MORPHISM-BASED EXPLANATIONS OF TSUMEGOS

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Abstract. Mathematics (unintentionally) has produced a grand theory of understanding based on *(homo)morphisms*, structure preserving maps. Having compatible operations in source and target domains allows us to move back and forth between those. We can work in the easier, more familiar domain and transfer the results to the domain we aim to understand. This mechanism goes beyond the mathematical examples. Here, as an illustration, we apply this morphic theory of understanding to explanations of tsumegos (puzzles for the game of Go).

This *meta-mathematical* essay explores the usefulness of constructing and using homomorphic images beyond the standard examples in Mathematics. Our approach is in line with the applied category theory movement, where categories (objects and their morphisms) are used as very general modeling tools [5]. In addition, we suggest that morphic relations provide the core mechanisms of understanding and explanations.

1. The Morphic Theory of Understanding

How can we describe the process of understanding in a precise language? What is its core mechanism? What clicks when something 'clicks'? A mathematical theory of understanding [3] proposes that the creation and utilization of an algebraic morphism between two domains can explain understanding in general. The theory can be summarized by the following thesis.

Thesis 1. *Finding compatible operations in two domains is not merely a tool in mathematics, but a universal mechanism for any understanding, reasoning and explanation, both for human and artificial intelligence.*

Formally, morphisms are described by the equation $\varphi(a) * \varphi(b) = \varphi(a \cdot b)$, where $\cdot$ and $*$ are the two operations in the different domains. Mathematicians are familiar with many examples. For instance, linear transformations, morphisms of formal languages, homomorphisms in universal algebra, order preserving maps, graph homomorphisms, relational morphisms of semigroups, just to name a few. Most people are familiar with analytic geometry and logarithms due to the compulsory studies in school. These are morphisms too, though not introduced as such. Beyond mathematics, any functional models, even analogies and metaphors [6] have this algebraic structure. Finding morphisms and building models are challenging tasks. The thesis implies that *understanding is indirect and rare.*

- **Indirect:** We do not understand things directly. We interpret something new in terms of something familiar. Thus, there are two domains in an explanation: the familiar and the new. Inference proceeds by mapping the familiar (its entities and operations) over the unknown.
- **Rare:** The choice for the familiar domain is not arbitrary. It has to be to some extent a model of what we want to understand, i.e., it should have *compatible operations*, which is a special algebraic property. We cannot understand something in terms of anything else.

We are already capable of understanding and explaining. The practical use of morphisms describing these processes lies in the clarification and in the improved quality. By spelling out the compatible operation we use we can be more certain about our comprehension.

2. Go Problems

It is a commonplace in computer science that board games are good testbed applications for algorithmic intelligence. The defining rules are few, performance measures are well-defined, but the challenge is far from trivial. Here, we use the game of Go as a case study for the algebraic theory of understanding and explanations. This choice involves several reductions in complexity compared to general scientific research:

- **real world:** (possibly) no ground truth, complex phenomena, incomplete knowledge both in common sense and in scientific understanding;

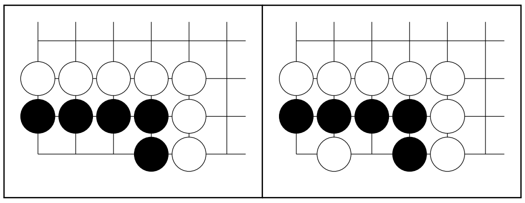

Figure 1. In the situation on the left Black surrounded the corner, but it is also surrounded by White from the outside. Who owns the corner? If White moves next, then Black can be captured. The solution move is shown on the right. Merely knowing the correct move does not give an explanation.

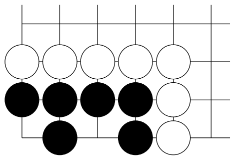

Figure 2. A living shape with two eyes (holes). Black is uncapturable as it would require two simultaneous moves by White to take away all empty neighbouring intersections from the Black stones. A single White move into a hole would be self-capture.

the game of Go: (inaccessible) ground truth, reduced complexity (only the things that can happen on the board), approximate knowledge (human expertise and deep learning systems).

We even go further with complexity reduction by concentrating on tsumegos, Go puzzles. They have further simplifying properties.

- The complete game tree for one problem is feasible to compute (the ground truth is accessible). This implies that the problem is limited to a small part of the board, it is *local*.
- The goal states are well-defined (e.g., capturing stones, connecting groups), i.e., we can decide in any position whether the goal is reached or not.
- In practice, there can be further conventions, e.g., insisting on a single-line solution, specifying the goal explicitly.

Solving tsumegos is still complex enough to study the process of understanding and the mechanism of explaining. Here, we are primarily interested in the explanations themselves, not in its effect on playing skill strength. On the contrary to our approach, in professional Go training, it is often advised to do as many tsumegos as possible to achieve statistical learning and not wasting time with crafting explanations. This is exactly how the deep learning AIs achieved the breakthrough [9]. We suggest that creating explanations is more interesting for humans, and that is where we are still ahead of the AIs.

3. Morphic Explanations for Go Problems

The goal of the game of Go is controlling more territory than the opponent. The players place their stones by taking alternating turns (for a complete description see for instance [11]). One method to gain territory is *capturing*, i.e., removing completely surrounded opponent stones. One of the very first capturing problems has a simple solution (Fig. 1), but why is that move the solution? What would be a good explanation?

One way to explain is to systematically explore all future possibilities, in other words, to generate the *game tree*. We enumerate each of White's legal moves and all possible Black countermoves. Then again White moves, and Black moves until the capturing is done or the impossibility of it becomes clear. The solution move leads to capture, while all other options do not. However, we would not consider this as a good explanation, even if it is indisputable evidence. We expect a good explanation to be short, which can be grasped in a single step.

A metaphorical approach provides a better explanation. A surrounded (small) territory can be divided into separate parts, called *eyes*. Two (or more) eyes mean unconditional life: two eyes cannot be filled by a single move, thus the group of stones in not capturable. Clearly, this is a topological idea: having holes in a shape.

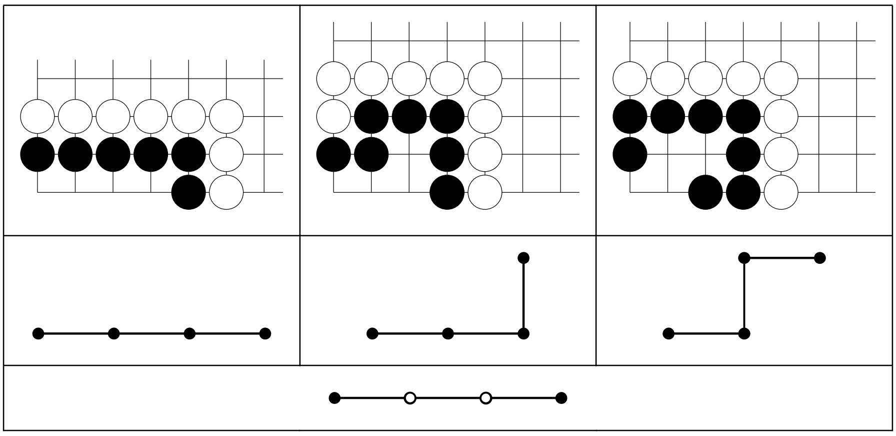

FIGURE 3. Four-point spaces and their graph representations. The layout of the graph does not matter. The empty circles denote the two cutting points, thus these shapes are unconditionally alive.

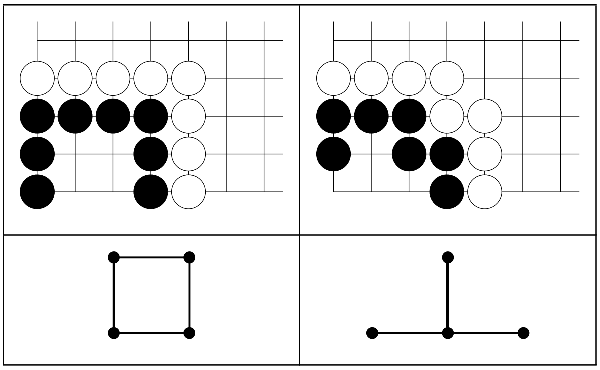

FIGURE 4. More 4-point spaces with different connection topologies. The 'square' is dead, since it takes two moves to disconnect the graph. Opponent moves do not divide the area. The shape on the right is conditionally alive. If Black moves first, the branching point can be taken, making three eyes.

We can further exploit topological connections. We explain the statuses of shapes (e.g., dead, alive) by converting them to graphs. Admittedly, the prime audience for this explanations are people familiar with graph theory.

3.1. **Life and death problems by graphs.** To make a living group, stones have to form a shape with two eyes. The possibility depends on the number points in the surrounded area and on its shape. However, we need to be more precise with the term shape. It turns out that connection structure of the points is what matters. Fig. 3 shows that the same graph can represent several different 4-point spaces. A *cutting point* is defined by the ability to separate the graph into disconnected components when removing that point. These correspond to the ways of dividing the territory into two parts. They are all *unconditionally* alive, since there are two cutting points on the graph. It does not matter whose turn is it. If the opponent occupies a cutting point, the other one can still disconnect the eye space. Placing a stone on the Go board and removing a vertex from the graph are the compatible operations in the morphism for this explanation. Fig. 4 shows different 4-point shapes. Now we can do the analysis without generating the game tree.

3.2. **Further morphic explanations.** A base explanation is the demonstration of the complete search tree. It clearly shows that the good moves lead to the desired result and all the other moves fail to achieve that. As we mentioned, this type of explanation is unsatisfactory. There is no *compression*, no surjective morphism to a smaller domain.

We prefer an explanation that groups some of the branches together in a pattern or a condition. This happened in the graph representation of eye spaces. Several stone configurations correspond to the same graph (see Fig. 3). Good explanations are compressions.

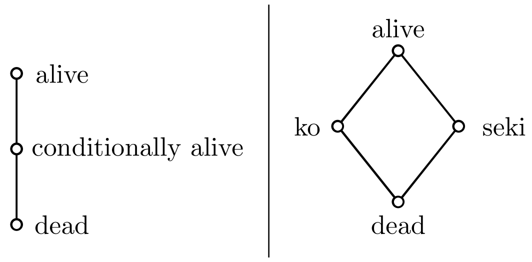

FIGURE 5. Partial orders that should be homomorphic images from any correct explanation of tsumegos. Ko refers to the rule that the same board position cannot repeat. This rule enables a player to use non-local threats to save a group. Seki refers to stalemate situations.

What other mathematical domains do we use when crafting explanations for solving tsumegos? The most immediate application is counting liberties. This is the number of empty neighboring intersections of a connected group of friendly stones. It tells how many moves needed to capture that group. To understand the safety of a group we map it into the natural numbers $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.

Rational numbers are also used in understanding puzzles and games. Efficiency of the play can be interpreted by the measure obtained territory per move. More importantly, evaluating endgame moves requires averaging over possible outcomes, thus rational numbers naturally appear in modern endgame theory [10].

How can an explanation be incorrect? The game tree-based algorithms (e.g., minimax [8]) assume that the opponent always plays optimally. Therefore, the expected score of a game or a puzzle can only decrease. In contrast, in beginner play, mistakes are made consecutively, thus the winning probabilities and expected scores can be swinging back and forth. To ensure that an explanation of a tsumego solution is correct, the sequence of moves should map onto the ordered set of possible outcomes (Fig. 5). These are examples of order-preserving maps.

To extend this idea, we can map the search tree to the set of future possibilities defined by the leaf nodes. Allowed variations of the same solution have a constant image set in the future possibilities.

4. Syntax – semantics

Turning more philosophical, here we contemplate on a parallel between mathematical and Go knowledge. In both, we can distinguish between syntactic and semantic levels. The *syntactic* level provides ground truth and it is described by few rules. The *semantic* level gives meaning by defining new objects regulated by many rules.

4.1. **Syntactic Level.**
In mathematics, the syntax is defined by a foundational axiomatic system: propositions connected by logical inference. The most common choice for foundations is set theory. The rules tell what follows from what by logical inference, but they do not say which propositions are interesting.

In Go, the syntactic level is the game tree: board positions connected by legal moves. The rules of the game tell what are the possible moves, but they do not say what moves are good.

4.2. **Semantic Level.**
In mathematics, the semantic level is comprised of mathematical objects (abstract spaces, algebraic structures, etc.). These are the things that the mind can "see" clearly by the so-called mathematical intuition. For instance, most people can answer this question (from [1]) *Can a straight line intersect a circle in three different points?* correctly with an immediate no. How do they do that? Most likely they imagine a circle and wave around a line in their heads to see how they can intersect. We can all do this since we built up good familiarity with these basic geometric objects (willingly or not) in school. A formal proof of this fact involves algebraic reasoning about the quadratic equation of the circle and the linear equation of the line. A mathematically trained person would see this immediately.

Similarly, an experienced player can just see the vital point in life and death problem, while a full 'proof' would involve counting liberties and traversing the game tree. A complete beginner simply does not see the same thing, the configuration of stones is meaningless.

A statement randomly chosen from the space of all logical consequences following from a set of axioms is of little interest for a mathematician. Similarly, a random board position is not useful for a human player, as there is very little to understand and reason about.

4.3. **Images of surjective morphisms.** The semantic level is formed by surjective morphisms from the syntactic level. The maps should group together elements of the syntactic level. The explanation should simplify the structure we need to think with. Note that this reduction is not limited to just choosing substructures. Morphic images can be structures that are not (literally) contained in the source domain [2].

Mathematical theories are homomorphic images of the underlying formal system. Operations in the theory and in the underlying formal system should be compatible to guarantee truth for the statements of the theory. Constructing these morphisms is non-trivial, amply demonstrated by the open problems and conjectures. Approximate knowledge can be defined as a partially checked morphism. This perspective is motivated by the more technically defined *functorial semantics* [7].

For the game of Go, semantic level concepts (shapes, strategies) should group together several board positions and moves. Different strategies about the game should be homomorphic images of the game tree, otherwise they would imply non-legal moves. The ultimate question about the game is *How compressible is Go knowledge?* [4]. If it is compressible, then we may find a computationally feasible theory that could explain and perform perfect play. If it is not compressible, then the only explanation for perfect play will be the explicitly generated (not feasible) vast subtrees of the game tree. In that case, the battle of approximate knowledge players, both human and AI, will continue indefinitely.

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