_q(\!(\varepsilon)\!)$ *is a field of equal characteristic and that* $x \in \widetilde{W}$ *is an element of geometric Coxeter type. Then for every* $b \in G(\breve{F})$ *such that* $X_x(b) \neq \varnothing$, *we get a direct product decomposition* $X_x(b) = X_1 \times X_2 \times X_3 \times X_4$ *as schemes over* $\overline{\mathbb{F}}_q$. *Here,* X_1 *is an affine space,* X_2 *is an affine space with the coordinate axes removed,* X_3 *is a classical Deligne–Lusztig variety of Coxeter type and* X_4 *is a discrete set of points.*

Some example of geometric Coxeter type come from cases discussed above. Combining the aforementioned papers with a bit of manual or computer calculation, it is possible to see that all elements $x \in \mathrm{Adm}(\mu)^K$ are of geometric Coxeter type whenever $1 < \mathrm{depth}(G, \mu) < 2$ and G is absolutely quasi–simple.

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