The syntomic realization functor for Shimura varieties of abelian type

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Abstract

This is a survey article on prismatic F-gauges and their application to Shimura varieties.

1 Introduction

Let (G,X) be a Shimura datum, and K a compact open subgroup of $G(\mathbb{A}_{\mathrm{f}})$. Let E be the reflex field of (G,X). We write $\mathrm{Sh}_K(G,X)$ for the Shimura variety over E attached to (G,X) and K. This should be "a moduli of G-motives of type X with K-level structure".

We define G^{c} as in [KSZ21, 1.5.8], which is a quotient of G by a subtorus of the center of G. We have $G^{\mathrm{c}} = G$ if (G,X) is of Hodge type (*cf.* [IKY23, Remark 2.6]).

Let p be a prime number. Assume that $G_{\mathbb{Q}_p}$ has a reductive model $\mathcal{G}$ over $\mathbb{Z}_p$. We assume that $K_p = \mathcal{G}(\mathbb{Z}_p)$. Let $\mathcal{G}^{\mathrm{c}}$ be an integral analogue of G^{c} define in [IKY23, §2.3]. Let v be a prime of E above p. We can construct a tensor functor

$$\omega_{\mathrm{et}} \colon \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}^{\mathrm{c}}) \to \mathrm{Loc}_{\mathbb{Z}_p}(\mathrm{Sh}_{K,E_v})$$

using a tower of Shimura varieties, where $\mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}^{\mathrm{c}})$ is the category of finite rank algebraic representations of $\mathcal{G}^{\mathrm{c}}$ over $\mathbb{Z}_p$ and $\mathrm{Loc}_{\mathbb{Z}_p}(\mathrm{Sh}_{K,E_v})$ is the category of etale $\mathbb{Z}_p$-local systems on Sh_{K,E_v}.

Assume (G,X) is of abelian type. Then there is a canonical integral model $\mathcal{S}_K$ of Sh_{K,E_v} constructed by Kisin in [Kis10]. In [Lov17], Lovering constructed a tensor functor

$$\omega_{\mathrm{crys}} \colon \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}^{\mathrm{c}}) \to \mathrm{FilF\text{-}Crys}(\widehat{\mathcal{S}}_K)$$

compatible with ω_{et}, where $\mathrm{FilF\text{-}Crys}(\widehat{\mathcal{S}}_K)$ is a category of filtered F-crystals on the p-adic completion $\widehat{\mathcal{S}}_K$ of $\mathcal{S}_K$.

If (G,X) is of Hodge type, a shtuka realization compatible with ω_{et} is constructed in [PR24].

In [IKY23], we give a refinement of these realizations in prismatic F-gauges.

2 Prismatic F-crystal

We recall the notion of prism and prismatic F-crystal from [BS22] and [BS23].

Definition 2.1. *A δ-ring is a commutative $\mathbb{Z}_{(p)}$-algebra A with a map $\delta\colon A \to A$ such that $\delta(1) = 0$ and*

$$\delta(x+y) = \delta(x) + \delta(y) + \frac{x^p + y^p - (x+y)^p}{p},$$
$$\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$$

for any $x, y \in A$.

For a δ-ring A with δ, we define a Frobenius lift $\phi\colon A \to A$ by $\phi(x) = x^p + p\delta(x)$. A morphism of δ-rings is a ring homomorphism that intertwines the δ-structures.

Example 2.2 ([Joy85])**.** *For a commutative ring R, let $W(R)$ be the ring of p-typical Witt vectors in R, and let*

$$\delta_{W(R)}\colon W(R) \to W(R);\ (x_0, x_1, \ldots) \mapsto (x_1, x_2, \ldots).$$

Then $W(R)$ with $\delta_{W(R)}$ is a δ-ring. The functor $R \mapsto (W(R), \delta_{W(R)})$ gives the right adjoint of the forgetful functor from the category of δ-rings to the category of commutative rings.

Definition 2.3. *A prism is a pair (A, I) where A is a δ-ring and $I \subset A$ is an invertible ideal such that A is derived (p, I)-adically complete, and $I + \phi(I)$ contains p.*

For a prism (A, I), we equip A with the (p, I)-adic topology.

Definition 2.4. *We say that a commutative ring R has bounded p^∞-torsion if there is some positive integer n such that $R[p^\infty] = R[p^n]$. We say that a prism (A, I) is bounded if A/I has bounded p^∞-torsion.*

Let $\mathfrak{X}$ be a p-adic formal scheme.

Definition 2.5. *The absolute prismatic site $\mathfrak{X}_{\Delta}$ of $\mathfrak{X}$ is the opposite category of the category of bounded prisms (A, I) with $\mathrm{Spf}(A/I) \to \mathfrak{X}$, with topology given by flat covers of prisms.*

We define presheaves $\mathcal{O}_{\Delta}$ and $\mathcal{I}_{\Delta}$ on $\mathfrak{X}_{\Delta}$ by $\mathcal{O}_{\Delta}(A, I) = A$ and $\mathcal{I}_{\Delta}(A, I) = I$. These are sheaves by [BS22, Corollary 3.12].

Definition 2.6. *A prismatic F-crystal on $\mathfrak{X}$ is a vector bundle $\mathcal{E}$ on $\mathfrak{X}_{\Delta}$ with an isomorphism $\varphi_{\mathcal{E}}\colon (\phi^*\mathcal{E})[\frac{1}{\mathcal{I}_{\Delta}}] \xrightarrow{\sim} \mathcal{E}[\frac{1}{\mathcal{I}_{\Delta}}]$.*

Let $\mathrm{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$ be the category of prismatic F-crystals on $\mathfrak{X}$.

Example 2.7. *If $\mathrm{Spa}(R, R^+)$ is an affinoid perfectoid space over $\mathbb{F}_p$ with an untilt $\mathrm{Spa}(R^\sharp, R^{+\sharp})$ and a morphism $\mathrm{Spf}(R^{+\sharp}) \to \mathfrak{X}$, then $(W(R^+), \mathrm{Ker}(W(R^+) \to R^{+\sharp}))$ with $\mathrm{Spf}(R^{+\sharp}) \to \mathfrak{X}$ gives an object of $\mathfrak{X}_{\Delta}$. Using the value of $(\mathcal{E}, \varphi_{\mathcal{E}})$ at this object, we can obtain a shtuka with one leg at $\mathrm{Spa}(R^\sharp, R^{+\sharp})$.*

The construction in Example 2.7 gives a connection between the prismatic theory and shtukas. See [IKY24, Theorem 3] for details.

3 Prismatic F-gauge

The notion of prismatic F-gauge is introduced by Bhatt–Lurie (*cf.* [Bha23]) based on previous works [Dri24], [BL22a], [BL22b] on the stacky approach to the integral p-adic Hodge theory.

Let W be the ring scheme of p-typical Witt vectors over $\mathbb{Z}_{(p)}$. We have the Frobenius morphism $F\colon W \to W$.

Definition 3.1. *Let R be a p-nilpotent ring. A Cartier–Witt divisor over R is a morphism $\alpha\colon I \to W(R)$ of $W(R)$-modules, where I is an invertible $W(R)$-module, such that the image of*

$$I \xrightarrow{\alpha} W(R) \xrightarrow{\gamma_0} R$$

is a nilpotent ideal, where γ_0 is the projection to the 0-th component, and the image of

$$I \xrightarrow{\alpha} W(R) \xrightarrow{\delta_{W(R)}} W(R)$$

generates the unit ideal.

We assume that $\mathfrak{X}$ is bounded in the sense that for any affine open U of $\mathfrak{X}$ the ring $\mathcal{O}_{\mathfrak{X}}(U)$ has bounded p^∞-torsion.

Definition 3.2. *The prismatization of $\mathfrak{X}$ is a formal stack $\mathfrak{X}^{\Delta}$ over $\mathbb{Z}_p$ defined as follows: For a p-nilpotent ring R, the groupoid $\mathfrak{X}^{\Delta}(R)$ consists of Cartier–Witt divisors $I \to W(R)$ with a morphism $\operatorname{Spec}(\operatorname{Cone}(I \to W(R))) \to \mathfrak{X}$ of derived formal schemes.*

For a p-nilpotent ring R and Cartier–Witt divisor $I \xrightarrow{\alpha} W(R)$ over R, the induced map $F^*I \xrightarrow{F^*\alpha} W(R)$ is also a Cartier–Witt divisor over R and we have the map

$$\operatorname{Cone}(I \xrightarrow{\alpha} W(R)) \to \operatorname{Cone}(F^*I \xrightarrow{F^*\alpha} W(R))$$

of animated rings induced from F. This gives a morphism $\phi_{\mathfrak{X}^{\Delta}}\colon \mathfrak{X}^{\Delta} \to \mathfrak{X}^{\Delta}$.

We define a group scheme $\mathbb{G}_a^\sharp$ over $\mathbb{Z}_{(p)}$ by

$$\mathbb{G}_a^\sharp = \operatorname{Ker}(F\colon W \to F_*W).$$

Definition 3.3. *Let R be a p-nilpotent ring.*

(1) A $\sharp$-invertible W-module over R is an affine W-module scheme over R that is fpqc locally isomorphic to $\mathbb{G}_a^\sharp$ as W-modules.

(2) An admissible W-module over R is an affine W-module scheme over R that can be realized as an extension of an invertible F_*W-module over R by a $\sharp$-invertible W-module over R.

Proposition 3.4 ([Bha23, Remark 5.2.5])**.** *Let R be a p-nilpotent ring. Let M and N be admissible W-modules over R which sit in short exact sequences*

$$0 \to L_M \to M \to F_*I_M \to 0, \qquad (*_M)$$
$$0 \to L_N \to N \to F_*I_N \to 0, \qquad (*_N)$$

where L_M, L_N are $\sharp$-invertible W-modules over R, and I_M and I_N are invertible W-module over R. Then any morphism $M \to N$ of W-modules induces a morphism of short exact sequences from $(*_M)$ to $(*_N)$.

In particular, the short exact sequence $(*_M)$ is unique up to unique isomorphism.

Definition 3.5. *Let R be a p-nilpotent ring. A filtered Cartier–Witt divisor over R is a morphism $d\colon M \to W$ of W-modules over R, where M is an admissible W-module over R, such that the induced morphism*

$$F_*I_M \to F_*W \tag{3.1}$$

comes from a Cartier–Witt divisor over R via F_*, where (3.1) is induced by d, $(*_M)$ and the sequence

$$0 \to \mathbb{G}_a^\sharp \to W \xrightarrow{F} F_*W \to 0$$

using Proposition 3.4.

Definition 3.6. *The filtered prismatization of $\mathfrak{X}$ is a formal stack $\mathfrak{X}^{\mathcal{N}}$ over $\mathbb{Z}_p$ defined as follows: For a p-nilpotent ring R, the groupoid $\mathfrak{X}^{\mathcal{N}}(R)$ consists of filtered Cartier–Witt divisors $M \to W$ with a morphism $\operatorname{Spec}((W/M)(R)) \to \mathfrak{X}$ of derived formal schemes, where*

$$(W/M)(R) = \mathrm{R}\Gamma(\operatorname{Spec}(R), \operatorname{Cone}(M \to W)).$$

Definition 3.7. *Let R be a p-nilpotent ring.*

Associating a filtered Cartier–Witt divisor $d\colon M \to W$ over R with the Cartier–Witt divisor over R giving (3.1), we obtain a morphism $\pi\colon \mathfrak{X}^{\mathcal{N}} \to \mathfrak{X}^{\mathbb{\Delta}}$.

Associating a Cartier–Witt divisor $I \xrightarrow{\alpha} W(R)$ with the filtered Cartier–Witt divisor

$$I \otimes_{W(R)} W \xrightarrow{\alpha \otimes \mathrm{id}_W} W(R) \otimes_{W(R)} W \cong W,$$

we obtain a morphism $j_{\mathrm{HT}}\colon \mathfrak{X}^{\mathbb{\Delta}} \to \mathfrak{X}^{\mathcal{N}}$.

Associating a Cartier–Witt divisor $I \xrightarrow{\alpha} W(R)$ with the filtered Cartier–Witt divisor M given by the following pullback

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{G}_a^\sharp & \longrightarrow & M & \longrightarrow & F_*(I \otimes_{W(R)} W) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow {\scriptstyle F_*(\alpha \otimes \mathrm{id}_W)} & & \\
0 & \longrightarrow & \mathbb{G}_a^\sharp & \longrightarrow & W & \xrightarrow{F} & F_*W & \longrightarrow & 0,
\end{array}$$

we obtain a morphism $j_{\mathrm{dR}}\colon \mathfrak{X}^{\mathbb{\Delta}} \to \mathfrak{X}^{\mathcal{N}}$.

By the construction, we have $\pi \circ j_{\mathrm{HT}} = \phi_{\mathfrak{X}^{\mathbb{\Delta}}}$ and $\pi \circ j_{\mathrm{dR}} = \mathrm{id}_{\mathfrak{X}^{\mathbb{\Delta}}}$. Further, j_{HT} and j_{dR} are open immersion with the disjoint images (*cf.* [Bha23, Remark 5.3.6]).

Example 3.8 ([Bha23, §5.5.1])**.** *Assume that $\mathfrak{X} = \operatorname{Spf} R$ where R is a perfectoid ring in the sense of [BMS18, Definition 3.5]. Then the natural homomorphism*

$$\theta\colon W(R^\flat) \to R$$

defined in [BMS18, §3.1] is surjective by [BMS18, Lemma 3.9]. We put $({\mathbb{\Delta}}_R, I) = (W(R^\flat), \ker\theta)$. Let ϕ act on ${\mathbb{\Delta}}_R = W(R^\flat)$ via the p-th power map on $R^\flat$. We define a filtration $\operatorname{Fil}^\bullet_{\mathcal{N}} {\mathbb{\Delta}}_R$ of ${\mathbb{\Delta}}_R$ by

$$\operatorname{Fil}^i_{\mathcal{N}} {\mathbb{\Delta}}_R = \begin{cases} \phi^{-1}(I^i) & \text{if } i \geq 0, \\ {\mathbb{\Delta}}_R & \text{if } i < 0. \end{cases}$$

We put

$$\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}} \mathbb{\Delta}_R) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Fil}^{i}_{\mathcal{N}} \mathbb{\Delta}_R t^{-i} \subset \mathbb{\Delta}_R[t, t^{-1}].$$

Let $\mathbb{G}_{\mathrm{m}}$ *act on this ring by* i*-th power on* $\mathrm{Fil}^{i}_{\mathcal{N}} \mathbb{\Delta}_R t^{-i}$*. Then*

$$\mathfrak{X}^{\mathbb{\Delta}} = \mathrm{Spf}\, \mathbb{\Delta}_R, \quad \mathfrak{X}^{\mathcal{N}} = [\mathrm{Spec}(\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}} \mathbb{\Delta}_R))/\mathbb{G}_{\mathrm{m}}] \times_{\mathrm{Spec}\, \mathbb{\Delta}_R} \mathrm{Spf}\, \mathbb{\Delta}_R,$$

and $\pi \colon \mathfrak{X}^{\mathcal{N}} \to \mathfrak{X}^{\mathbb{\Delta}}$ *is the natural projection. Further,* j_{HT} *and* j_{dR} *are induced from*

$$\mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}} \mathbb{\Delta}_R) \xrightarrow{\phi} \bigoplus_{i \in \mathbb{Z}} I^i t^{-i} \quad \textit{and} \quad \mathrm{Rees}(\mathrm{Fil}^{\bullet}_{\mathcal{N}} \mathbb{\Delta}_R) \hookrightarrow \mathbb{\Delta}_R[t^{\pm 1}]$$

respectively.

Definition 3.9. *We define the syntomification* $\mathfrak{X}^{\mathrm{syn}}$ *of* $\mathfrak{X}$ *by the cocartesian diagram*

$$\begin{array}{ccc} \mathfrak{X}^{\mathbb{\Delta}} \coprod \mathfrak{X}^{\mathbb{\Delta}} & \xhookrightarrow{j_{\mathrm{HT}} \coprod j_{\mathrm{dR}}} & \mathfrak{X}^{\mathcal{N}} \\ \downarrow & & \downarrow \\ \mathfrak{X}^{\mathbb{\Delta}} & \longrightarrow & \mathfrak{X}^{\mathrm{syn}}. \end{array}$$

The category of prismatic F*-gauges in vector bundles is defined as the category of vector bundles on* $\mathfrak{X}^{\mathrm{syn}}$*, for which we write* $\mathrm{Vect}(\mathfrak{X}^{\mathrm{syn}})$*.*

4 Tannakian framework

Let k be a perfect field of characteristic p. Assume that $\mathfrak{X}$ is a smooth p-adic formal scheme over $W(k)$.

For an object $(\mathcal{E}, \varphi_{\mathcal{E}})$ of $\mathrm{Vect}^{\varphi}(\mathfrak{X}_{\mathbb{\Delta}})$, we define the filtration $\mathrm{Fil}^{\bullet}_{\mathrm{Nyg}}(\phi^* \mathcal{E}) \subset \phi^* \mathcal{E}$ by $\mathcal{O}_{\mathbb{\Delta}}$-submodules, which we call the Nygaard filtration, so that

$$\mathrm{Fil}^{r}_{\mathrm{Nyg}}(\phi^* \mathcal{E})(A, I) = \{ x \in \phi^* \mathcal{E}(A, I) \mid \varphi_{\mathcal{E}}(x) \in I^r \mathcal{E}(A, I) \}$$

for an object (A, I) of $\mathfrak{X}_{\mathbb{\Delta}}$.

Let $\mathrm{Vect}^{\varphi, \mathrm{lff}}(\mathfrak{X}_{\mathbb{\Delta}})$ be the category of prismatic F-crystal on $\mathfrak{X}$ with locally filtered free Nygaard filtration.

Let $\mathscr{G}$ be a smooth group scheme over $\mathbb{Z}_p$. For a $\mathbb{Z}_p$-linear tensor category $\mathcal{C}$, let $\mathscr{G}$-$\mathcal{C}$ denote the category of $\mathbb{Z}_p$-linear exact tensor functors $\mathrm{Rep}_{\mathbb{Z}_p}(\mathscr{G}) \to \mathcal{C}$. Using [GL23, Theorem 2.31], we can show the following:

Proposition 4.1 ([IKY23, §1.3])**.** *There is a bi-exact* $\mathbb{Z}_p$*-linear tensor-equivalence*

$$\mathrm{Vect}(\mathfrak{X}^{\mathrm{syn}}) \cong \mathrm{Vect}^{\varphi, \mathrm{lff}}(\mathfrak{X}_{\mathbb{\Delta}}).$$

In particular, we have an equivalence

$$\mathscr{G}\text{-}\mathrm{Vect}(\mathfrak{X}^{\mathrm{syn}}) \cong \mathscr{G}\text{-}\mathrm{Vect}^{\varphi, \mathrm{lff}}(\mathfrak{X}_{\mathbb{\Delta}}).$$

Definition 4.2. *An analytic prismatic F-crystal on $\mathfrak{X}$ is a compatible system of pairs $(\mathcal{E}_{(A,I)}, \varphi_{(A,I)})$ of a vector bundle $\mathcal{E}_{(A,I)}$ on* $\operatorname{Spec}(A) \setminus V(p,I)$ *and* $\varphi_{(A,I)} \colon (\phi^*\mathcal{E}_{(A,I)})[\frac{1}{I}] \xrightarrow{\sim} \mathcal{E}_{(A,I)}[\frac{1}{I}]$ *for $(A,I) \in \mathfrak{X}_{\Delta}$, where $V(p,I)$ is the closed subscheme of* $\operatorname{Spec} A$ *defined by the ideal (p,I).*

Let $\mathrm{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\Delta})$ be the category of analytic prismatic F-crystals on $\mathfrak{X}$. We write $\mathfrak{X}_\eta$ for the generic fiber of $\mathfrak{X}$. Let $\mathrm{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p}(\mathfrak{X}_\eta)$ be the category of crystalline $\mathbb{Z}_p$-local systems on $\mathfrak{X}_\eta$.

Theorem 4.3 ([DLMS24, Theorem 3.46], [GR24, Theorem A])**.** *There is an equivalence*

$$\mathrm{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\Delta}) \xrightarrow{\sim} \mathrm{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p}(\mathfrak{X}_\eta)$$

of categories.

The restriction functor $\mathrm{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \to \mathrm{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\Delta})$ is fully faithful by [GR24, Proposition 3.7].

We say that an object of $\mathrm{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p}(\mathfrak{X}_\eta)$ is prismatically good reduction if it comes from $\mathrm{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$ under the equivalence in Theorem 4.3. Let $\mathrm{Loc}^{\Delta\text{-gr}}_{\mathbb{Z}_p}(\mathfrak{X}_\eta) \subset \mathrm{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p}(\mathfrak{X}_\eta)$ be the exact full subcategory of prismatically good reduction objects. By the definition, we have an equivalence

$$\mathrm{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \xrightarrow{\sim} \mathrm{Loc}^{\Delta\text{-gr}}_{\mathbb{Z}_p}(\mathfrak{X}_\eta) \tag{4.1}$$

of categories.

Remark 4.4. *The equivalence in Theorem 4.3 is bi-exact, but the equivalence* (4.1) *may not be bi-exact. More precisely, a quasi-inverse of* (4.1) *may not be exact. The problem is that there is a non-exact sequence in* $\mathrm{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$ *whose image in* $\mathrm{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\Delta})$ *is exact.*

In spite of Remark 4.4, we still can show the following:

Theorem 4.5 ([IKY24, §2.4])**.** *Assume that $\mathscr{G}$ is a reductive group scheme over $\mathbb{Z}_p$. Then there is an equivalence*

$$\mathscr{G}\text{-}\mathrm{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \xrightarrow{\sim} \mathscr{G}\text{-}\mathrm{Loc}^{\Delta\text{-}gr}_{\mathbb{Z}_p}(\mathfrak{X}_\eta)$$

of categories.

5 Integral crystalline functor

In this section, we construct the integral crystalline functor

$$\mathbb{D}_{\mathrm{crys}} \colon \mathrm{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \to \mathrm{VectNF}^{\varphi}(\mathfrak{X}_{\mathrm{crys}}),$$

where $\mathrm{VectNF}^{\varphi}(\mathfrak{X}_{\mathrm{crys}})$ is the category of naive filtered F-crystals on $\mathfrak{X}$ in the sense of [IKY25, §2.1.1]. In [BS23, Construction 4.12], Bhatt–Scholze constructed

$$\mathrm{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \to \mathrm{Vect}^{\varphi}(\mathfrak{X}_{\mathrm{crys}});\ \mathcal{E} \mapsto \mathcal{E}^{\mathrm{crys}}.$$

Therefore it is enough to define a filtration on $\mathcal{E}^{\mathrm{crys}}$. Let

$$\mathbb{D}^+_{\mathrm{dR}} \colon \mathrm{Vect}^{\varphi}(\mathfrak{X}_{\Delta}) \to \mathrm{FilVect}(\mathfrak{X})$$

be the de Rham realization functor defined in [IKY25, §1.2]. As explained there, the filtration of $\mathbb{D}^+_{\mathrm{dR}}(\mathcal{E})$ can be described using the Nygaard filtration of $\phi^*\mathcal{E}$.

Theorem 5.1 ([IKY25, §1.3])**.** *For $\mathcal{E} \in \mathrm{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$, there is a canonical crystalline-de Rham comparison isomorphism $\mathcal{E}^{\mathrm{crys}} \cong \mathbb{D}^{+}_{\mathrm{dR}}(\mathcal{E})$.*

For $\mathcal{E} \in \mathrm{Vect}^{\varphi}(\mathfrak{X}_{\Delta})$, we put $\mathbb{D}_{\mathrm{crys}}(\mathcal{E}) = \mathcal{E}^{\mathrm{crys}}$ and define the filtration on $\mathbb{D}_{\mathrm{crys}}(\mathcal{E})$ by the filtration on $\mathbb{D}^{+}_{\mathrm{dR}}(\mathcal{E})$ using the crystalline-de Rham comparison isomorphism in Theorem 5.1.

The functor $\mathbb{D}_{\mathrm{crys}}$ induces an equivalence between the category of prismatic F-crystals on $\mathfrak{X}$ with Hodge–Tate weights in $[0, p-2]$ and the category of Fontaine–Laffaille modules on $\mathfrak{X}$. See [IKY25, §3] for details.

6 Syntomic realization

Let $\mu_h \colon \mathbb{G}_{\mathrm{m},\mathcal{O}_{E_v}} \to \mathcal{G}_{\mathcal{O}_{E_v}}$ be the extension of a Hodge cocharacter of (G, X). Let $\mu_h^{\mathrm{c}} \colon \mathbb{G}_{\mathrm{m},\mathcal{O}_{E_v}} \to \mathcal{G}^{\mathrm{c}}_{\mathcal{O}_{E_v}}$ be the cocharacter induced by μ_h. We assume that p is odd.

Theorem 6.1 ([IKY23, §2.4 and §2.6])**.** *There is a tensor functor*

$$\omega_{\mathrm{syn}} \colon \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}^{\mathrm{c}}) \to \mathrm{Vect}(\widehat{\mathcal{S}}_K^{\mathrm{syn}})$$

of type μ_h^{c} that is compatible with ω_{crys} and ω_{et}.

For construction of ω_{syn}, first we show that ω_{et} belongs to $\mathcal{G}^{\mathrm{c}}\text{-}\mathrm{Loc}_{\mathbb{Z}_p}^{\Delta\text{-}\mathrm{gr}}(\widehat{\mathcal{S}}_{K,\eta})$. Then we obtain an object of $\mathcal{G}^{\mathrm{c}}\text{-}\mathrm{Vect}^{\varphi}(\widehat{\mathcal{S}}_{K,\Delta})$ by Theorem 4.5. Next we show that the obtained object belongs to $\mathcal{G}^{\mathrm{c}}\text{-}\mathrm{Vect}^{\varphi,\mathrm{lff}}(\widehat{\mathcal{S}}_{K,\Delta})$. Then we can obtain ω_{syn} using Proposition 4.1.

We note that to make sense of the compatibility with ω_{crys} in Theorem 6.1, we need the functor $\mathbb{D}_{\mathrm{crys}}$ constructed in §5.

See [IKY23, §2.6.2] for cohomological consequences of Theorem 6.1.

7 Characterization of the integral model

Definition 7.1. *A potentially crystalline locus of $\mathrm{Sh}^{\mathrm{an}}_{K,E_v}$ is a quasi-compact open subspace U of $\mathrm{Sh}^{\mathrm{an}}_{K,E_v}$ uniquely determined by the following condition:*

> *For any classical point x of $\mathrm{Sh}^{\mathrm{an}}_{K,E_v}$, the point x is in U if and only if $\omega_{\mathrm{et}}(V)_x$ is potentially crystalline for any $V \in \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}^{\mathrm{c}})$.*

Let $\mathrm{Sh}^{\mathrm{pc}}_{K,E_v} \subset \mathrm{Sh}^{\mathrm{an}}_{K,E_v}$ be the potentially crystalline locus constructed in [IM20, Theorem 5.17] (*cf.* [IM20, Remark 2.12] and [LZ17, Theorem 1.2]). We write ω_{pc} for the restriction of $\omega^{\mathrm{an}}_{\mathrm{et}}$ to $\mathrm{Sh}^{\mathrm{pc}}_{K,E_v}$.

We write $\mathrm{BT}^{\mathcal{G}^{\mathrm{c}},-\mu_h^{\mathrm{c}}}$ for the moduli stack of prismatic F-gauges with $\mathcal{G}^{\mathrm{c}}$-structure of type $-\mu_h^{\mathrm{c}}$ constructed by [GM24], confirming a conjecture in [Dri23]. Then we have $\widehat{\mathcal{S}}_K \to \mathrm{BT}^{\mathcal{G}^{\mathrm{c}},-\mu_h^{\mathrm{c}}}$ given by ω_{syn}.

Definition 7.2. *A syntomic integral canonical model of Sh_{K,E_v} is a smooth, separated integral model $\mathcal{X}$ of Sh_{K,E_v} over $\mathcal{O}_{E_v}$ satisfying the following conditions:*

(1) $\widehat{\mathcal{X}}_{\eta} = \mathrm{Sh}^{\mathrm{pc}}_{K,E_v}$.

(2) There is a tensor functor

$$\mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}^{\mathrm{c}}) \to \mathrm{Vect}(\widehat{\mathcal{X}}^{\mathrm{syn}})$$

of type $-\mu_h^{\mathrm{c}}$ whose etale realization is equal to ω_{pc} such that the resulting morphism $\widehat{\mathcal{X}} \to \mathrm{BT}^{\mathcal{G}^{c},-\mu_h^{\mathrm{c}}}$ is formally etale.

Remark 7.3. *To make sense of a condition like Definition 7.2 (2), the moduli of shtukas is not enough because the theory of v-sheaves can not distinguish nilpotent thickenings.*

Theorem 7.4 ([IKY23, §3.4]). *The scheme $\mathcal{S}_K$ over $\mathcal{O}_{E_v}$ is the unique syntomic integral canonical model.*

In the proof of Theorem 7.4, confirming the condition (1) in Definition 7.2 generalizes a result in [IM13, §7] for the PEL type case, and confirming the condition (2) in Definition 7.2 amounts to proving a Serre–Tate theorem for Shimura varieties of abelian type.

Remark 7.5. *The characterization in Theorem 7.4 works for each level K contrary to previously known characterization in [Kis10].*

We write $\overline{\mathcal{S}}_K$ and $\overline{\mathcal{G}}^{\mathrm{c}}$ for the special fibers of $\mathcal{S}_K$ and $\mathcal{G}^{\mathrm{c}}$ respectively. Let $\overline{\mathcal{G}}^{\mathrm{c}}\text{-Zip}^{-\mu_h^{\mathrm{c}}}$ be the moduli stack of $\overline{\mathcal{G}}^{\mathrm{c}}$-zips of type $-\mu_h^{\mathrm{c}}$. As an application of the syntomic realization functor, we can obtain the following theorem:

Theorem 7.6 ([IKY23, §3.5]). *There is a natural smooth morphism*

$$\overline{\mathcal{S}}_K \to \overline{\mathcal{G}}^{\mathrm{c}}\text{-Zip}^{-\mu_h^{\mathrm{c}}}.$$

Theorem 7.6 generalizes [Zha18, Theorem 4.12] in the Hodge type case to the abelian type case. This allows us to apply results in [IK24] to Shimura varieties of abelian type as well.

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