The pro-modularity in the residually reducible case

Xinyao Zhang
Graduate School of Mathematical Sciences, University of Tokyo

1 Introduction

This article is based on the author's work [1].

Let $\bar{\rho}_0 : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F})$ be a continuous odd representation, where $\mathbb{F}$ is finite field with characteristic $p \geq 3$. Let $R_{\bar{\rho}_0}$ be the universal deformation ring of $\bar{\rho}_0$. It is conjectured by Gouvêa that $R_{\bar{\rho}_0}$ is isomorphic to some p-adic big Hecke algebra.

Recently, Deo has explored the reducible case in [2]. More precisely, suppose that $\bar{\rho}_0^{\mathrm{ss}} = 1 \oplus \bar{\chi}$ for some odd character $\bar{\chi}$ unramified outside Np, where N is a positive integer. Let $R_{\bar{\rho}_x}$ be the universal deformation ring of ρ_x, where $0 \neq x \in H^1(G_{\mathbb{Q},Np}, \mathbb{F}(\bar{\chi}))$ and ρ_x is the unique reducible Galois representation determined by x such that $\rho_x^{\mathrm{ss}} = 1 \oplus \bar{\chi}$. Let $R^{\mathrm{pd}}_{\bar{\rho}_0}$ be the universal pseudo-deformation ring of $\bar{\rho}_0^{\mathrm{ss}}$. Deo proves the following results.

Theorem 1.0.1. *[2, Theorem A, Theorem 4.5] Let N_0 be the tame Artin conductor of $\bar{\rho}_0$. Suppose $N_0|N$, $\bar{\chi}|_{G_{\mathbb{Q}_p}} \neq \mathbf{1}, \omega_p^{-1}$, $\dim_{\mathbb{F}} H^1(G_{\mathbb{Q},Np}, \mathbb{F}(\bar{\chi})) = 1$ and $p \nmid \phi(N)$. Then*

1) There is an isomorphism $(R^{\mathrm{pd}}_{\bar{\rho}_0})^{\mathrm{red}} \cong T(N)_{\bar{\rho}_0}$, where $T(N)_{\bar{\rho}_0}$ is the localization of the big Hecke algebra $T(N)$ (of level N) at the maximal ideal $\mathfrak{m}_0$ determined by $\bar{\rho}_0$.

2) There exists an isomorphism $R_{\bar{\rho}_x} \twoheadrightarrow T(N)_{\bar{\rho}_0}$.

In his proof, a crucial assumption is the cyclicity of $H^1(G_{\mathbb{Q},Np}, \mathbb{F}(\bar{\chi}))$. In this case, the natural map $R^{\mathrm{pd}}_{\bar{\rho}_0} \to R_{\bar{\rho}_x}$ is actually a surjection. However, such a surjection may not hold in general, and the cyclicity assumption is possible only for $\mathbb{Q}$ rather than any other totally real field F due to the global characteristic formula. To give an analogue result for totally real fields, we follow the strategy of the proof of the Fontaine-Mazur conjecture from [3] and [4] to study the pro-modularity of deformation rings.

Let p be an odd prime. Let F be an abelian totally real field of even degree p splits completely in F. Let Σ be a finite set of finite places of F containing Σ_p consisting of all the places $v \mid p$. Let $\mathcal{O}$ be the ring of integers of some local field E with a uniformizer π and residue field $\mathbb{F}$, and $\chi : \mathrm{Gal}(\bar{F}/F) \to \mathcal{O}^{\times}$ a totally odd character unramified outside Σ, and $\bar{\chi} = \chi \mod \pi$. Let F_{Σ} be the maximal extension of F unramified outside Σ and all infinite places. Let ϵ, ω_p be the p-adic, mod p cyclotomic character, respectively.

Let $R^{\rm ps}_{\rm aux}$ be the universal pseudo-deformation ring of the residual pseudo-representation $1+\bar{\chi}$ with fixed determinant χ. Let $\mathbb{T}_\xi$ be the big p-adic Hecke algebra, and assume that $1+\bar{\chi}$ determines a maximal ideal $\mathfrak{m}_\xi$. For a prime $\mathfrak{p}$ of $R^{\rm ps}_{\rm aux}$, we say that it is *pro-modular* if it comes from a prime of $(\mathbb{T}_\xi)_{\mathfrak{m}_\xi}$. The main theorem is the following.

Theorem 1.0.2. *Keep notations as above. For any place $v \in \Sigma \setminus \Sigma_p$, assume that $p \mid {\rm Nm}(v) - 1$. Assume that the character χ satisfies the following conditions:*

1) χ is unramified outside Σ_p,

2) $\chi({\rm Frob}_v) \equiv 1 \mod \pi$ for $v \in \Sigma \setminus \Sigma_p$.

3) $\bar{\chi}$ can be extended to a character of $G_{\mathbb{Q}}$.

4) $\bar{\chi}\mid_{G_{F_v}} \neq 1$ for any $v \mid p$.

5) $\chi\mid_{G_{F_v}}$ is de Rham for any $v \mid p$.

If $[F:\mathbb{Q}] \geq 7|\Sigma \setminus \Sigma_p| + 4$, then for any irreducible component of $R^{\rm ps}_{\rm aux}$ of dimension at least $1 + 2[F:\mathbb{Q}]$, it is pro-modular of dimension $1 + 2[F:\mathbb{Q}]$.

2 Preliminaries

2.1 Pseudo-deformations

In this subsection, we recall some basic results about two-dimensional pseudo-deformations.

Definition 2.1.1. For a profinite group G and a topological commutative ring R in which 2 is invertible, a 2-dimensional *pseudo-representation* is a continuous function $T : G \to R$ such that

1) $T(1) = 2$,

2) there exists an order 2 element $c \in G$ such that $T(c) = 0$,

3) $T(\sigma\tau) = T(\tau\sigma)$ for all $\sigma, \tau \in G$,

4) $T(\gamma\delta\eta) + T(\gamma\eta\delta) - T(\gamma\eta)T(\delta) - T(\eta\delta)T(\gamma) - T(\delta\gamma)T(\eta) + T(\gamma)T(\delta)T(\eta) = 0$, for any $\delta, \gamma, \eta \in G$.

The *determinant* $\det(T)$ of a pseudo-representation T is a character (using 4)) defined by

$$\det(T) : G \to R^\times, \ \ \det(T)(\sigma) = \frac{1}{2}(T(\sigma)^2 - T(\sigma^2)), \sigma \in G.$$

For a pseudo-representation T and $\sigma, \tau \in G$, define:

1) $a(\sigma) = \frac{1}{2}(T(c\sigma) + T(\sigma))$,

2) $d(\sigma) = \frac{1}{2}(-T(c\sigma) + T(\sigma))$,

3) $y(\sigma, \tau) = a(\sigma\tau) - a(\sigma)a(\tau)$.

The continuous functions $\{a, d, y\}$ satisfy the following equations.

(1) $y(\sigma, \tau) = d(\tau\sigma) - d(\tau)d(\sigma)$.

(2) $y(\sigma\tau,\delta)=a(\sigma)y(\tau,\delta)+y(\sigma,\delta)d(\tau)$.
(3) $y(\sigma,\tau\delta)=a(\delta)y(\sigma,\tau)+y(\sigma,\delta)d(\tau)$.
(4) $y(\alpha,\beta)y(\sigma,\tau)=y(\alpha,\tau)y(\sigma,\beta)$.

If $\rho : G \to \mathrm{GL}_2(R)$ is a continuous representation with an element $c \in G$ satisfying $\rho(c)=\begin{pmatrix}1 & \\ & -1\end{pmatrix}$, then $\mathrm{tr}(\rho)$ is a pseudo-representation. More explicitly, if $\rho(\sigma)=\begin{pmatrix}a_\sigma & b_\sigma \\ c_\sigma & d_\sigma\end{pmatrix}$, then $a(\sigma)=a_\sigma$, $d(\sigma)=d_\sigma$ are as defined above and $y(\sigma,\tau)=b_\sigma c_\tau$. Thus, from a representation ρ, we always get the corresponding pseudo-representation $\mathrm{tr}(\rho)$.

Conversely, from a pseudo-representation, we may also get a representation in some special cases. Assume that R is either a field or a discrete valuation ring. For a pseudo-representation $T : G \to R$, a representation ρ associated to T is in the following form.

1) $\rho(\sigma)=\begin{pmatrix}a(\sigma) & \\ & d(\sigma)\end{pmatrix}$, if all $y(\sigma,\tau)=0$. We call this case reducible.

2) Choose σ_0,τ_0 such that $\frac{y(\sigma,\tau)}{y(\sigma_0,\tau_0)}\in R$ for any σ,τ, if $y(\sigma,\tau)\neq 0$ for some σ,τ. Define $\rho(\sigma)=\begin{pmatrix}a(\sigma) & \frac{y(\sigma,\tau_0)}{y(\sigma_0,\tau_0)} \\ y(\sigma_0,\sigma) & d(\sigma)\end{pmatrix}$. We call this case irreducible.

Lemma 2.1.2. *Let R be a CNL domain with maximal ideal $\mathfrak{m}_R$ and residue field $\mathbb{F}$. Let G be a profinite group and $T : G \to R$ be a pseudo-representation of G. Assume that $T \bmod \mathfrak{m}_R$ is reducible. Let S be a finite subset (not necessarily a group) of G . Then there exist a partition of $S = S_1 \amalg S_2$, a positive integer $n > 1$ satisfying $(n,p)=1$ and a CNL domain R' satisfying the following conditions.*

a) R' is a quotient of R.

b) For any $\theta\in S_1$, we have $y(\theta,\alpha)=0$ for any $\alpha\in G$ in R'. For any $\theta',\theta''\in S_2$, we have $y(\theta',\alpha)^n=y(\theta'',\alpha)^n$ for any $\alpha\in G$ in R'.

c) For any $\alpha\in G$, $\theta\in S_2$, either $y(\theta,\beta)=0$ for any $\beta\in G$ in R' or $u(\alpha,\theta)$ is well-defined and integral over R'. Here, we define $u(\alpha,\beta)(\tau)=\frac{y(\alpha,\tau)}{y(\beta,\tau)}$ for some $\tau\in G$, and it is independent of τ.

d) We have $\dim R'\geq \dim R-|S|$. If S_2 is not empty, then further $\dim R'\geq \dim R-|S|+1$.

Now let F be a totally real field. Let p be an odd rational prime and $\mathbb{F}$ be a finite field of characteristic p. Let Σ be a finite set of finite places of F containing all the places $v \mid p$. Recall that $\mathcal{O}$ is the ring of integers of some local field E (a finite extension of $\mathbb{Q}_p$) with a uniformizer π and residue field $\mathbb{F}$. Let $\chi : \mathrm{Gal}(\bar{F}/F)\to\mathcal{O}^\times$ be a continuous totally odd character unramified outside Σ, and $\bar{\chi}=\chi \bmod \pi$. Let F_Σ be the maximal extension of

F unramified outside Σ and all infinite places.

We focus on the case $G = \mathrm{Gal}(F_\Sigma/F)$. By Galois deformation theory, we have a universal pseudo-deformation ring $R^{\mathrm{ps}}_{\mathrm{aux}}$ of the pseudo-representation $1 + \bar{\chi}$ with determinant χ.

Lemma 2.1.3. *Suppose $\mathfrak{p}$ is a one-dimensional prime of $R^{\mathrm{ps}}_{\mathrm{aux}}$ whose corresponding represenation is absolutely irreducible.*

1) $\dim(R^{\mathrm{ps}}_{\mathrm{aux}})_{\mathfrak{p}} \geq 2[F:\mathbb{Q}]$.

2) The connectedness dimension $c((R^{\mathrm{ps}}_{\mathrm{aux}})_{\mathfrak{p}})$ is at least $2[F:\mathbb{Q}]-1$.

3) Every irreducible component of $R^{\mathrm{ps}}_{\mathrm{aux}}$ containing $\mathfrak{p}$ is of dimension at least $1+2[F:\mathbb{Q}]$.

Lemma 2.1.4. *Assume F is abelian over $\mathbb{Q}$.*

1) Let $R^{\mathrm{ps,red}}$ be the reducible locus of $R^{\mathrm{ps}}_{\mathrm{aux}}$. We have $\dim(R^{\mathrm{ps,red}}) \leq 2$.

2) Assume $\bar{\chi}$ is quadratic and can be extended to $G_{\mathbb{Q}}$. Then the dihedral locus C^{dih} of $\mathrm{Spec}\, R^{\mathrm{ps}}_{\mathrm{aux}}$, i.e., for any point $\mathfrak{p}$ in C^{dih}, $\rho(\mathfrak{p}) \cong \mathrm{Ind}^{G_F}_{G_L}\theta$ for some character θ and the splitting field $L = F(\bar{\chi})$, is of dimension at most $2+[F:\mathbb{Q}]$.

2.2 Pro-modularity

In this subsection, we recall some modularity results in [4].

Assume F is an abelian totally real number field of even degree over $\mathbb{Q}$ in which p splits completely. Let Σ be a finite set of finite places of F containing all places $v \mid p$ (consisting of Σ_p), and assume that for all $v \in \Sigma \setminus \Sigma_p$, we have $p \mid \mathrm{Nm}(v)-1$. Let $\xi_v : k(v)^\times \to \mathcal{O}^\times$ be characters of p-power order for $v \in \Sigma \setminus \Sigma_p$, and we can view them as characters of I_v by local class field theory. Let $\chi : \mathrm{Gal}(F_\Sigma/F) \to \mathcal{O}^\times$ be a continuous totally odd character such that

1) χ is unramified outside Σ_p,

2) $\chi(\mathrm{Frob}_v) \equiv 1 \mod \pi$ for $v \in \Sigma \setminus \Sigma_p$.

Let D be a quaternion algebra over F ramified exactly at all infinite places, and we fix an isomorphism between $(D \otimes_F \mathbb{A}^\infty_F)^\times$ and $\mathrm{GL}_2(\mathbb{A}^\infty_F)$. Let $\psi = \chi\epsilon$ be a character of $(\mathbb{A}^\infty_F)^\times/F^\times_+$ via global class field theory. Let $U^p = \prod_{v\nmid p} U_v$ be a tame level such that $U_v = \mathrm{GL}_2(\mathcal{O}_{F_v})$ if $v \notin \Sigma$ and $U_v = \mathrm{Iw}_v := \{g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod \pi_v, g \in \mathrm{GL}_2(\mathcal{O}_{F_v})\}$ otherwise. For any $v \in \Sigma \setminus \Sigma_p$, the map $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \xi_v(a/d \mod \pi_v)$ defines a character of U_v and the product of ξ_v can be viewed as a character ξ of U^p by projecting to $\prod_{v\in\Sigma\setminus\Sigma_p} U_v$. From our setting, we can define a Hecke algebra $\mathbb{T}_\xi := \mathbb{T}_{\psi,\xi}(U^p)$ for quaternionic forms (see [4, Section 3.1] for details).

Suppose that $\bar{\rho}_0 = 1 \oplus \bar{\chi}$ is modular, i.e. $T_v - (1 + \chi(\mathrm{Frob}_v)), v \notin \Sigma$ and π generate a maximal ideal $\mathfrak{m}_\xi$ of $\mathbb{T}_\xi$. Then by the universal property, we get a natural surjection $R^{\mathrm{ps},\{\xi_v\}} \twoheadrightarrow (\mathbb{T}_\xi)_{\mathfrak{m}_\xi}$, where $R^{\mathrm{ps},\{\xi_v\}}$ is quotient of $R^{\mathrm{ps}}_{\mathrm{aux}}$ such that for all $v \in \Sigma \setminus \Sigma_p$, the universal deformation T_{aux} satisfies $T_{\mathrm{aux}} \mid_{I_v} = \xi_v + \xi_v^{-1}$.

Theorem 2.2.1. *[4, Theorem 3.6.1] Each irreducible component of $(\mathbb{T}_\xi)_{\mathfrak{m}_\xi}$ is of characteristic zero and of dimension at least $1 + 2[F : \mathbb{Q}]$.*

Definition 2.2.2. (1) We say that a prime of $R^{\mathrm{ps},\{\xi_v\}}$ is *pro-modular* if it comes from a prime of $(\mathbb{T}_\xi)_{\mathfrak{m}_\xi}$. We say that a prime of $R^{\mathrm{ps}}_{\mathrm{aux}}$ is *pro-modular* if it comes from a pro-modular prime of some $R^{\mathrm{ps},\{\xi_v\}}$.

(2) Let $\mathfrak{q}$ be a prime of $(\mathbb{T}_\xi)_{\mathfrak{m}_\xi}$ and A be the normal closure of $(\mathbb{T}_\xi)_{\mathfrak{m}_\xi}/\mathfrak{q}$ in $k(\mathfrak{q})$. We say that $\mathfrak{q}$ is a *nice* prime if $\mathfrak{q}$ contains p and $\dim(\mathbb{T}_\xi)_{\mathfrak{m}_\xi}/\mathfrak{q} = 1$ and there exists a two-dimensional representation $\rho(\mathfrak{q})^o : \mathrm{Gal}(F_\Sigma/F) \to \mathrm{GL}_2(A)$ satisfying:

1) $\rho(\mathfrak{q})^o \otimes k(\mathfrak{q}) \cong \rho(\mathfrak{q})$ is irreducible.

2) The mod $\mathfrak{m}_A$ reduction $\bar{\rho}_b$ of $\rho(\mathfrak{q})^o$ is a non-split extension and has the form $\bar{\rho}_b(g) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, $g \in \mathrm{Gal}(F_\Sigma/F)$. Here $\mathfrak{m}_A$ is the maximal ideal of A.

3) (dihedral condition) If $\rho(\mathfrak{q})$ is dihedral, namely isomorphic to $\mathrm{Ind}_{G_L}^{G_F} \theta$ for some quadratic extension L of F and continuous character $\theta : G_L \to k(\mathfrak{q})^\times$, then $L \cap F(\zeta_p) = F$, where ζ_p is a primitive p-th root of unity.

4) $\rho(\mathfrak{q})^o \mid_{G_{F_v}} = \bar{\rho}_b \mid_{G_{F_v}}$ for any $v \in \Sigma \setminus \Sigma_p$.

(3) We say that a prime of $R^{\mathrm{ps},\{\xi_v\}}$ is *nice* if it comes from a nice prime of $(\mathbb{T}_\xi)_{\mathfrak{m}_\xi}$.

The following patching theorem is crucial for our pro-modularity result.

Theorem 2.2.3. *Let $\mathfrak{q}$ be a nice prime of $(\mathbb{T}_\xi)_{\mathfrak{m}_\xi}$ and $\mathfrak{q}^{\mathrm{ps}} = \mathfrak{q} \cap R^{\mathrm{ps},\{\xi_v\}}$. Then the natural surjective map $(R^{\mathrm{ps},\{\xi_v\}})_{\mathfrak{q}^{\mathrm{ps}}} \twoheadrightarrow (\mathbb{T}_\xi)_{\mathfrak{q}}$ has nilpotent kernel.*

2.3 The ordinary Fontaine-Mazur conjecture

In this subsection, we recall the ordinary Fontaine-Mazur conjecture in [3] and [4].

Assume that F is an abelian totally real number field over $\mathbb{Q}$ in which p is unramified. Let Σ be a finite set of primes of F containing all places above p. Let $\chi : \mathrm{Gal}(F_\Sigma/F) \to \mathcal{O}^\times$ be a continuous totally odd character such that

1) $\bar{\chi}$ can be extended to a character of $G_{\mathbb{Q}}$.

2) $\bar{\chi} \mid_{G_{F_v}}$ is not trivial for any $v \mid p$.

3) $\chi \mid_{G_{F_v}}$ is de Rham for any $v \mid p$.

Let $(R^{\mathrm{ps,ord}}, T^{\mathrm{ord}})$ be the universal pseudo-deformation ring of $\mathrm{tr}(\bar{\rho}_0)$ of auxiliary type such that $T^{\mathrm{ord}} \mid_{G_{F_v}}$ is reducible for any $v \mid p$, i.e. $y(\sigma, \tau) = 0$ for any $\sigma, \tau \in G_{F_v}$, $v \mid p$. Since $\bar{\chi} \mid_{G_{F_v}}$ is not trivial, we have $T^{\mathrm{ord}} \mid_{G_{F_v}} = \psi_{v,1}^{\mathrm{ord}} + \psi_{v,2}^{\mathrm{ord}}$ for some characters $\psi_{v,1}^{\mathrm{ord}}, \psi_{v,2}^{\mathrm{ord}} : G_{F_v} \to (R^{\mathrm{ps,ord}})^{\times}$ which are liftings of $\mathbf{1}, \bar{\chi} \mid_{G_{F_v}}$ respectively.

Define the *Iwasawa algebra*

$$\Lambda_F := \widehat{\bigotimes}_{v \mid p} \mathcal{O}[[\mathcal{O}_{F_v}^{\times}(p)]],$$

and we get a natural map $\Lambda_F \to R^{\mathrm{ps,ord}}$ induced by the characters $\psi_{v,1}^{\mathrm{ord}}$.

Theorem 2.3.1. *[4, Theorem 5.1.1] (1) $R^{\mathrm{ps,ord}}$ is a finite Λ_F-algebra.*

(2) For any maximal ideal $\mathfrak{p}$ of $R^{\mathrm{ps,ord}}[\frac{1}{p}]$, we denote the associated semi-simple representation $\mathrm{Gal}(F_{\Sigma}/F) \to \mathrm{GL}_2(k(\mathfrak{p}))$ by $\rho(\mathfrak{p})$. Assume that

i) $\rho(\mathfrak{p})$ is irreducible.

ii) For any $v \mid p$, $\rho(\mathfrak{p}) \mid_{G_{F_v}} \cong \begin{pmatrix} \chi_{v,1} & * \\ 0 & \chi_{v,2} \end{pmatrix}$ such that $\chi_{v,1}$ is de Rham and has strictly less Hodge-Tate number than $\chi_{v,2}$ for any embedding $F_v \hookrightarrow \overline{\mathbb{Q}}_p$.

Then $\rho(\mathfrak{p})$ comes from a twist of Hilbert modular form.

The following result is a direct consequence of the previous theorem.

Corollary 2.3.2. *Let C^{ord} be an irreducible component of $R^{\mathrm{ps,ord}}$ of dimension at least $1 + [F : \mathbb{Q}]$. Let $C^{\mathrm{ord,aut}}$ be the set of regular de Rham primes in C^{ord}. Then $C^{\mathrm{ord,aut}}$ is dense in C^{ord}. Here a prime $\mathfrak{q}$ is called regular de Rham prime if*

i) $p \notin \mathfrak{q}$ and $R^{\mathrm{ps,ord}}/\mathfrak{q}$ is one-dimensional.

ii) The semi-simple representation $\rho(\mathfrak{q}) \otimes k(\mathfrak{q}) : \mathrm{Gal}(F_{\Sigma}/F) \to \mathrm{GL}_2(k(\mathfrak{q}))$ (in the sense of Definition 2.1.1) is irreducible.

iii) For any $v \mid p$, $\rho(\mathfrak{q}) \mid_{G_{F_v}} \cong \begin{pmatrix} \chi_{v,1} & * \\ 0 & \chi_{v,2} \end{pmatrix}$ such that $\chi_{v,1}$ is de Rham and has strictly less Hodge-Tate number than $\chi_{v,2}$ for any embedding $F_v \hookrightarrow \overline{\mathbb{Q}}_p$.

3 Main steps of the proof

In this section, we sketch the proof of the main theorem. For simplicity, we only discuss the generic case, i.e., $\bar{\chi} \mid_{G_{F_v}} \neq \mathbf{1}, \omega_p^{\pm 1}$ for any $v \mid p$.

Proposition 3.0.1. Let $\mathfrak{r}$ be a one-dimensional irreducible pro-modular prime of $R^{\mathrm{ps}}_{\mathrm{aux}}$. Then every irreducible component of $R^{\mathrm{ps}}_{\mathrm{aux}}$ containing $\mathfrak{r}$ contains a nice prime, and hence is pro-modular.

Sketch of the proof. Assume that there exists an irreducible component of $R^{\mathrm{ps}}_{\mathrm{aux}}$ containing $\mathfrak{r}$ is pro-modular.

1) Consider the partition $Z_1 \amalg Z_2$ of the set of irreducible components of $(R^{\mathrm{ps}}_{\mathrm{aux}})_{\mathfrak{r}}$ such that Z_1 consists of pro-modular ones. Then Z_1 is non-empty, and by Lemma 2.1.3, there exist $C_1 \in Z_1, C_2 \in Z_2$ and a pro-modular prime $\mathfrak{q} \in C_1 \cap C_2$ of dimension at least $2[F : \mathbb{Q}] - 1$.

2) Choose $S = \{\sigma_v \in I_v, \mathrm{Frob}_v : v \in \Sigma \setminus \Sigma_p\}$ as generators at all $v \in \Sigma \setminus \Sigma_p$. By Lemma 2.1.2, we can find a "nice" quotient R' of $R^{\mathrm{ps}}_{\mathrm{aux}}/\mathfrak{q}$ of dimension at least $[F : \mathbb{Q}] + 2$.

3) By Lemma 2.1.4, we can find an irreducible and non-dihedral one-dimensional prime $\mathfrak{q}_0$ of R'. From our construction, $\mathfrak{q}_0$ is a nice prime, which implies that C_2 is pro-modular by Theorem 2.2.3. Hence, Z_2 is empty.

Next, we show that Z_1 is non-empty. As $\mathfrak{r}$ is pro-modular, combining Theorem 2.2.1, we can find a pro-modular prime $\mathfrak{r}_0$ contained in $\mathfrak{r}$ of dimension at least $1 + 2[F : \mathbb{Q}]$. Then we can find a nice prime containing $\mathfrak{r}_0$ as above, which implies that Z_1 is non-empty. ☐

By the previous proposition, we just need to find an irreducible one-dimension pro-modular prime in each irreducible component of $R^{\mathrm{ps}}_{\mathrm{aux}}$ of dimension at least $1 + 2[F : \mathbb{Q}]$. We fix such an irreducible component C.

As $\bar{\chi} \mid_{G_{F_v}} \neq \mathbf{1}, \omega_p^{\pm 1}$ for any $v \mid p$, we can show that the kernel of the natural surjection $R^{\mathrm{ps}}_v \twoheadrightarrow R^{\mathrm{ps,ord}}_v$ is principal. Here R^{ps}_v and $R^{\mathrm{ps,ord}}_v$) are the universal pseudo-deformation ring and its ordinary locus of $1 + \bar{\chi}|_{G_{F_v}}$. Then by Krull's principal ideal theorem, we have $\dim C^{\mathrm{ord}} \geq \dim C - |\Sigma_p| \geq [F : \mathbb{Q}] + 1$. Here C^{ord} is the ordinary locus of C. Then by Corollary 2.3.2, we can find an irreducible one-dimensional pro-modular prime in C. Hence, we get the main result.

4 An application to the non-ordinary Fontaine-Mazur conjecture

Let p be an odd prime and $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}_p})$ be a continuous, irreducible representation such that

1) ρ is almost unramified,

2) $\rho|_{D_p}$ is absolutely irreducible, potentially semi-stable with distinct Hodge-Tate weights,

3) ρ is odd,

4) its residual representation $\bar{\rho}$ is reducible.

We can show that ρ is modular (especially the case $p = 3$ and $\bar{\chi}|_{D_p} = \omega_p$). First, by Grunwald-Wang's theorem, we can find an abelian base change $F/\mathbb{Q}$ satisfying all of the assumptions of our main theorem. Then by our big $R = \mathbb{T}$ theorem, we know that the prime corresponding to the representation $\rho|_{G_F}$ is pro-modular. Hence, by the local-global compatibility results at p, $\rho|_{G_F}$ is modular, which implies ρ is modular by soluble base change. See [4] and [5] for more details.

Combining the previous work on the Fontaine-Mazur conjecture, the following result has been given in [5].

Theorem 4.0.1. *Let p be an odd prime number and*

$$\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}_p})$$

be a continuous, irreducible representation such that

- *ρ is only ramified at finitely many places,*
- *$\rho|_{G_{\mathbb{Q}_p}}$ is de Rham of distinct Hodge-Tate weights,*
- *ρ is odd,*

Then ρ arises from a cuspidal eigenform up to twist.

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Graduate School of Mathematical Sciences
University of Tokyo
3-8-1 Komaba Meguro-ku Tokyo 153-8914
JAPAN
E-mail address: `zhangxy96@g.ecc.u-tokyo.ac.jp`