On Reynolds-Hurkens-Coquand paradox and T-algebras

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Abstract

Recently, Coquand [2] presented a variant on the paradox, as a variation of Reynolds "paradox" [5] in terms of weakly initial T-algebras. We first formalize the notion of the powerful universe U in λHOL such as $\Gamma_U := \{U:\Box, in:\mathtt{T}(U) \to U, out:U \to \mathtt{T}(U)\}$ with a functor $\mathtt{T}$ where $\mathtt{T}(U) := \mathtt{Pow}(\mathtt{Pow}(U))$ and $\mathtt{Pow}(U) := U \to *$. Then rather than a retraction such that $out \circ in$ is the identity on $\mathtt{T}(U)$, we suppose logical equivalence with kind $*$ as follows:

- $Ret_1 := \Pi X:\mathtt{T}(U).\Pi p:\mathtt{Pow}(U).(out \circ in\, X\, p) \leftrightarrow X\, p$,
- $Ret_2 := \Pi X:\mathtt{T}(U).\Pi p:\mathtt{Pow}(U).(\mathtt{T}\,\delta\, X\, p) \leftrightarrow X\, p$,
- $Weak_1 := \Pi X:\mathtt{T}(U).\Pi p:\mathtt{Pow}(U).(out \circ in\, X\, p) \leftrightarrow \mathtt{T}\,\delta\, X\, p$, and
- $Weak_2 := \Pi X:\mathtt{T}(U).\Pi p:\mathtt{Pow}(U).(out \circ in\, X\, p) \leftrightarrow X\,(p \circ \delta)$,

where $\delta := (in \circ out)$. Now, λHOL can prove that $\Gamma_U \vdash Weak_1 \leftrightarrow Weak_2$; $\Gamma_U \vdash Ret1 \wedge Ret2 \to Weak$; $\Gamma_U \vdash Ret1 \wedge Weak \to Ret2$; $\Gamma_U \vdash Ret2 \wedge Weak \to Ret1$; $\Gamma_U \vdash \neg Ret_1$; and $\Gamma_U \vdash \neg Weak$.

Next, in $\lambda\mathrm{U}^-$, we show a dual one with respect to the weakly initial T-algebra [2], i.e., a weakly final T-coalgebra as another instance of the powerful universe.

1. Introduction

Let λHOL (Church 1940) be a PTS (Pure Type System) with the specification [1], consisting of $*:\Box, \Box:\Delta$ for axioms; and $(*,*), (\Box,*), (\Box,\Box)$ for rules. That is, λHOL is Girard's system F_ω with the axiom $\Box:\Delta$, which is a consistent type system and can introduce variables of kinds, e.g., $X:\Box$. It is well-known that in System F, weakly initial T-algebras and final T-coalgebras are impredicatively represented as the categorical data types.

Let $\lambda\mathrm{U}^-$ be λHOL together with the rule $(\Delta,\Box)$, which can introduce polymorphic kinds like polymorphic types in system F. The use of the rule makes it possible to define powerful universes. Moreover, we work with the rules $(\Box,\Box)$ and $(\Delta,\Box)$ to represent categorical data types, just like system F of $(*,*)$ and $(\Box,*)$.

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2. Encoding powerful universe in λHOL

As usual, define logical connectives for types and formulae with the kind $*$. Also define power sets and data types as kinds with the super kind $\Box$.

1. Bottom $\bot \ := \ \Pi X : *.X$ with kind $*$ by $(\Box, *)$.
2. Negation $\neg \ := \ \lambda X{:}*.X \to \bot$ with kind $(* \to *)$ by $(\Box, *), (\Box, \Box)$.
3. Conjunction $\wedge \ := \ \lambda A, B{:}*.\Pi X{:}*.(A \to B \to X) \to X$ with kind $(* \to * \to *)$ by $(\Box, \Box)$.

   Write $A \leftrightarrow B$ for $(A \to B) \wedge (B \to A)$ with $pair, fst, snd$ where $A, B{:}*$.
4. For $X{:}\Box$, power set $\mathtt{Pow}(X) \ := \ X \to *$ with super kind $\Box$ by $(\Box, \Box)$.

   Write $\mathtt{T}(X)$ for $\mathtt{Pow}(\mathtt{Pow}(X))$ where $X{:}\Box$.
5. Composition $g \circ f := \lambda x{:}X.g(fx)$

   $X, Y, Z{:}\Box, f{:}(X \to Y), g{:}(Y \to Z) \vdash \lambda x{:}X.g(fx) : X \to Z$ by $(\Box, \Box)$.
6. Contravariant functor $\mathtt{mon}_{X \to Y}(f) := \lambda q{:}\mathtt{Pow}(Y).q \circ f$:

   $X, Y{:}\Box, f{:}X \to Y \vdash \lambda q{:}\mathtt{Pow}(Y).q \circ f : \mathtt{Pow}(Y) \to \mathtt{Pow}(X)$ by $(\Box, \Box)$.
7. Covariant functor $\mathtt{T}_{X \to Y}(f) := \mathtt{mon}_{\mathtt{Pow}(Y) \to \mathtt{Pow}(X)}(\mathtt{mon}_{X \to Y} f)$

   $X, Y{:}\Box, f{:}X \to Y \vdash \lambda F{:}\mathtt{T}(X).F \circ (\mathtt{mon}_{X \to Y} f) : \mathtt{T}(X) \to \mathtt{T}(Y)$ by $(\Box, \Box)$.
8. Powerful universe $U{:}\Box$ and a well-formed context Γ by $\Box{:}\Delta$ and $(\Box, \Box)$:

   $\Gamma_U := \{U{:}\Box, in{:}(\mathtt{T}(U) \to U), out{:}(U \to \mathtt{T}(U))\}$

   $\delta := in \circ out$
9. Retraction by $(\Box, *)$:

   $Ret1 := (\Pi X{:}\mathtt{T}(U).\Pi p{:}\mathtt{Pow}(U).(out \circ in)\, X\, p \leftrightarrow X\, p)$

   $\Gamma_{ret1} := \Gamma_U \cup \{ret1{:}(\Pi X{:}\mathtt{T}(U).\Pi p{:}\mathtt{Pow}(U).(out \circ in)\, X\, p \leftrightarrow X\, p)\}$

   $\alpha_e\, X\, p := fst(ret1\, X\, p) : (out \circ in)\, X\, p \to X\, p$

   $\beta_e\, X\, p := snd(ret1\, X\, p) : X\, p \to (out \circ in)\, X\, p$

   $Ret2 := (\Pi X{:}\mathtt{T}(U).\Pi p{:}\mathtt{Pow}(U).\mathtt{T}\,\delta\, X\, p \leftrightarrow X\, p)$

   $\Gamma_{ret2} := \Gamma_U \cup \{ret2{:}(\Pi X{:}\mathtt{T}(U).\Pi p{:}\mathtt{Pow}(U).\mathtt{T}\,\delta\, X\, p \leftrightarrow X\, p)\}$
10. U is weakly initial (final) by $(\Box, *)$:

    $\Gamma_U \vdash \Pi X{:}\mathtt{T}(U).\Pi p{:}\mathtt{Pow}(U).(out \circ in) X\, p \leftrightarrow \mathtt{T}(in \circ out) X\, p : *$ by $(\Box, *)$.

    $Weak1 := \Pi X{:}\mathtt{T}(U).\Pi p{:}\mathtt{Pow}(U).(out \circ in) X\, p \leftrightarrow \mathtt{T}(\delta) X\, p$

    $Weak2 := (\Pi X{:}\mathtt{T}(U).\Pi p{:}\mathtt{Pow}(U).(out \circ in) X\, p \leftrightarrow X\,(p \circ \delta)$

    $\Gamma_{weak1} := \Gamma_U \cup \{w1{:}(\Pi X{:}\mathtt{T}(U).\Pi p{:}\mathtt{Pow}(U).(out \circ in) X\, p \leftrightarrow \mathtt{T}(\delta) X\, p)\}$

    $\Gamma_{weak2} := \Gamma_U \cup \{w2{:}(\Pi X{:}\mathtt{T}(U).\Pi p{:}\mathtt{Pow}(U).(out \circ in) X\, p \leftrightarrow X\,(p \circ \delta)\}$

    $\alpha_w\, X\, p := fst(weak2\, X\, p) : (out \circ in)\, X\, p \to X\,(p \circ \delta)$

    $\beta_w\, X\, p := snd(weak2\, X\, p) : X\,(p \circ \delta) \to (out \circ in)\, X\, p$

Lemma 1 *Let* $X, Y, Z : \square, f : X \to Y, g : Z \to X, h : W \to Z, p : \mathtt{Pow}(Y), P : \mathtt{T}(X)$. *Then we have the following elementary facts under* $=_\beta$.

1. $(f \circ g) \circ h =_\beta f \circ (g \circ h)$
2. $(\mathtt{mon}\, g) \circ (\mathtt{mon}\, f) =_\beta \mathtt{mon}(f \circ g)$
3. $(\mathtt{T}\, f) \circ (\mathtt{T}\, g) =_\beta \mathtt{T}(f \circ g)$
4. $\mathtt{mon}\, f\, p =_\beta p \circ f$
5. $\mathtt{T}\, f\, P =_\beta P \circ (\mathtt{mon}\, f)$

We write simply $\Gamma \vdash A$ if $\Gamma \vdash M : A$ for some M.

2.1. Retraction and Γ_{ret} in λHOL

Definition 1 1. $X_0 := \lambda p : (\mathtt{Pow}\, U).\Pi u : U.p\, u \to \neg(out\, u\, p)$ *with* $\mathtt{T}(U)$.

2. $q_0 := \lambda u : U.\Pi p : (\mathtt{Pow}\, U).p\, u \to \neg(out\, u\, p)$ *with* $\mathtt{Pow}(U)$.

3. $x_0 := in\, X_0$ *with* U.

Under Γ_{ret1}, λHOL proves the following proposition.

Proposition 1 1. $\Gamma_{ret1} \vdash l_1 : X_0 q_0$ *for some* l_1.

2. $\Gamma_{ret1} \vdash l_2 : q_0 x_0$ *for some* l_2.

Proof. (1) $l_1 := \lambda u : U.\lambda h : q_0 u.h q_0 h$ proves $X_0\, q_0$, i.e., $\Pi u : U.q_0\, u \to \neg(out\, u\, q_0)$.

(2) $l_2 := \lambda p : \mathtt{Pow}(U).\lambda h : p x_0.\lambda h_1 : X_0 p.(\alpha_e\, X_0\, p)\, h_1\, x_0\, h\, h_1$ proves $q_0\, x_0$, i.e., $\Pi p : \mathtt{Pow}(U).p\, x_0 \to \neg(out\, x_0\, p)$.

Proposition 2 1. λHOL *proves that* $\Gamma_{ret1} \vdash \bot$, *and then* $\Gamma_U \vdash \neg Ret1$ *and* $U : \square \vdash \Pi\, in : (\mathtt{T}U \to U).\Pi\, out : (U \to \mathtt{T}U).\neg Ret1$.

2. *Let* $F := (\alpha_e\, X_0\, q_0) \circ (\beta_e\, X_0\, q_0) : X_0 q_0 \to X_0 q_0$. λHOL *never enjoy the reduction such that* $F\, l \to_\beta l$.

Proof. (1) λHOL proves that $\Gamma_{ret1} \vdash l_1\, x_0\, l_2\, (\beta_e\, X_0\, q_0\, l_1) : \bot$, and hence $\Gamma_U \vdash \lambda ret1 : Ret1.l_1\, x_0\, l_2\, (\beta_e\, X_0\, q_0\, l_1) : \neg Ret1$, and then $\Pi\, in : (\mathtt{T}U \to U).\Pi\, out : (U \to \mathtt{T}U).\neg Ret1$ by $(\square, *)$.

(2) Towards a contradiction, suppose that $\alpha_e\, X_0\, q_0\, (\beta_e\, X_0\, q_0\, l) \to_\beta l$, i.e., type isomorphism $(out \circ in)\, X\, p \simeq X\, p$. Then the head reduction gives

$l_1\, x_0\, l_2\, (\beta_e\, X_0\, q_0\, l_1)$
$\to_\beta^* l_2\, q_0\, l_2\, (\beta_e\, X_0\, q_0\, l_1)$
$\to_\beta^* \alpha_e\, X_0\, q_0\, (\beta_e\, X_0\, q_0\, l_1)\, x_0\, l_2\, (\beta_e\, X_0\, q_0\, l_1)$
$\to_\beta^* l_1\, x_0\, l_2\, (\beta_e\, X_0\, q_0\, l_1) \to_\beta^* \cdots$,
leading to a contradiction to the fact that λHOL is strongly normalizing.

It should be remarked that we had $F\, l_1\, x_0\, l_2\, (\beta_e\, X_0\, q_0\, l_1) =_\beta l_1\, x_0\, l_2\, (\beta_e\, X_0\, q_0\, l_1)$ in the above so that the proof of the retract gave a certain fixed point of $F : X_0 q_0 \to X_0 q_0$. This simple example shows that an existence of type isomorphism $(out \circ in)\, X_0\, p_0 \simeq X_0\, p_0$ leads to a fixed point of functions with $X_0 q_0 \to X_0 q_0$.

2.2. Weak initiality (cofinality) and Γ_{weak} in λHOL

Under Γ_{weak_i}, λHOL proves the following proposition. Simply write Γ_{weak} for Γ_{weak_i} with $i = 1, 2$

Proposition 3 *1.* $\Gamma_{weak1} \vdash Weak2$, *and* $\Gamma_{weak2} \vdash Weak1$.

2. $\Gamma_{weak}, X : \mathtt{T}(U), p : \mathtt{Pow}(U) \vdash L_1 : \neg(out \circ in\, X\, p) \to \neg(X(p \circ \delta))$ *for some* L_1.

3. $\Gamma_{weak}, X : \mathtt{T}(U), p : \mathtt{Pow}(U) \vdash L_2 : \neg(X(p \circ \delta)) \to \neg(out \circ in\, X\, p)$ *for some* L_2.

Proof. (1) $\mathtt{T}(in \circ out)\, X\, p =_\beta X \circ \mathtt{mon}(in \circ out)\, p =_\beta X(\mathtt{mon}(in \circ out)\, p) =_\beta X(p \circ \delta)$.
(2) From the assumption $Weak$, $L_1 := \lambda z : \neg(out \circ in\, X\, p).\lambda b : X(p \circ \delta).z(\beta_w\, X\, p\, b)$.
(3) $L_2 := \lambda z : \neg(X(p \circ \delta)).\lambda a : (out \circ in\, X\, p).z(\alpha_w\, X\, p\, a)$.

Definition 2 *1.* $X_0 := \lambda p : (\mathtt{Pow}\, U).\Pi u : U.p\, u \to \neg(out\, u\, p)$ *with* $\mathtt{T}(U)$.

2. $p_0 := \lambda u : U.\Pi p : (\mathtt{Pow}\, U).p(\delta u) \to \neg(out\, u\, p)$ *with* $\mathtt{Pow}(U)$.

3. $x_0 := in\, X_0$ *with* U.

Under Γ_{weak}, λHOL proves the following proposition.

Proposition 4 *1.* $\Gamma_{weak} \vdash stableX_0 : (\Pi p : (\mathtt{Pow}\, U).X_0\, p \to X_0(\mathtt{mon}\, \delta\, p))$

2. $\Gamma_{weak} \vdash stablep_0 : (\Pi z : U.p_0\, z \to p_0(\delta\, z))$

3. $\Gamma_{weak} \vdash lem1 : (\Pi p : (\mathtt{Pow}\, U).p\, x_0 \to \neg(X_0\, p))$

4. $\Gamma_{weak} \vdash lem2 : (p_0\, x_0) = (\Pi p : (\mathtt{Pow}\, U).p(\delta\, x_0) \to \neg(out\, x_0\, p))$

5. $\Gamma_{weak} \vdash lem3 : (X_0\, p_0) = (\Pi u : U.p_0\, u \to \neg(out\, u\, p_0))$

Proof. We show here the cases of 1 and 5. Leave it to the reader to prove the remaining cases.
(1) $stableX_0 : (\Pi p : (\mathtt{Pow}\, U).X_0\, p \to X_0(\mathtt{mon}\, \delta\, p))$

$$\dfrac{\dfrac{\dfrac{\dfrac{L_1 : \neg(out\, (\delta\, u)\, p) \to \neg(out\, u\, (p \circ \delta)) \qquad \dfrac{\dfrac{[p : \mathtt{Pow}(U) \vdash h_p : X_0\, p]}{h_p : \Pi u : U.p\, u \to \neg(out\, u\, p)} =_\beta}{h_p(\delta\, u) : p(\delta\, u) \to \neg(out\, (\delta\, u)\, p)}\; u := (\delta\, u)}{\dfrac{L_1 \circ (h_p(\delta\, u)) : p(\delta\, u) \to \neg(out\, u\, (p \circ \delta))}{L_1 \circ (h_p(\delta\, u)) : (\mathtt{mon}\, \delta\, p)\, u \to \neg(out\, u\, (\mathtt{mon}\, \delta\, p))} =_\beta}\; Prop3}{\dfrac{\lambda u : U.L_1 \circ (h_p(\delta\, u)) : \Pi u : U.(\mathtt{mon}\, \delta\, p)\, u \to \neg(out\, u\, (\mathtt{mon}\, \delta\, p))}{\lambda u : U.\lambda x.L_1(h_p(\delta\, u)) : X_0(\mathtt{mon}\, \delta\, p)} =_\beta}\; (\Box, *)}{\lambda h_p : X_0 p.\lambda u : U.L_1 \circ (h_p(\delta\, u)) : X_0\, p \to X_0(\mathtt{mon}\, \delta\, p)}\; (*, *)}{\lambda p : \mathtt{Pow}(U).\lambda h_p : X_0 p.\lambda u : U.L_1 \circ (h_p(\delta\, u)) : \Pi p : \mathtt{Pow}(U).X_0\, p \to X_0(\mathtt{mon}\, \delta\, p)}\; (\Box, *)$$

(5) $lem3 : (X_0\, p_0) = (\Pi u : U.p_0\, u \to \neg(out\, u\, p_0))$

$$\dfrac{\dfrac{\dfrac{\dfrac{[h_u : p_0\,u] \quad \dfrac{stablep_0 : \Pi u{:}U.p_0u \to p_0(\delta\,u)}{stablep_0\,u : p_0u \to p_0(\delta\,u)}}{stablep_0\,u\,h_u : p_0(\delta\,u)} \quad \dfrac{\dfrac{[h_z : p_0\,u]}{h_z : \Pi p{:}\mathtt{Pow}(U).p(\delta\,u) \to \neg(out\,u\,p)}\,=_\beta}{h_z p_0 : p_0(\delta\,u) \to \neg(out\,u\,p_0)}\,p := p_0}{h_u\,p_0\,(stablep_0\,u\,h_u) : \neg(out\,u\,p_0)}}{\lambda h_u{:}p_0u.h_u\,p_0\,(stablep_0\,u\,h_u) : p_0\,u \to \neg(out\,u\,p_0)}\,(*,*)}{\lambda u{:}U.\lambda h_u{:}p_0u.h_u\,p_0\,(stablep_0\,u\,h_u) : \Pi u{:}U.p_0\,u \to \neg(out\,u\,p_0)}\,(\Box,*)$$

Let $loop := lem1\,p_0\,lem2\,lem3$.

Proposition 5 1. λHOL *proves that* $\Gamma_{weak} \vdash loop : \bot$.

2. λHOL *proves that* $\Gamma_U \vdash \neg Weak$ *and then* $U : \Box \vdash \Pi\, in : \mathtt{T}U \to U.\Pi\, out : U \to \mathtt{T}U.\neg Weak$.

3. Encoding of sets and algebras in λU$^-$

We define instances of the powerful universe, i.e., a weakly initial T-algebra $\mathtt{A}$ as in [2] and in addition here, a weakly final T-coalgebra $\mathtt{B}$, as follows.

1. For $A, B{:}\Box$, product $A \times B := \Pi X{:}\Box.(A \to B \to X) \to X$ with $\Box$.
   $\langle M, N\rangle := \lambda X{:}\Box.\lambda z{:}(A \to B \to X).zMN$ with $A \times B$
   $\pi_1 M := MA(\lambda x{:}A.\lambda y{:}B.x)$ with A where $M{:}A \times B$
   $\pi_2 M := MB(\lambda x{:}A.\lambda y{:}B.y)$ with B where $M{:}A \times B$
2. For $A, B{:}\Box$, coproduct $A + B := \Pi X{:}\Box.(A \to X) \to (B \to X) \to X$ with $\Box$.
3. For $G(X){:}\Box$ with $X{:}\Box$,
   existential quantifier $\Sigma X{:}\Box.G(X) := \Pi Y{:}\Box.(\Pi X{:}\Box.G(X) \to Y) \to Y$ with $\Box$.
   $[B, M] := \lambda Y{:}\Box.\lambda z{:}(\Pi X{:}\Box.G(X) \to Y).zBM$ with $\Sigma X{:}\Box.G(X)$.
   $(\mathtt{let}\,[X, z] = M\,\mathtt{in}\,N) := MC(\lambda X{:}\Box.\lambda z{:}G(X).N$ with $C{:}\Box$,
   where $N{:}C$ and $M{:}(\Sigma X{:}\Box.G(X))$.
   Note that $(\mathtt{let}\,[X, z] = [B, M]\,\mathtt{in}\,N) \to_\beta N[X := B, z := M]$.

Now the context Γ_U in λHOL can be established by the following algebras in λU$^-$, so that the condition $Weak$ holds in λU$^-$ under the weakly initial and cofinal algebras, respectively.

3.1. Weakly initial algebra

We introduce the powerful universe $\mathtt{A}$ with $\mathtt{fold}$, $\mathtt{intro}$, and $\mathtt{match}$. Then the following Proposition 6 can be established [2].

1. Powerful universe $\mathtt{A} := \Pi X{:}\Box.(\mathtt{T}(X) \to X) \to X$ with super kind $\Box$.
   Let $\mathtt{fold}^{\mathtt{A}}_X := \lambda f.\lambda a^{\mathtt{A}}.aXf$ with $(\mathtt{T}(X) \to X) \to (\mathtt{A} \to X)$.
   Let $\mathtt{intro} := \lambda u{:}\mathtt{T}(\mathtt{A}).\lambda X{:}\Box.\lambda f.f(\mathtt{T}\,\mathtt{A}\,X(\mathtt{fold}^{\mathtt{A}}_X\,f)\,u)$ with $\mathtt{T}(\mathtt{A}) \to \mathtt{A}$.
   Let $\mathtt{match} := \mathtt{fold}^{\mathtt{A}}_{\mathtt{T}(\mathtt{A})}(\mathtt{T}\,(\mathtt{T}\,\mathtt{A})\,\mathtt{intro})$ with $\mathtt{A} \to \mathtt{T}(\mathtt{A})$, i.e., $\mathtt{match}(a) = a(\mathtt{T}\,\mathtt{A})(\mathtt{T}\,\mathtt{intro})$.
2. **Proposition 6** $\langle \mathtt{A}, \mathtt{intro}\rangle$ *is weakly initial: for any* $f{:}\mathtt{T}(X) \to X$, *the following is commutative.*

$$((\mathtt{fold}^{\mathtt{A}}_X f) \circ \mathtt{intro}) =_\beta (f \circ (\mathtt{T}(\mathtt{fold}^{\mathtt{A}}_X\,f)))$$
$$(\mathtt{match} \circ \mathtt{intro}) =_\beta ((\mathtt{T}\,\mathtt{intro}) \circ (\mathtt{T}\,\mathtt{match}))$$

3.2. Weakly final co-algebra

We here define the powerful universe $\mathtt{B}$ together with $\mathtt{unfold}$, $\mathtt{in}$, and $\mathtt{out}$ as well. Then the following Proposition 7 introduces a weakly final T-coalgebra.

1. Powerful universe $\mathtt{B} := \Sigma X : \Box.(X \to \mathtt{T}(X)) \times X$ with super kind $\Box$.

   Let $\mathtt{unfold}^{\mathtt{B}}_X := \lambda f.\lambda z : X.[X, \langle f, z\rangle]$ with $(X \to \mathtt{T}(X)) \to (X \to \mathtt{B})$.

   Let $\mathtt{out} := \lambda b{:}\mathtt{B}.(\mathtt{let}\,[X, \beta] = b\,\mathtt{in}\,\mathtt{T}(\mathtt{unfold}_X\,(\pi_1\beta))\,(\pi_1\beta(\pi_2\beta)))$
   with $\mathtt{B} \to \mathtt{T}(\mathtt{B})$.

   Let $\mathtt{in} := \mathtt{unfold}^{\mathtt{B}}_{\mathtt{TB}}(\mathtt{T}\,(\mathtt{T}\,\mathtt{B})\,\mathtt{out})$ with $\mathtt{T}\,\mathtt{B} \to \mathtt{B}$, i.e., $\mathtt{in}(u) = [\mathtt{T}\,\mathtt{B}, \langle \mathtt{T}\,\mathtt{out}, u\rangle]$.

2. **Proposition 7** $\langle \mathtt{B}, \mathtt{out}\rangle$ *is weakly final: for any* $f : X \to \mathtt{T}(X)$*, the following is commutative.*

$$(\mathtt{out} \circ (\mathtt{unfold}^{\mathtt{B}}_X f)) =_\beta ((\mathtt{T}(\mathtt{unfold}^{\mathtt{B}}_X\, f)) \circ f)$$

$$(\mathtt{out} \circ \mathtt{in}) =_\beta ((\mathtt{T}\,\mathtt{in}) \circ (\mathtt{T}\,\mathtt{out}))$$

3.3. Computational behaviour of the proof of $\vdash \bot$ in λU^-

Now for both algebras, we obtain the same computational behaviour of the proof, which do not produce a term that reduces to itself. In the proof Proposition4, the use of Prop3 can be simply replaced with $=_\beta$ by removing L_1, L_2, and then reset $\mathtt{loop} := \mathtt{lem1}\,\mathtt{p_0}\,\mathtt{lem2}\,\mathtt{lem3}$. Then $\mathtt{loop}$ has type $\bot$, and there exists an infinite reduction path with head redexes of $\mathtt{lem1}$, $\mathtt{2}$, $\mathtt{3}$ [3], which means that redexes (cut-formulae) cannot be removed as follows.

$$\begin{array}{lll}
\mathtt{loop} & \to^+_\beta & \mathtt{lem3}\,\mathtt{x_0}\,\mathtt{lem2}\,(\mathtt{stableX_0}\,\mathtt{p_0}\,\mathtt{lem3}) \\
& \to^+_\beta & \mathtt{lem2}\,\mathtt{p_0}\,(\mathtt{stablep_0}\,\mathtt{x_0}\,\mathtt{lem2})\,(\mathtt{stableX_0}\,\mathtt{p_0}\,\mathtt{lem3}) \\
& \to^+_\beta & \mathtt{lem1}\,(\mathtt{p_0} \circ \delta)\,(\mathtt{stablep_0}\,\mathtt{x_0}\,\mathtt{lem2})\,(\mathtt{stableX_0}\,\mathtt{p_0}\,\mathtt{lem3}) \\
& \to^+_\beta & \mathtt{lem1}\,(\mathtt{p_0} \circ \delta \circ \delta)\,(\mathtt{stablep_0}\,(\delta\,\mathtt{x_0})\,(\mathtt{stablep_0}\,\mathtt{x_0}\,\mathtt{lem2})) \\
& & \qquad (\mathtt{stableX_0}\,(\mathtt{p_0} \circ \delta)\,(\mathtt{stableX_0}\,\mathtt{p_0}\,\mathtt{lem3})) \\
& \to^+_\beta & \cdots\cdots\cdots
\end{array}$$

4. Concluding Remarks

Independently of the concrete structure of the powerful universe, λHOL can prove $\bot$ under the assumption Γ_{weak} or Γ_{ret}, so that λHOL derives $\Gamma_U \vdash \neg Weak$; $\Gamma_U \vdash \neg Ret1$; $\Gamma_U \vdash Ret1 \wedge Ret2 \to Weak$; $\Gamma_U \vdash Ret1 \wedge Weak \to Ret2$; $\Gamma_U \vdash Ret2 \wedge Weak \to Ret1$.

λU^- introduces not only the weakly initial T-algebra [2], but also the dual one, i.e., a weakly final T-coalgebra as instances of the powerful universe.

It should be remarked that natural transformations can be used to characterize functions on final coalgebras having a unique fixed point [6], and this property should be applied to our final T-coalgebra.

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