REDUCTIVE MODIFICATIONS OF LOCALLY SYMMETRIC SPACES

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Abstract. Let $X = G/A_GK$, resp. $Y = \Gamma\backslash G/A_GK$, be a symmetric space, resp. a locally symmetric space, associated to a split reductive group $G/\mathbb{Q}$, its maximal compact subgroup K and an arithmetic subgroup Γ. In this paper, motivated by Zucker's reductive Borel-Serre compactification $\overline{Y}^{\mathrm{rBS}}$ of Y, we construct new but genuine topological spaces X^{red} and Y^{red}, together with their natural compactifications $\overline{X^{\mathrm{red}}}$ and $\overline{Y^{\mathrm{red}}}$ respectively, for the purpose to uniformly understand reductive structures involved at different parabolic levels, based on Saper's tiling theory. We show that for a regular representation $\rho : G \to \mathrm{GL}(V)$, there is a natural isomorphism

$$IH^*(\overline{Y^{\mathrm{red}}}, \mathbb{V}) \simeq IH^*(\overline{Y}^{\mathrm{rBS}}, \mathbb{V}),$$

where $\mathbb{V}$ denotes the natural associated local systems on the associated spaces induced by ρ. Furthermore, when X is of equal rank, we show that, for the interiors Y_0^o of the central tile $Y_0 = \overline{Y}_G^{\mathrm{red}}$ of a tiling $\overline{Y^{\mathrm{red}}} = \bigsqcup_P \overline{Y}_P^{\mathrm{red}}$, and for the equal rank Satake compactification Y^* of Y, there are natural isomorphisms

$$H^*_{(2)}(Y_0^o, \overline{\mathbb{V}}_\rho) \simeq H^*_{(2)}(Y, \overline{\mathbb{V}}_\rho) \simeq IH^*(\overline{Y}^{\mathrm{rBS}}, \mathbb{V}) \simeq IH^*(Y^*, \mathbb{V})$$

based on Looijenga, Saper-Stone's solution to the (Borel-)Zucker conjecture and Saper's confirmation to the Rapoport/Goresky-MacPherson conjecture.

1. Borel-Serre Compactification and Reductive Borel-Serre Compactification

Let $G/\mathbb{Q}$ be a split reductive group and let K be a maximal compact subgroup of G. Denote by $X := G/A_GK$ the associated symmetric space, where A_G denotes the identity (connected) component of a maximal split torus in the center Z_G of G. And for an arithmetic subgroup Γ of G, denote by $Y = \Gamma\backslash X$ the induced locally symmetric space.

To facilitate our ensuing , we first recall some details on the Borel-Serre compactifications $\overline{X}^{\mathrm{BS}}$ and $\overline{Y}^{\mathrm{BS}}$ of X and Y, respectively, and Zucker's reductive Borel-Serre compactifications $\overline{X}^{\mathrm{rBS}}$ and $\overline{Y}^{\mathrm{rBS}}$ of X and Y, respectively.

For a parabolic ($\mathbb{Q}$-)subgroup P of G, denote its associated structural (split) exact sequence by

$$1 \to N_P \to P \to M_P \to 1,$$

where N_P denotes the unipotent radical of P and M_P the reductive Levi quotient of P, which, via the splitting (associated to the Cartan involution

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with respect to K), will also be viewed as a Levi subgroup, often denoted by $\widetilde{M}_P$, of P.

As usual, let A_P be the lift to P of the identity (connected) component of the maximal split torus Z_{M_P} of M_P. Since $G = PK$, the action of P on the symmetric space G/K is transtive. Following Borel-Serre [4], define the *geodesic action* of A_P on X via

$$a \circ (pk) := pka = pak \quad (\forall p \in P,\ a \in A_P,\ k \in K_P = P \cap K)$$

since A_P and K_P commute.

It is well known that the simple roots occurring in N_P defines an isomorphism

$$A_P \simeq (0, \infty)^{r(P)}$$

where $r(P)$ denotes the parabolic rank of P. Let $\overline{A}_P$ be the enlargement of A_P obtained by transporting of structures from the naturally embedding $(0, \infty) \subset (0, \infty]$. Set $X(P) = X \times_{A_P} \overline{A}_P$.

Let ∞_P be the zero dimensional A_P-orbit in $\overline{A}_P$ corresponding to the point $(\infty, \ldots, \infty) \in (0, \infty]^{r(P)}$. Accordingly, it maps canonically to $X/A_P \simeq e(P) \subset X(P)$. We know that $e(P)$ is homogeneous under ${}^0P := \big\{p \in P : |p^\chi| = 1\ \forall \chi \in \mathrm{Mor}_{\mathbb{Q}}(P, \mathbb{G}_m)\big\}$, isomorphic to P/A_P. Denote by

$$\pi_P : X(P) \to e(P),$$

the geodesic projections. There is a natural P-action on $e(P)$ with A_P acting trivially. In particular, $A_P \times {}^0P$ acts on X by the product of the geodesic action and the usual multiplication of ${}^0P$, from which we obtain an analytic isomorphism

$$(a_P, \pi_P) : X \xrightarrow{\simeq} A_P \times e(P)$$

of $A_P \times {}^0P$-homogeneous spaces. We normalize a_P such that $a_P(x_0) = 1$, where x_0 denote the base points of X corresponding to K. In this way, X is trivialized as a principal A_P-bundle with *canonical cross sections* $e(P)$ given by the orbits of ${}^0P$.

For parabolic subgroups P, Q of G satisfying $Q \subseteq P$, there is a canonical embedding of $X(P)$ in $X(Q)$. Note that there is a natural decomposition $A_Q = A_P \times A_{Q,P}$ where $A_{Q,P} \subset A_Q$ denotes the intersection of kernels of simple roots for A_P. Then there is an embedding

$$\begin{aligned} X(P) &= X \times_{A_P} \overline{A}_P \\ &\simeq \big(X(P) \times_{A_{Q,P}} A_{Q,P}\big) \times_{A_P} \overline{A}_P \\ &\subseteq \big(X(P) \times_{A_{Q,P}} \overline{A}_{Q,P}\big) \times_{A_P} \overline{A}_P \\ &\subseteq X \times_{A_Q} \overline{A}_Q = X(Q) \end{aligned} \tag{1}$$

Accordingly, we can view $e(Q)$ as a part of the boundary of $e(P)$. This is achieved by considering the geodesic action of $A_{Q,P}$ on $e(P)$ (A_P acts trivially), so that $e(Q) \simeq e(P)/A_{Q,P} = e(P)/A_Q$.

By definition, the (partial) Borel-Serre compactification of the symmetric space $X := G/K$ is given by

$$\overline{X}^{\mathrm{BS}} := \bigcup_P X(P) = \bigsqcup_P e(P) = X \bigsqcup \Big(\bigsqcup_{P : P \subsetneq G} e(P) \Big),$$

where P runs over Para(G), the collection of all parabolic $\mathbb{Q}$-subgroups of G. By Borel-Serre [4], for the weak topology from the $X(P)$'s, there is a manifold-with-corner structure on the Borel-Serre compactification $\overline{X}^{\mathrm{BS}}$ which naturally becomes a stratified manifold with strata $e(P)$'s.

Clearly this construction is compatible with the action by the arithmetic subgroup $\Gamma \subseteq G$. In fact the action by an element $\gamma \in \Gamma$ yields a homeomorphism of $\overline{X}^{\mathrm{BS}}$. Accordingly, passing to the quotient, we obtain the so-called Borel-Serre compactification $\overline{Y}^{\mathrm{BS}}$ of the locally symmetric space Y:

$$Y \subseteq \overline{Y}^{\mathrm{BS}} = Y \bigsqcup \Big(\bigsqcup_{P:P \subsetneq G} \Gamma_P \backslash e(P) \Big),$$

where $\Gamma_P := \Gamma \cap P$. In particular, for $\Gamma_P \subset {}^0P$, its action commutes with the geodesic action of A_P. Denote by $Y_P := \Gamma_P \backslash e(P)$ the corresponding faces of $\overline{Y}^{\mathrm{BS}}$. By §9 of [4], there is a neighborhood of Y_P in $\overline{Y}^{\mathrm{BS}}$ on which geodesic projection π_P descends. For general background, please refer to [3].

For many purposes, the Borel-Serre compactification $\overline{Y}^{\mathrm{BS}}$ of Y is not fine enough. Indeed, in this parabolic reduction, passing from the reductive group G to parabolic $\mathbb{Q}$-subgroups P yields structural discrepancies, in addition to the fact that at infinity, what really added is a stratified nilmanifold fibration. In the sequel, for our limited purpose, we introduce a much better reduction via the process of starting with a reductive G and ending with a lower rank reductive M_P, or better, the Levi subgroup $\widetilde{M}_P$ of P. To understand this, let us first recall some details on Zucker's reductive Borel-Serre compactification $\overline{X}^{\mathrm{rBS}}$ and $\overline{Y}^{\mathrm{rBS}}$ of X and Y, respectively.

Recall that, for $P \in$ Para(G), its unipotent radical N_P acts naturally on $X(P) \simeq e(P) \times \overline{A}_P$ via multiplication $u \cdot (p, a) := (up, a)$ and that this action commutes with the action of $K_P \cdot A_P$ since $N_P \cap (K_P \cdot A_P) = \{1\}$. Consequently, there is a canonical projection $X(P) \to N_P \backslash X(P)$. Set now $e'(P) := N_P \backslash e(P)$, and form the space

$$\overline{X}^{\mathrm{rBS}} := \bigsqcup_P e'(P) = X \bigsqcup \Big(\bigsqcup_{P:P \subsetneq G} e'(P) \Big).$$

Since $N_Q \supseteq N_P$ whenever $Q \subseteq P$, there is a globally defined quotient map $\pi : \overline{X}^{\mathrm{BS}} \longrightarrow \overline{X}^{\mathrm{rBS}}$. Clearly there is an induced action of Γ on $\overline{X}^{\mathrm{rBS}}$ which is compatible with the stratifications since Γ takes the stratum X_P onto that of a conjugate parabolic subgroup, with P preserves $e'(P)$. Consequently, we get a quotient map

$$\pi : \overline{Y}^{\mathrm{BS}} \to \overline{Y}^{\mathrm{rBS}} := \Gamma \subseteq G \backslash \overline{X}^{\mathrm{rBS}} = \bigsqcup_P \widehat{Y}_P = Y_G \bigsqcup \Big(\bigsqcup_{P:P \subsetneq G} \widehat{Y}_P \Big),$$

where $\widehat{Y}_P := \Gamma_P \backslash e'(P)$. This $\overline{Y}^{\mathrm{rBS}}$ is Zucker's *reductive Borel-Serre compactification* of the locally symmetric space Y ([20]). In particular, π is continuous with respect to the induced quotient topology, and hence, similar to $\overline{Y}^{\mathrm{BS}}$, $\overline{Y}^{\mathrm{rBS}}$ becomes Hausdorff. Clearly, if we denote by $\overline{X}^{\mathrm{rBS}}(P)$ the image of $X(P)$ in $\overline{X}^{\mathrm{rBS}}$, then we get an open neighborhood $Y(P) := \Gamma_P \backslash \overline{X}^{\mathrm{rBS}}(P)$ of $e'(P) = N_P \backslash e(P)$. For recent developments on this reductive compactifications, please refer to [6] and [8].

2. Saper's Tiling Theory

To construct our reductive modifications X^{red} and Y^{red} of X and Y, respectively, we need to make modifications within X with respect to parabolic subgroups P, by cutting the P-part off first, then gluing back the $\widetilde{M}_P$-part. To make this operation as canonical as possible, we use Saper's tiling theory [12], a geometric realization of Arthur's analytic truncation, based on Langlands's cone decompositions for root spaces.

Following Saper ([12]), a *tiling* of $\overline{X}^{\mathrm{BS}}$ is a cover $\overline{X}^{\mathrm{BS}} = \bigsqcup_P \overline{X}_P$ by disjoint sets (called *tiles*), having the following properties

(i) The *central tile* $X_0 := \overline{X}_G$ is a closed, codimension 0 submanifold with corners contained in X.
(ii) The closed *boundary faces* $\{\partial^P X_0\}_P$ of X_0 are indexed by $P \in \mathrm{Para(G)}$ so that $P \mapsto \partial^P X_0$ is an incusion preserving bijection.
(iii) Each boundary face $\partial^P X_0$ lies in a canonical cross-section $\{b_P\} \times e(P)$.
(iv) The P-*tile* $\overline{X}_P$ $(P \neq G)$ is obtained from $\partial^P X_0$ by flowing out under the geodesic action of the strictly dominant cone $A_P(1) = \exp\big(\mathfrak{a}_P^+\big)$, that is $\overline{X}_P = \overline{A}_P(b_P) \circ \partial^P X_0$, where $\mathfrak{a}_P^+$ denotes the positive Weyl chamber in $\mathfrak{a}_P$ and $\overline{A}_P(b_P) := b_P \cdot \overline{A}_P(1)$.

Furthermore, the tiling is called Γ-*invariant* if $\gamma \cdot \overline{X}_P = \overline{X}_{\,^\gamma P}$ for all $\gamma \in \Gamma$ and $P \in \mathrm{Para(G)}$, where $\,^\gamma P := \gamma P \gamma^{-1}$. Directly from the definition, it is not difficult to conclude that for a fixed tiling $\overline{X}^{\mathrm{BS}} = \bigsqcup_P \overline{X}_P$ we have

(a) Each P-tile $\overline{X}_P$ is a codimension 0 submanifold with corners.
(b) The closure of any two tiles are either disjoint or intersect in a common closed boundary face. That is, for P, $P' \in \mathrm{Para(G)}$,

$$\mathrm{cl}(\overline{\mathrm{X}}_{\mathrm{P}}) \cap \mathrm{cl}(\overline{\mathrm{X}}_{\mathrm{P'}}) = \begin{cases} \mathrm{cl}(\overline{\mathrm{A}}_{(\mathrm{P}\vee\mathrm{P'})}(\mathrm{b}_{\mathrm{P}\vee\mathrm{P'}}) \circ \partial^{\mathrm{P}\cap\mathrm{P'}}\mathrm{X}_0, & P \cap P' \in \mathrm{Para(G)} \\ \varnothing, & \text{else} \end{cases}$$

Here $P \vee P'$ denotes the smallest parabolic subgroup of G containing $P \cup P'$.

Furthermore, if the tiling is Γ-invariant, then

(c) The central tile $\Gamma \backslash X_0$ is compact.
(d) For all $P \in \mathrm{Para(G)}$, the natural projection $\Gamma_P \backslash \mathrm{cl}(\overline{\mathrm{X}}_{\mathrm{P}}) = \overline{\mathrm{A}}_{\mathrm{P}}(\mathrm{b}_{\mathrm{P}}) \circ \Gamma_{\mathrm{P}} \backslash \mathrm{X}_0 \to \pi\big(\mathrm{cl}(\overline{\mathrm{X}}_{\mathrm{P}})\big)$ is a homeomorphism.

Note that for any $P \in \mathrm{Para(G)}$, the associated b_P in the corresponding canonical cross-section $\{b_P\} \times e(P)$ is determined as the intersection of the canonical cross-section $\{b_Q\} \times e(Q)$ for $Q \supseteq P$. Therefore among all b_P's, only $\{b_Q : Q \text{ maximal}\}$ are essential. For this reason, we call $\mathbf{b} = \{b_Q\}_{Q \in \mathrm{Max(G)}}$ the *parameter* of the tiling. In fact, a tiling is uniquely determined by its parameter $\mathbf{b}$ as we can apply the following discussion to $\overline{X}^{\mathrm{BS}}$: For any open subset $M \subset \overline{X}^{\mathrm{BS}}$, there exists at most one decomposition $M = \bigsqcup_R \overline{M}_R$ for which $\overline{M}_R \subseteq X(R)$ is $\mathrm{cl}(\overline{\mathrm{A}}_{\mathrm{R}}(1))$-invariant and satisfies

$$\overline{M}_R = \Big(\overline{A}_R(b_R) \times \pi_R(\overline{M}_R)\Big) \cap M.$$

In fact if $P \subseteq R$, $\overline{M}_P \cap \overline{e(R)} = \pi_R(\overline{M}_P)$ and hence

$$M \cap \overline{e(R)} = \bigsqcup_P \overline{M}_P \cap \overline{e(R)} = \bigsqcup_{P \subseteq R} \pi_R(\overline{M}_P).$$

Therefore

$$\pi_R(\overline{M}_R) = \Big(M \cap \overline{e(R)}\Big) \smallsetminus \bigcup_{P \subsetneq R} \pi_R(\overline{M}_P).$$

Therefore $\overline{M}_R$ is uniquely determined if R is minimal, and by recursion on parabolic rank in general.

In other words, from stratified manifold point of view, we may reconstruct a tiling by first determining the most deepest tile for minimal parabolic subgroup P_0 by setting

$$\pi_{P_0}(\overline{X}_{P_0}) = \overline{e(P_0)},$$

then use recursion on parabolic ranks to recover outer tiles $\overline{X}_R$ via

$$\pi_R(\overline{X}_R) = \overline{e(R)} \smallsetminus \bigcup_{P \subsetneq R} \pi_R(\overline{X}_P).$$

In particular, the central tile is given by

$$X_0 = \overline{X}_G = \mathrm{cl}\Big(\mathrm{X} \smallsetminus \bigcup_{\mathrm{P}:\mathrm{P} \subsetneqq \mathrm{G}} \pi_{\mathrm{G}}(\overline{\mathrm{X}}_{\mathrm{P}})\Big).$$

Saper ([12]) divides his tiling constructions in three steps. Namely, the first on the root space $\mathfrak{a}_P$, and hence on

$$\overline{A}_P := \bigsqcup_{R:R \supseteq P} \langle \overline{A}_P \rangle_R \qquad \text{with} \quad \langle \overline{A}_P \rangle_R := \exp\big(\langle \overline{\mathfrak{a}}_P \rangle_R\big)$$

by using the famous Langlands's cone decomposition for the root space $\mathfrak{a}_P$ in terms of parabolic subgroups R ($P \subseteq R$) of G:

$$\mathfrak{a}_P = \bigsqcup_{R:R \supseteq P} \langle \mathfrak{a}_P \rangle_R,$$

where

$$\langle \mathfrak{a}_P \rangle_R := \Big\{ v \in \mathfrak{a}_P : \gamma_R(v) > 0 \ \forall \gamma_R \in \Delta_R \ \& \ \varpi_\alpha^P(v) \leqslant 0 \ \forall \varpi_\alpha^R \in \widehat{\Delta}_P^R \Big\};$$

the second on $X(P) = \overline{A}_P \times e(P)$ by using the first step to get

$$X(P) := \bigsqcup_{R:R \supseteq P} X(P)_R \qquad \text{with} \quad X(P)_R := b_P \cdot \langle \overline{A}_P \rangle_R \times e(P).$$

In particular, with $X(P)_0 = X(P)_G$,

$$\partial^R X(P)_0 := b_P \cdot \partial^R \langle A_P \rangle_0 \times e(P) \subseteq \partial^R X(R)_0 = \{b_R\} \times e(R)$$

then finally on X, by setting

$$\overline{X}_R := \bigcap_{P:P \subseteq R} X(P)_R \qquad \text{and} \qquad \partial^R X_0 := \bigcap_{P:P \subseteq R} \partial^R X(P)_0.$$

Ideally, this three-step construction would offer us a tiling of $\overline{X}^{\mathrm{BS}}$. However, in Saper's discussion, say in Theorem 5.7 of [12], there is a very important technical condition: The parameter $\mathbf{b}$ involved should be *sufficiently regular* in the sense of Arthur ([1]). Indeed, whether the $\overline{X}_R$'s above cover $\overline{X}^{\mathrm{BS}}$ depends heavily on the classical reduction theory, for which to work, the associated Siegel sets should be chosen to be sufficiently large, say approaching to the P-cusps for each P. This is quite similar to the discussion in trace formula using Arthur's analytic truncation Λ^T, where T is always assumed to be sufficiently regular.

However, there is an alternative modern treatment to the classical reduction theory using stability. This new stability approach has proven to be very powerful, and indeed significantly simplifies the matters involved: After all to get a reduction theory, as what was done in Lafforgue [9], there is no need to require the parameter T to be sufficiently large: $T = 0$ is already sufficient! This Lafforgue's work for $G = \mathrm{SL}_n$, i.e., for vector bundles on curves over function fields, was generalized to number fields first for $\mathcal{O}_F$-lattices by myself in [16]. Later, I was able to treat general split reductive groups G/F to obtain Theorem 16.3 in [17]. Back to the point whether the $\overline{X}_R$'s cover $\overline{X}^{\mathrm{BS}}$, it suffices to show that Arthur's truncation $\Lambda^{T_0}\mathbf{1}$ for the constant function $\mathbf{1}$ on Y yields a compact subset Σ_{T_0} for a fixed $T_0 \in \overline{\mathfrak{a}}_0^+$, the positive Weyl chamber in $\mathfrak{a}_0$. Once this is achieved, then we can form a tiling of Saper, by letting Σ_{T_0} to be the central tile X_0: Indeed, despite of the fact that in Saper's paper [12], Arthur's truncation was not used, but in essence, Saper's tiling theory is a geometric companion of Arthur's truncation theory. All tiling constructions of Saper can be carried out using Arthur's truncation theory: This is certainly the case for the A_P level tiling in the first step – Saper's construction is simply equivalent to (Arthur-)Langlands cone decomposition of $\mathfrak{a}_P$. (See e.g. Osborne–Warner [11].) On the other hand, Arthur's Λ^T is a well-designed device taking care of each P-level carefully so as to offer a uniform treatment for all P's. Viewing from this, it is only natural to assume the sufficiently regularity for the parameter $\mathbf{b}$ in Saper's titling theory – After all, in Arthur's truncation theory, the parameters T are always assumed to be sufficiently regular, to make sure that $\Lambda^T\mathbf{1}$ becomes a compact subset in X after carefully removing from X the P-contributions for all P's, with the help from the classical reduction theory. Once this central tile is obtained, the P-direction extension using geodesic action becomes rather direct following Saper.

We will leave our further discussions in this direction to some other occasions (see e.g. [18]). Instead, for our limited purpose here, let us next recall the following qualitative main theorem of [12], besides the existence result mentioned above.

Theorem 1 (Theorem 6.1 of [12])**.** *For Γ-invariant parameter $\mathbf{b}$ and $t \in \mathrm{cl}(\mathrm{A_G}(1))$, let $\{\overline{X}_{P,t}\}$ be a tiling of $\overline{X}$ with parameter $t \cdot \mathbf{b}$.*

(1) *For all $t \in \mathrm{cl}(\mathrm{A_G}(1))$, there exists a unique Γ-equivariant piecewise-analytic retraction $r_t : \overline{X}^{\mathrm{BS}} \to X_{0,t}$ satisfying $r_t(\overline{A}_P(1) \circ y) = y$ for $y \in \partial^P X_{0,t}$ and $P \in \mathrm{Para(G)}$.*

(2) *For all $t \in \mathrm{cl}(\mathrm{A_G}(1))$, there exists a unique Γ-equivariant piecewise-analytic diffeomorphism $s_t : \overline{X}^{\mathrm{BS}} \to X_{0,t}$ such that for all $P \in \mathrm{Para(G)}$*

   (i) *s_t preserves the $\overline{A}_P(1)$-orbits in $\overline{X}_{P,1}$.*

   (ii) *The family of diffeomorphisms induced on the $\overline{A}_P(1)$-orbits in $\overline{X}_{P,1}$ is constant with respect to the canonical cross-sections.*

(iii) *In terms of the coordinates* $a \mapsto (a^{-\alpha})_{\alpha\in\Delta_P}$*, each coordinate function of the diffeomprphism induced on* $\overline{A}_P(1)$ *is the exponential of a polynomial having degree at most 1 in each variable.*

Both r_t *and* s_t *depend piecewise-analytically on* t*. As* t *tends to infinity under the action of a strictly dominant 1-parameter subgroup,* r_t *and* s_t *converge to the identity; and as* t *tends to 1,* s_t *converges to* r_1*.*

3. Reductive Modifications X^{red} and Y^{red} and their Compactifications $\overline{X^{\mathrm{red}}}$ and $\overline{Y^{\mathrm{red}}}$

For Zucker's reductive Borel-Serre compactification $\overline{X}^{\mathrm{rBS}}$, working over various boundaries, the P-level reductive structure ${}^0M_P/K_{M_P} \subset N_P\backslash e(P)$ can be glued with the R-level reductive structure $M_R/A_{M_R}K_{M_R} \subset N_R\backslash e(R)$ even when $P \subsetneq R$. This is because the fact that, by the construction, the P-boundary $e(P) \simeq e(P) \times (\{\infty\}_{r_P})$ is disjoint from the P-boundary $e(R) \simeq e(R) \times (\{\infty\}_{r_R})$. However, within the symmetric space X, since $e(R)$ and $e(P)$ are well organized, a simple gluing does not work. To solve this, we examine the structure on Saper's tiling associated to the central tile X_0 first, and then make crucial reductive modifications on each P-tiles when flowing out under the geodesic action of the cone $\overline{A}_P(1)$ for the canonical cross-sections.

Recall that the P-boundary $\partial^P X_0$ of X_0 is contained in $\partial^P X(P)_0 = \{b_P\} \times e(P)$, and inside X

$$\bigcap_{P:P\subseteq R} \partial^R X(P)_0 = \partial^R X_0 \qquad \text{with} \quad \partial^R X(P)_0 = b_P \cdot \partial^R \langle A_P\rangle_0 \times e(P).$$

As remarked after (1), we view $e(P)$ as a part of the well-organized part of $e(R)$ when $P \subseteq R$. This is achieved by considering the geodesic action of $A_{P,R}$ on $e(R)$ (on which A_R acts trivially), so that $e(P) \simeq e(R)/A_{P,R} = e(R)/A_P$. Accordingly there is a natural stratified structure when taking the N_P-quotient: Since $N_P \supseteq N_R$, there is a natural morphism

$$N_P\backslash e(P) \to N_R\backslash e(R)$$

induced by the projection $e(P) \to e(R)$. We may identify $N_P\backslash e(P)$ with $A_{P,R} \times \Big(N_R\backslash e(R)\Big)$. Thus for $x \in e(R)$, the fiber over $N_R\backslash x \in N_R\backslash e(R)$ in $N_P\backslash e(P)$ includes the $A_{P,R}$-directions, ensuring smooth transition. In other words, if we set

$$\partial^R X(P)_0^{\mathrm{red}} := b_P \cdot \partial^R \langle A_P\rangle_0 \times N_P\backslash e(P),$$

then $\partial^P X(P)_0^{\mathrm{red}} = \{b_P\} \times N_P\backslash e_P$. And for $P \subseteq R$, we may and hence will view $\partial^P X(P)_0^{\mathrm{red}}$ and $\partial^R X(R)_0^{\mathrm{red}}$ as the subspace contained in different strata. Then it makes sense to talk about

$$\partial^R X_0^{\mathrm{red}} := \bigcap_{P:P\subseteq R} \partial^R X(P)_0^{\mathrm{red}},$$

and hence introduce a stratified manifold X_0^{red} by

$$X_0^{\mathrm{red}} := X_0^o \bigsqcup \bigcup_{P\neq G} \partial^P X_0^{\mathrm{red}} \qquad \text{with} \quad X_0^o := X_0 \smallsetminus \bigcup_{P:P\neq G} \partial^P X_0.$$

Accordingly, for each $P \in \mathrm{Para(G)}$, we modify the corresponding P-tile $\overline{X}_P = \overline{A}_P(b_p) \times e(P)$ to $X_P^{\mathrm{red}} := A_P(b_p) \times N_P\backslash e(P)$ and its partial reductive Borel-Serre compactification $\overline{X}_P^{\mathrm{red}} := \overline{A}_P(b_p) \times N_P\backslash e(P)$. And finally, from a new stratified topological space X^{red} by setting

$$X^{\mathrm{red}} := X_0^o \bigsqcup \bigsqcup_{P \neq G} X_P^{\mathrm{red}} \qquad \text{and} \qquad \overline{X^{\mathrm{red}}} := X_0^o \bigsqcup \bigsqcup_{P \neq G} \overline{X}_P^{\mathrm{red}},$$

we obtain a commutative diagrams with horizontal maps being natural stratified quotients

$$\begin{array}{ccc} X = X_0^o \bigsqcup \bigsqcup_{P \neq G} X_P & \longrightarrow & X_0^o \bigsqcup \bigsqcup_{P \neq G} X_P^{\mathrm{red}} = X^{\mathrm{red}}, \\ \big\downarrow & & \big\downarrow \\ \overline{X}^{\mathrm{BS}} = X_0 \bigsqcup \bigsqcup_{P \neq G} \overline{X}_P & \longrightarrow & X_0^{\mathrm{red}} \bigsqcup \bigsqcup_{P \neq G} \overline{X}_P^{\mathrm{red}} = \overline{X^{\mathrm{red}}}, \end{array}$$

for which the bottom morphism factorizes through $\overline{X}^{\mathrm{BS}} \to \overline{X}^{\mathrm{rBS}}$. In particular, X^{red} and $\overline{X^{\mathrm{red}}}$ are Hausdorff.

Theorem 2. *Let $\{\overline{X}_P\}_{P \in \mathrm{Para(G)}}$ be a Γ-invariant tiling of Saper associated to parameter $\mathbf{b}$. Then, with the same notation as above,*

(1) *$\{X_0^{\mathrm{red}}\} \cup \{\overline{X}_P^{\mathrm{red}}\}_P$ form a tiling of the stratified space $\overline{X^{\mathrm{red}}}$. That is,*
 (i) *The central tile $X_0^{\mathrm{red}} = \overline{X}_G^{\mathrm{red}}$ is a closed, codimension 0 subspace of the stratified topological space $\overline{X^{\mathrm{red}}}$.*
 (ii) *The closed boundary faces $\{\partial^P X_0^{\mathrm{red}}\}_P$ of X_0^{red} are indexed by $P \in \mathrm{Para(G)}$ so that $P \mapsto \partial^P X_0^{\mathrm{red}}$ is an inclusion preserving bijection.*
 (iii) *Each boundary face $\partial^P X_0^{\mathrm{red}}$ lies in a canonical reductive cross-section $\{b_P\} \times N_P\backslash e(P)$.*
 (iv) *The P-tile $\overline{X}_P^{\mathrm{red}}$ $(P \neq G)$ is obtained from $\partial^P X_0^{\mathrm{red}}$ by flowing out under the geodesic action of the strictly dominant cone $\overline{A}_P(1) = \exp\left(\overline{\mathfrak{a}}_P^+\right)$, that is $\overline{X}_P^{\mathrm{red}} = \overline{A}_P(b_P) \circ \partial^P X_0^{\mathrm{red}}$.*

(2) *For Γ-invariant parameter $\mathbf{b}$ and $t \in \mathrm{cl}(\mathrm{A_G}(1))$, let $\{\overline{X}_{P,t}\}$ be a tiling of $\overline{X}$ with parameter $t \cdot \mathbf{b}$.*
 (a) *For all $t \in \mathrm{cl}(\mathrm{A_G}(1))$, there exists a unique Γ-equivariant piecewise-analytic retraction*

$$r_t^{\mathrm{red}} : \overline{X}^{\mathrm{rBS}} = X \bigsqcup \Big(\bigsqcup_{P: P \neq G} \overline{X}_P \Big) \to X_{0,t}^o \bigsqcup \Big(\bigsqcup_{P: P \neq G} \partial^P X_{0,t}^{\mathrm{red}} \Big) = X_{0,t}^{\mathrm{red}}$$

 satisfying $r_t^{\mathrm{red}}|_X = r_t|_X$, and

$$r_t^{\mathrm{red}}(A_P(1) \circ N_P y) = N_P \backslash N_P y \quad (\forall y \in \partial^P X_{0,t}^{\mathrm{red}},\ P \in \mathrm{Para(G)}).$$

 (b) *For all $t \in \mathrm{cl}(\mathrm{A_G}(1))$, there exists a unique Γ-equivariant piecewise-analytic diffeomorphism $s_t : \overline{X}^{\mathrm{rBS}} \to X_{0,t}^{\mathrm{red}}$ such that for all $P \in \mathrm{Para(G)}$*
  (i) *s_t preserves the $\overline{A}_P(1)$-orbits in $\overline{X}_{P,1}^{\mathrm{red}}$.*
  (ii) *The family of diffeomorphisms induced on the $\overline{A}_P(1)$-orbits in $\overline{X}_{P,1}^{\mathrm{red}}$ is constant with respect to the canonical reductive cross-sections.*

(iii) *In terms of the coordinates* $a \mapsto (a^{-\alpha})_{\alpha \in De_P}$*, each coordinate function of the diffeomprphism induced on* $\overline{A}_P(1)$ *is the exponential of a polynomial having degree at most 1 in each variable.*

Both r_t *and* s_t *depend piecewise-analytically on* t*. As* t *tends to infinity under the action of a strictly dominant 1-parameter subgroup,* r_t *and* s_t *converge to the identity; as* t *tends to 1,* s_t *converges to* r_1*.*

(3) *Let* $\rho : G \to \mathrm{GL}(V)$ *be a regular representation fo* G*. Denote by* $\mathbb{V}$ *the induced local system on* $\overline{X^{\mathrm{red}}}$*. Then*

$$IH^*(\overline{Y^{\mathrm{red}}}, \mathbb{V}) \simeq IH^*(\overline{Y}^{\mathrm{rBS}}, \mathbb{V}).$$

(4) *If* Y *is of equal rank, then*

$$IH^*(\overline{Y^{\mathrm{red}}}, \mathbb{V}) \simeq IH^*(Y^*, \mathbb{V}) \simeq H^*_{(2)}(Y_0^o, \overline{\mathbb{V}}).$$

Here, Y^* *denotes an equal rank Satake compactification of* Y *and we have use the diffeomorphism between* Y_0^o *and* Y *in (2) to pull back the metrics of* $(\overline{\mathbb{V}}, Y)$ *to* $(\overline{\mathbb{V}}, Y_0^o)$*.*

Proof. (Sketched) By our construction, the same proof for (1) and (2) in Saper [12] works here as well, since Saper's arguments are concentrated in the A_P's, for which our constructions remain the same–the only part we altered is about the nilpotent radical, which has nothing to do with the torus parts A_P's, being nilpotent versus torous. Hence it suffices to treat (3) and (4). We first deal with (3). Recall that the intersection cohomology is determined by the perverse (intersection cohomology) sheaf IC^*, which depends on the stratification and codimensions of singular strata. To show that $\overline{Y^{\mathrm{red}}}$ and $\overline{Y}^{\mathrm{rBS}}$ have the same intersection cohomologies, we first note that the only difference between $\overline{Y}^{\mathrm{rBS}}$ and $\overline{Y^{\mathrm{red}}}$ comes from P-level nilmanifold fibration $e(P) \longrightarrow N_P\backslash e(P)$ or better $\Gamma_P\backslash e(P) \longrightarrow \Gamma_P\backslash M_P/K_P$ with nilmanifold fibers $\Gamma_{N_P}\backslash N_P$. But it is well known that such a nilmanifold fibration collapses interior topology appropriately to the boundaries, since nilmanifolds have cohomology concentrated in degree up to their dimension. In other words, relative to codimension $\dim N_P + \dim A_P$ of the P-stratum, the nilmanifold's contribution is trivial in high degrees, not affecting the perverse sheaf IC^*. Therefore, this modification from $e(P)$ to $N_P\backslash e(P)$ in $\overline{X^{\mathrm{red}}}$ and $\overline{X}^{\mathrm{rBS}}$, or better in $\overline{Y^{\mathrm{red}}}$ and $\overline{Y}^{\mathrm{rBS}}$, does not alter the perverse sheaves IC^*. This essentially gives (3).

In a more concrete term, associated to the nilmanifold fibration $e(P) \to N_P\backslash e(P)$, the fibration is rooted from $e(P) \simeq X/A_P \simeq N_P \times M_P/K_P$ so that the map $e(P) \to N_P\backslash e(P)$ coincides with the quotient $N_P \times M_P/K_P \to M_P/K_P$ by the action of N_P. But N_P is a unipotent group and hence admits a filtration whose graded pieces are isomorphic to $\mathbb{R}$, accordingly $\Gamma_{N_P} := \Gamma_P \cap N_P$ may be viewed a successive quotient lattice in $\mathbb{R}$, so the fiber over points in $\Gamma_P\backslash(M_P/K_P)$ is a nilmanifold, i.e. a quotient $\Gamma_{N_P}\backslash N_P$.

Now on $\overline{Y}^{\mathrm{rBS}}$, the P-tile in Y is given by $X(P) \cap Y = A_P \times e(P) \simeq A_P \times (N_P \times M_P/K_P)$, while in the reductive Borel-Serre compactification $\overline{Y}^{\mathrm{rBS}}$ itself, the first A_P-direction is compactified to $\overline{A}_P$ via $(0,\infty)^{r_P} \hookrightarrow (0,\infty]^{r_P}$ and the N_P-direction is quotient out to form $\Gamma_P\backslash(M_P/K_P)$. Clearly, the map $e(P) \to N_P\backslash e(P)$ reflects the collapse of the N_P (nilmanifold) to a point in the P-boundary stratum.

Turning to $\overline{Y^{\mathrm{red}}}$, the P-tile is given by $\overline{A}_P \times \Gamma_P\backslash(M_P/K_P)$ for which the N_P-factor is absent in the interior, so that Y^{red} is constructed to approach $\Gamma_P\backslash M_P/K_P$ directly. That is, the compactification only involves $A_P \to \overline{A}_P$, while mapping $\Gamma_P\backslash M_P/K_P$ identically to itself.

What is the impact of our construction to the intersection cohomology: The key point is the nilmanifold fibration $e(P) \to N_P\backslash e(P)$ or better $\Gamma_{N_P}\backslash N_P \times \Gamma_P\backslash M_P/K_P \to \Gamma_P\backslash M_P/K_P$ in $\overline{Y}^{\mathrm{rBS}}$ describes the topology of the interior P-tile approaching the boundary. But the nilmanifodls are homologically trivial in high dimensions relative to their codimension. So the fibration $N_P \to e(P) \to N_P\backslash e(P)$ implies that the N_P-directions contribute no additional singularities to the P-boundary stratum $\Gamma_P\backslash M_P/K_P$. Consequently, in the perverse sheaf, the cohomology of the nilmanifold fiber does not alter the truncation conditions, which are set by the codimension of $\Gamma_P\backslash(M_P/K_P)$. Moving to $\overline{Y^{\mathrm{red}}}$, the absence of the N_P-factor simplifies the interior: Within Y^{red} the P-tile is given by $A_P(b_P) \times \Gamma_P\backslash(M_P/K_P)$ lacking the nilmanifodl fiber, but the boundary stratum is the same as in $\overline{Y}^{\mathrm{rBS}}$, while the compactification $\overline{A}_P$ ensures the same topological collapse to $\Gamma_P\backslash(M_P/K_P)$.

Clearly, the intersection cohomology depends on the stratification at the boundary, not the interior topology, as long as the interior is homologically compatible with the boundary. Thus the nilmanifold fibration in $\overline{Y}^{\mathrm{rBS}}$ ensures that the N_P-dierctions do not introduce additional perversity constraints beyond of $\Gamma_P\backslash(M_P/K_P)$, and similarly, in $\overline{Y^{\mathrm{red}}}$, the direct approach to M_P/K_P skips the N_P-fibration but lands on the same boundary. This then clearly preserves the perverse condition and hence the perverse sheaf. That is to say, the nilmanifold fibration $e(P) \to N_P\backslash e(P)$ in $\overline{Y}^{\mathrm{rBS}}$ ensures that the interior P-tile's topology (with N_P) collapses homologically trivially to the boundary, matching the way how $\overline{Y^{\mathrm{red}}}$ proceed under a much simpler interior structure.

To complete our proof of (3), let us finally check that in both spaces, the P-stratum codimensions are given by $\dim N_P + \dim A_P$ even with different interiors. Indeed, both $\overline{Y}^{\mathrm{rBS}}$ and $\overline{Y^{\mathrm{red}}}$ admit the same interior X_0^o for the central tile X_0 and with the same P-boundary $\Gamma_P\backslash M_P/K_P$. So the perverse sheaf truncations are identical

$$\dim(\mathrm{supp}(\xi) \cap \overline{\Gamma_P\backslash(M_P/K_P)} \leqslant i - \left[\frac{\dim N_P + \dim A_P}{2}\right]$$

Consequently, we obtain an identification on perverse sheaves

$$IC(\overline{Y}^{\mathrm{rBS}}) \simeq IC(\overline{Y^{\mathrm{red}}}),$$

since the nilmanifold fibration in $\overline{Y}^{\mathrm{rBS}}$ ensures that the N_P-factor does not alter the perverse conditions, to give the same structure using $\overline{Y^{\mathrm{red}}}$ with the much simpler interior. This proves (3).

Finally let us verify (4). But this is a direct consequence of (3) and the well-known conjecture of Rapoport and Goresky-MacPherson, verified by Saper in [13], and the famous conjecture of Zucker for Hermitian type X, proved by Looijenga [10] and Saper-Stein [15] independently, and generalized by Borel, based on Borel-Casselman [2] for equal rank X, proved by Saper-Stein [15]. Indeed, by (3) we have $IH^*(\overline{Y^{\mathrm{red}}},\mathbb{V}) \simeq IH^*(\overline{Y}^{\mathrm{rBS}},\mathbb{V})$. So it suffices to verify

$$IH^*(\overline{Y}^{\mathrm{rBS}},\mathbb{V}) \simeq IH^*(Y^*,\mathbb{V}) \simeq H^*_{(2)}(Y_0^o,\overline{\mathbb{V}})$$

under the condition of equal rank. As explained above, this becomes rather direct as to be indicated below.

We first treat the isomorphism $IH^*(\overline{Y}^{\mathrm{rBS}},\mathbb{V}) \simeq IH^*(Y^*,\mathbb{V})$. Under the equal rank condition, this is exactly the content of the well-known conjecture of Rapoport and Goresky-MacPherson, verified by Saper in [**?**] using his $\mathcal{L}$-module theory.

Then, let us verify the second isomorphism. Note that, by Theorem 1(2), there is a diffeomorphism contraction $\overline{Y}^{\mathrm{BS}} \to X_0$ which implies that there is a diffeomorphism contraction $Y \to Y_0^o$. Therefore, we have an isomorphism $H^*_{(2)}(Y_0^o,\overline{\mathbb{V}}) \simeq H^*_{(2)}(Y,\overline{\mathbb{V}})$. Now under the equal rank condition, by Borel-Zucker conjecture, verified by Saper-Stein in equal rank situation, we have $H^*_{(2)}(Y,\overline{\mathbb{V}}) \simeq IH^*(Y^*,\mathbb{V})$. Therefore, $IH^*(Y^*,\mathbb{V}) \simeq H^*_{(2)}(Y_0^o,\overline{\mathbb{V}})$. □

Remark 1. Under the equal rank condition, using the natural rational structure on the intersection cohomology, we can and hence will define a new type of *regulator*, unique up to rational factors, as the determinant of the isomorphism in Theorem 2(4). Many examples suggest that for SL_2 over real quadratic fields F, or the same over the assocaited Hilbert modular surfaces, these regulatore are essentially the Beilinson regulators associated to Hilbert modular form f of parallel weight (2,2) in $S_{2,2}(\mathrm{SL}_2(\mathcal{O}_F))$ via the special values $L(f,(1,1))$. We leave the details to the reader.

We end this paper by mentioning that there is a further refinement on the reductive modification, the so-called *total reductive modifications* of $\overline{Y}^{\mathrm{rBS}}$, in terms of manifolds with horn type ends. For details, please refer to [18].

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