On symmetric composition algebras and Lie algebras

– 対称的合成代数とリー代数–

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Abstruct: この小論ではある 8 次元代数 (symmetric composition algebra) の all subalgebras and its Lie subalgebras induced from their basis を考察します. そしてこの 8 次元代数は Lie algebras, Jordan algebras, alternative algebras and structurable algebras を含む normal triality algebras と呼ばれる代数系の中に含まれます. これらの応用として bisymmetric spaces を論究します.

§0. Introduction (Motivation)

This article is to introduce a new concept called a normal triality algebra. To describe the concept, first in the case of the complex number **C**, we denote the standard product by $x * y$ and the conjugation by $\overline{x}$, respectively, and define a new product defined by $xy = \overline{x * y}$, here $\sqrt{-1}$ is denoted by i. Also, the notations $Trid\ and\ Trig$ are described in the next section. Then, for $j = 0, 1, 2,$ here the index j is defined by modulo 3 and with respect the product xy, we have

$$(\spadesuit)\ \ Der\ \mathbf{C} \cong \{d \in End\ \mathbf{C} \mid d = \frac{2}{3}n\pi i,\ n = 0, 1, 2, \cdots\},$$

$$Trid\ \mathbf{C} \cong \{(d_0, d_1, d_2) \in (End\ \mathbf{C})^3 \mid\ d_j = \theta_j i,\ \Sigma_{j=0}^{2}\theta_j = 0,\ \text{and}\ \theta_j \in Re\ \mathbf{C}\},$$

$$\text{and we have } Aut\ \mathbf{C} = \{g \in End\ \mathbf{C} \mid g = exp(\frac{2}{3}n\pi i), n = 0, 1, 2, \cdots\} \cong S_3,$$

$$Trig\ \mathbf{C} \cong \{(g_0, g_1, g_2) \in (End\ \mathbf{C})^3 \mid g_j = exp(i\theta_j),\ \theta_1 + \theta_2 + \theta_3 = 0, \theta_j \in Re\ \mathbf{C}\}.$$

Next in a general linear group $A = GL(n, F) := \{x \mid det\ x \neq 0,\ x \in Mat(n, F)\}$ with the standard product $x * y$ of the matrix product over a field F, we introduce g_j with $g_j(a)x = a_{j+1}^{-1} * x * a_{j+2}$, and with respect to the new product $xy = (x * y)^{-1}$, then for $a = (a_1, a_2, a_3) \in\ A^3$ we obtain

$$g_j(a)(xy) = (g_{j+1}(a)x)(g_{j+2}(a)y),$$

where $j = 0, 1, 2,$ This g_j is denoted by

$$g_j(a) = R(a_{j+1})R(a_{j+2}) \text{ and } R(a_j)x = xa_j = (x * a_j)^{-1}.$$

On the basis of ([K.1],[K-O.1]), we call a *"generalized structurable algebra"* with respect to a nonassociative algebra A equipped with endmorphisms $D(x, y)$ as follows:

1) $D(x, y) = -D(y, x),$

2) $D(xy,z)+D(yz,x)+D(zx,y)=0,\ \forall x,y,z\in A$,

3) $D(x,y)$ is a derivation.

Remark. If $A=Fe$, that is, $dim_F\ A=1$, where e is the unit element of A, then the triality group of A is $Trig_F\ A\cong K_4$ (Klein's 4 group), and if $\pi,\ \sqrt{-1}\in F$ with a conjugation, then $Trid_F\ A=\{(0,\pi\sqrt{-1},\pi\sqrt{-1})\}$.

Remark. 微分の一般化の我々の概念が適用可能な Tits による 8 元数の三対原理 (♠♠) については ([S],[T],[Tô] and [K.3] 等) を参照してください.

これらの概念を一般の非結合的代数で以下考察します, リー代数などの代数系も考えますので, 勿論単位元をもたない場合も存在します.

§1. Normal triality algebra

Let A be an algebra over a field F with $ch\ F\neq 2,3$ and the algebra which possess $d_j(x,y)\in End\ A$ for $j=0,1,2,\ \forall x,y\in A$, satisfying $d_j(y,x)=-d_j(x,y)$, $d_1(x,y)=R(y)L(x)-R(x)L(y)$, and $d_2(x,y)=L(y)R(x)-L(x)R(y)$, where $L(x)y=xy,\ R(x)y=yx$, furtheremore, "*principle of triality*";

$$(\spadesuit\spadesuit)\qquad d_j(a,b)(xy)=(d_{j+1}(a,b)x)y+x(d_{j+2}(a,b)y).$$

We call A then be a regular triality algebra, and d_j is called a *triality derivation.* Note that a explicit form for $d_0(x,y)$ is not specified at all. If A satisfies further

$$[d_j(u,v),d_k(x,y)]=d_k(d_{j-k}(u,v)x,y)+d_k(x,d_{j-k}(u,v)\ and$$

$$d_0(x,y)z+d_0(y,z)x+d_0(z,x)y=0,$$

in addition for any $j,k=0,1,2$, then A is called a pre-normal triality algebra.

Next we introduce $Q(x,y,z)\in End\ A$ by

$$Q(x,y,z):=d_0(x,yz)+d_1(z,xy)+d_2(y,zx).$$

We call the pre-normal triality algebra be a **normal triality algebra** if it satisfies $Q(x,y,z)=0\ \forall x,y,z\in A$ in addition.

We denote $Trid\ A=span<d_j(x,y),\forall\ j=0,1,2>$.

The conjugation algebra $A^*$of a normal triality algebra A defined by $x*y=\overline{xy}$ with involution $\overline{xy}=\overline{y}\overline{x}$ satisfying $\overline{d_j(x,y)}=d_{3-j}(\overline{x},\overline{y})$ is called a normal Lie-related triality algebra ([K-O.1][K-O.2]).

Note that if this A^* has an unit element, then A^* is a structurable algebra ([K.1],[K-O.1]), and putting $D=d_0+d_1+d_2$, then D is a derivation.

Proposition 1. *For a normal triality algebra A , if we define*

$$\xi_j=exp\ d_j=\sum_{k=0}^{\infty}\frac{(d_j)^k}{k!}=Id+d_j+\frac{(d_j)^2}{2}+\cdots\ (well-defined)$$

then we have

$$\xi_j(xy)=(\xi_{j+1}x)(\xi_{j+2}y)\quad (global\ triality)$$

$$[\frac{d}{dt}((exp\ td_j)d_k(exp\ td_j)^{-1})]_{t=0}=[d_j,d_k]\in Trid\ A,$$

that is, as a generalization of the derivation of A, this means

$$d_j(xy) = (d_{j+1}x)y + x(d_{j+2}y) \quad (\textit{local triality})$$

Example. The Jordan and Lie algebras are a normal triality algebra, with $\overline{x} = x$ and $\overline{x} = -x$, respectively. The case of Lie algebra with the product $[x, y]$ is $d_j(x, y) = L([x, y]), j = 0, 1, 2$. This triality derivation is the inner derivation.

Example. For the matrix algebra, an alternative algebra and a structurable algebra $(A, x * y)$ with involution, the conjugation algebra is a normal triality algebra, because $x * y = \overline{xy}$ and so $\overline{x * y} = xy$. Here a definition of the structurable algebra is defined by a ternary product (triple system)as follows;

$$< xyz >= (x * \overline{y}) * z + (z * \overline{y}) * x - (z * \overline{x}) * y$$

and it is a (-1,1)Freudenthal-Kantor triple system ([K.1],[K-O.1],[K-O.4]).

§2. Symmetric composition algebra

この章では normal triality algebra ([K-O.1],[K-O.2],[K-O.3]) の典型的な実例である symmetric composition algebra について論究します. 以後, ノーベル賞受賞者の Gell-Mann(quark theory の創始者) の単位元をもたない 8 次元代数 (pseudo octonion) を含む $< x, y >$ を持つこの代数系 ([O],[K.3]) を考えます.

§2.1. Definitions and Results

Let A be an algebra with symmetric bilinear nondegenerate form $< | >$ over the field F of characteristic $\neq 2$. Suppose that we have

$$(\sharp) \quad x(yx) = (xy)x =< x|x > y, \ \ \forall x, y \in A.$$

Then A is known as a *symmetric composition algebra* (SCA), since it satisfies

$$(\sharp\sharp) \quad < xy|xy >=< x|x >< y|y > \ \textit{and} \ < xy|z >=< x|yz >, \ \forall x, y, z \in A.$$

Remark. Note that $(\sharp) \Longleftrightarrow_{iff} (\sharp\sharp)$, that is, this means that the relation of composition law $||xy|| = ||x||||y||$ is equivalent to $< xy|xy >=< x|x >< y|y >$, because $< x|x >= ||x||^2$.

For symmetric composition algebra, we have two Propositions as follows.

Proposition 2.1([O],[K.3]). *Any symmetric composition algebra over a field of ch* $F \neq 2$, 3 *is limited to be either*

(a) a para-Hurwitz algebra, or

(b) an eight dimensional pseudo octonion algebra,

where the para-Hurwitz algebra is the conjugation algebra of Hurwitz algebra (i.e., **R**, **C**, **H** *(quaternion),* **O** *(octonion) if ch* $F = 0$*), and the pseudo octonion algebra is induced from Gell-Mann's any traceless* 3×3 *matrix ([G],[K.3]).*

Proposition 2.2([K-O.2],[K.3]). *Any symmetric algebra A is a normal triality algebra. In particular, we have relations;* $d_1(x, y) = R(y)L(x) - R(x)L(y)$, $d_2(x, y) = L(y)R(x) - L(x)R(y)$ *and* $d_0(x, y)z = 2\{[L(x), L(y)] - R[x, y]\}z = 4\{< x, z > y- < y, z > x\}$, *furthermore,* $< d_j(x, y) >_{span} \cong D_4$, *and the local triality relation* (♠♠) *that is,* $d_j(xy) = (d_{j+1}x)y + x(d_{j+2}y), \ j = 0, 1, 2.$

Note that the *"principle of triality"* due to Tits is a special case of this proposition. These imply that there is a generalization of the derivation.

For a generalization of the symmetric composition algebra, without assuming the nondegenerate property of $< x|y >$, it is said to be a *generalized symmetric composition algebra*, we denote by GSCA.

Example ([K.2],[K.4]). Examples of GSCA over a field $\mathbf{Z}_p$ (finite field) are $\mathbf{Z}_p[\sqrt{q}], \mathbf{Z}_p[\sqrt{q}, \sqrt{r}]\ (qr \equiv 1\ mod\ p), \mathbf{Z}_p[i,j,k], and\ \mathbf{Z}_p[e_1, e_2, \cdots, e_7]$.

§2.2. Multiplication tables

SCA の８次元代数の基底に関する乗積表 (para and pseudo octonion algebras) はそれぞれ次の表で与えられ ([K.4]), 内積は $x = \sum_j x_j e_j,\ y = \sum_j y_j e_j$ のとき $< x|y >= \sum_j x_j y_j$ で定義されます, ただし $< e_i|e_j >= \delta_{i,j}$ (クロネッカーの δ).

$(*)$ para octonion の基底: $e_1, \cdots, e_7$ の積の例, $e_1e_2 = -e_3, e_7e_1 = -e_6, e_6e_7 = -e_1$.

para case	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	$-e_3$	e_2	$-e_5$	e_4	$-e_7$	e_6
e_2	e_3	-1	$-e_1$	e_6	$-e_7$	$-e_4$	e_5
e_3	$-e_2$	e_1	-1	$-e_7$	$-e_6$	e_5	e_4
e_4	e_5	$-e_6$	e_7	-1	$-e_1$	e_2	$-e_3$
e_5	$-e_4$	e_7	e_6	e_1	-1	$-e_3$	$-e_2$
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	$-e_6$	$-e_5$	$-e_4$	e_3	e_2	e_1	-1

where $e_0(= 1)$ is the unit element, and omit from the multiplicative table, also since $\forall x,\ e_0x = \overline{x},\ e_0$ is said to be a *para-unit*. 上の表は para octonion の conjugation algebra が octonion（Cayley number) であることを示しています.

Note that the product $[e_j, e_k] := e_je_k - e_ke_j,\ (j,\ k = 1, \cdots, 7)$ makes a Malcev algebra with seven dimension ([O]). Indeed, $(e_ie_j)e_i = e_j$ in SCA.

$(**)$ pseudo octonion の基底: $e_1, \cdots, e_8$ の積の例, $e_1e_2 = e_3,\ e_8e_1 = e_1, e_8e_8 = -e_8$.

pseudo case	e_1	e_2	e_3	e_4
e_1	e_8	e_3	$-e_2$	$\frac{\sqrt{3}}{2}e_6 + \frac{1}{2}e_7$
e_2	$-e_3$	e_8	e_1	$\frac{1}{2}e_6 - \frac{\sqrt{3}}{2}e_7$
e_3	e_2	$-e_1$	e_8	$\frac{\sqrt{3}}{2}e_4 + \frac{1}{2}e_5$
e_4	$\frac{\sqrt{3}}{2}e_6 - \frac{1}{2}e_7$	$-\frac{1}{2}e_6 - \frac{\sqrt{3}}{2}e_7$	$\frac{\sqrt{3}}{2}e_4 - \frac{1}{2}e_5$	$\frac{\sqrt{3}}{2}e_3 - \frac{1}{2}e_8$
e_5	$\frac{1}{2}e_6 + \frac{\sqrt{3}}{2}e_7$	$\frac{\sqrt{3}}{2}e_6 - \frac{1}{2}e_7$	$\frac{1}{2}e_4 + \frac{\sqrt{3}}{2}e_5$	$-\frac{1}{2}e_3 - \frac{\sqrt{3}}{2}e_8$
e_6	$\frac{\sqrt{3}}{2}e_4 - \frac{1}{2}e_5$	$\frac{1}{2}e_4 + \frac{\sqrt{3}}{2}e_5$	$-\frac{\sqrt{3}}{2}e_6 + \frac{1}{2}e_7$	$\frac{\sqrt{3}}{2}e_1 - \frac{1}{2}e_2$
e_7	$\frac{1}{2}e_4 + \frac{\sqrt{3}}{2}e_5$	$-\frac{\sqrt{3}}{2}e_4 + \frac{1}{2}e_5$	$-\frac{1}{2}e_6 - \frac{\sqrt{3}}{2}e_7$	$-\frac{1}{2}e_1 - \frac{\sqrt{3}}{2}e_2$
e_8	e_1	e_2	e_3	$-\frac{1}{2}e_4 + \frac{\sqrt{3}}{2}e_5$

pseudo case	e_5	e_6	e_7	e_8
e_1	$-\frac{1}{2}e_6 + \frac{\sqrt{3}}{2}e_7$	$\frac{\sqrt{3}}{2}e_4 + \frac{1}{2}e_5$	$-\frac{1}{2}e_4 + \frac{\sqrt{3}}{2}e_5$	e_1
e_2	$\frac{\sqrt{3}}{2}e_6 + \frac{1}{2}e_7$	$-\frac{1}{2}e_4 + \frac{\sqrt{3}}{2}e_5$	$-\frac{\sqrt{3}}{2}e_4 - \frac{1}{2}e_5$	e_2
e_3	$-\frac{1}{2}e_4 + \frac{\sqrt{3}}{2}e_5$	$-\frac{\sqrt{3}}{2}e_6 - \frac{1}{2}e_7$	$\frac{1}{2}e_6 - \frac{\sqrt{3}}{2}e_7$	e_3
e_4	$\frac{1}{2}e_3 + \frac{\sqrt{3}}{2}e_8$	$\frac{\sqrt{3}}{2}e_1 + \frac{1}{2}e_2$	$\frac{1}{2}e_1 - \frac{\sqrt{3}}{2}e_2$	$-\frac{1}{2}e_4 - \frac{\sqrt{3}}{2}e_5$
e_5	$\frac{\sqrt{3}}{2}e_3 - \frac{1}{2}e_8$	$-\frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2$	$\frac{\sqrt{3}}{2}e_1 + \frac{1}{2}e_2$	$\frac{\sqrt{3}}{2}e_4 - \frac{1}{2}e_5$
e_6	$\frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2$	$-\frac{\sqrt{3}}{2}e_3 - \frac{1}{2}e_8$	$-\frac{1}{2}e_3 + \frac{\sqrt{3}}{2}e_8$	$-\frac{1}{2}e_6 - \frac{\sqrt{3}}{2}e_7$
e_7	$\frac{\sqrt{3}}{2}e_1 - \frac{1}{2}e_2$	$\frac{1}{2}e_3 - \frac{\sqrt{3}}{2}e_8$	$-\frac{\sqrt{3}}{2}e_3 - \frac{1}{2}e_8$	$\frac{\sqrt{3}}{2}e_6 - \frac{1}{2}e_7$
e_8	$-\frac{\sqrt{3}}{2}e_4 - \frac{1}{2}e_5$	$-\frac{1}{2}e_6 + \frac{\sqrt{3}}{2}e_7$	$-\frac{\sqrt{3}}{2}e_6 - \frac{1}{2}e_7$	$-e_8$

Here for the pseudo case, assume that $\sqrt{3} \in F$.

以後 para octonion を $\mathbf{O}$, pseudo octonion (or Okubo algebra) を $\mathbf{O}_p$ と書く.

Remark. Note that $Der\ \mathbf{O} \cong G_2$ and $Der\ \mathbf{O}_p \cong A_2$, furthermore with respect to the product $[x, y] = xy - yx$, $(\mathbf{O}, [x, y])$ is said to be a *Malcev admissible algebra.* Also, $(\mathbf{O}_p, [x, y])$ is said to be a *Lie admissible algebra*, and the structure is an isomorphism to A_2 (eight dimensional Lie algebra). These (the cases of para and pseudo) have an application to mathematical physics ([O]).

§2.3. Linear dependency of triality derivations $d_j(x, y),\ j = 0, 1, 2$

この節では symmetric composition algebra の triality derivation の $d_1(x, y)$ により 他の $d_2(x, y)$ and $d_0(x, y)$ が 1 次結合で表されることを述べます.

8 次元代数の para octonion を $\mathbf{O}$, pseudo octonion を $\mathbf{O}_p$ の記号で表し, 同様に 4 次元の para quaternion, pseudo quaternion も定義されます, $\mathbf{O}$ を (a)type, $\mathbf{O}_p$ を (b)type と呼ぶ. また b) type is called a *Okubo algebra*.

$d_1(x, y),\ d_2(x, y),\ d_0(x, y)$ の関係については $d_2(e_i, e_k)$ and $d_0(e_i, e_k)$ が $d_1(e_i, e_k)$ の一次結合 で表示可能を示します. $\forall x, y \in \mathbf{O}_p$ (pseudo octonion) のとき $d_1(x, y)$ は $d_1(e_i, e_j)$ $(1 \leq i \neq j \leq 8)$ なる ${}_8C_2 = 28$ 個の基底を持つ $End\ \mathbf{O}_p$ の元であり normal triality algebra $([d_1(a, b), d_2(c, d)] \in < d_1(e_i, e_j) >_{span})$ の性質より Lie algeba D_4 with 28 dim を生成します. 従って $d_2(x, y)$ も 28 次元の $End\ \mathbf{O}_p$ の元ですので $d_1(x, y)$ の基底の 1 次結合により表示可能です. そして normal triality algebra なので $d_0(x, yz) + d_1(y, zx) + d_2(z, xy) = 0$ が成り立ち, この式より, $d_0(e_i, e_k e_8) + d_1(e_k, e_8 e_i) + d_2(e_8, e_i e_k) = 0$ となり, 乗積表より $1 \leq k \leq 3$ ならば $e_k e_8 = e_8 e_k = e_k$ and $4 \leq k \leq 7$ ならば $e_k e_8 + e_8 e_k = -e_k$, $e_8 e_8 = -e_8$ より $d_0(e_i, e_k)$ は $d_0(e_i, e_8 e_k + e_k e_8)$ で表されてこれらは $d_1(e_i, e_j)$ の一次結合で表示可能です, つまり $d_2(x, y), d_0(x, y)$ が $d_1(e_i, e_j)$ で表せます.
para case of (a): e_8 の代わりに e_0 を用いれば同様に計算が可能ですので省略.

以上をまとめると次の定理が示せます.

Proposition 2.3. *Let A be any eight dimensional symmetric composition algebra. For $(d_0, d_1, d_2) \in Trid\ A$, then we have*
$d_0(x, y), d_1(x, y), d_2(x, y)$ are linearly dependent in End A.
That is, $d_2(e_i, e_k)$ and $d_0(e_i, e_k)$ are represented by the linear combinations of $d_1(e_i, e_k)$, and dim $< d_j(x, y) | x, y \in A >_{span} = 28$ (j=0,1,2), these imply, $< d_j(x, y) >_{span} \cong$ D_4(28 dimensional simple Lie algebra) and satisfies $< d_j(x, y)u | v > + < u | d_j(x, y)v >= 0$. Furthermore, $< d_j(x, y) >_{span}$ is an ideal of $Trid\ A$.

Remark. *If A is a normal triality algebra satisfying $AA = A$ and involutive, then, $d_0(x, y)$ is unique determined by $d_1(x, y),\ d_2(x, y),\ \forall\ x, y \in A$.*

§3. All subalgebras of symmetric composition algebra

§3.1. All subalgebras of symmetric composition algebras

この節では *Symmetric composition algebra A over a field F with $dim_F\ A = 8$* の *trivial* でない *subalgebra* B を考察する, 特に *para case* を B *and pseudo case* を B_p と区別してかく. その時,$1 \leq dim\ B \leq 7$, *and* $1 \leq\ dim\ B_p \leq 7$ であり, 乗積表より次の結果が成り立つ. ここでは正規直交基底で考察しています.

Proposition 3.1. *Let A be a symmetric composition algebra over a field F. Then there is not a subalgebra of A with dimension 3, 5, 6 and 7.*

実際 (♯) para type (a): $dim\ B = 1$ のとき　$B \cong Fe_0$, e_0 は単位元です.

$dim\ B = 2$ のとき　B は $< e_0, e_j >_{span}$, and e_j は $(1 \leq j \leq 7)$ の一つで, 同型な 7 組が存在, 基礎体 F が実数の場合は para complex number であり Introduction の複素数の new product の場合 (example) and $e_j^2 = -e_0$ です.

$dim\ B = 4$ のとき　B は $< e_0, e_1, e_2, e_3 >_{span}$, $< e_0, e_1, e_4, e_5 >_{span}$, $< e_0, e_1, e_6, e_7 >_{span}$, $< e_0, e_2, e_5, e_7 >_{span}$, $< e_0, e_2, e_4, e_6 >_{span}$, $< e_0, e_3, e_4, e_7 >_{span}$, ($< e_0, e_3, e_5, e_6 >_{span}$, これらは a para quaternion algebra (the conjugation algebra of quaternion number) と呼ばれる同型な 7 組が乗積表より得られます.

(♯♯) pseudo type (b): $dim\ B_p = 1$ のとき $B_p \cong Fe_8$, where $(-e_8)(-e_8) = -e_8$.
$dim\ B_p = 2$ のとき　$< e_1, e_8 >_{span}$,$< e_2, e_8 >_{span}$,$< e_3, e_8 >_{span}$, 同型な 3 組が乗積表より得られます (called a *pseudocomplexnumber*).

$dim\ B_p = 4$ のとき　2 次元の subalgebra $< e_3, e_8 >_{span}$ を持つ, $< e_1, e_2, e_3, e_8 >_{span}$,$< e_4, e_5, e_3, e_8 >_{span}$, $< e_6, e_7, e_3, e_8 >_{span}$, の同型な 3 組が得られます (called pseudo quaternion by the author's naming).□

§3.2. All Lie subalgebras induced from symmetric composition algebras

para case (a) のときは　**O** は $[x, y] = xy - yx$ の積で　Lie でなく,Malcev algebra になりますのであまり興味がないかもしれませんので別の機会に述べます.

Proposition 3.2. *Let A be a pseudo octonion algebra (or Okubo algebra) and for the Lie admissible algebra $(A, [x, y])$, then we have, there is not a Lie subalgebra with dimension 5,6,7. That is, there only exist 1,3,4 dimension.*

実際, $dim\ (B_p, [\ ,\ ]) = 1$ のとき $B_p \cong Fe_8$.
$dim\ (B_p, [\ ,\ ]) = 2$ のとき乗積表より　B_p は $< e_1, e_8 >_{span}$,$< e_2, e_8 >_{span}$, $< e_3, e_8 >_{span}$, から生成される 2 次元の abel Lie subalgebra of $(\mathbf{O}_p, [\ ,\ ])$ です.
$dim\ (B_p, [\ ,\ ]) = 3$ のとき B_p は　$< e_1, e_2, e_3 >_{span}$, $< e_1, e_4, e_7 >_{span}$, $< e_1, e_5, e_6 >_{span}$, $< e_2, e_4, e_6 >_{span}$, $< e_2, e_5, e_7 >_{span}$, $< e_3, e_4, e_5 >_{span}$, $< e_3, e_6, e_7 >_{span}$, から生成される 7 組の 3 次元の Lie subalgebra of $(\mathbf{O}_p, [\ ,\ ])$ です, そしてそれらは $\Lambda(x, y, z) :=< x, y, z | [x, y] = z, [y, z] = x, [z, x] = y >_{span}$ なる外積より定義される 3 次元の Lie algebra と同型です.
$dim\ (B_p, [\ ,\ ]) = 4$ のとき B_p は　$< e_1, e_2, e_3, e_8 >_{span}$, $< e_3, e_4, e_5, e_8 >_{span}$, $< e_3, e_6, e_7, e_8 >_{span}$, 3 個の 4 次元の Lie subalgebra of $(\mathbf{O}_p, [\ ,\ ])$ が存在します. $< e_1, e_2, e_3, e_8 >_{span}$ のときのみ, $[e_8, e_1] = [e_8, e_2] = [e_8, e_3] = 0$, だから, e_8 が center であり, それは $Fe_8 \oplus \Lambda(x, y, z)$ と同型です.□

Remark. (b)pseudo case: $(\mathbf{O}_p, [\ ,\ ]) \simeq A_2$. We create the Lie subalgebras.
(a)para case:　(Lie algebra を作るとき)　$dim\ (B, [\ ,\ ]) = 1$ のときは Fe_0 です. $dim\ (B, [\ ,\ ]) = 2$ のときは $< e_0, e_i >_{span}$, e_i は $(1 \leq i \leq 7)$ のどれかです.
$dim\ (B, [\ ,\ ]) = 4$ のときは $< e_0, e_1, e_2, e_3 >_{span}$, $< e_0, e_1, e_4, e_5 >_{span}$, $< e_0, e_1, e_6, e_7 >_{span}$, $< e_0, e_2, e_4, e_6 >_{span}$, $< e_0, e_2, e_5, e_7 >_{span}$, $< e_0, e_3, e_4, e_7 >_{span}$, $< e_0, e_3, e_5, e_6 >_{span}$, が作る 7 個の Lie algebra です. $Fe_0 \oplus \Lambda(x, y, z)$ と同型です. 勿論 e_0 is the center, since $[e_{0,}, e_i] = 0$.
$dim\ (B, [\ ,\ ]) = 3$ のときは　上の 4 次元の場合で e_0 を除いたものが作る 7 個の Lie algebra で $\Lambda(x, y, z)$ と同型です. $(\mathbf{O}, [\ ,\ ])$ は Lie algebra を作らない.
A Cayley algebra is the conjugation algebra of a para-Hurwitz algebra with eight dimension（[O],[K-O.2]). そして 8 次元の交代代数の一種である Cayley 代数については E. Artin による次の定理が知られています ([S]).

Proposition 3.3. *The subalgebra generated by any two elements of alternative algebra* $\mathfrak{A}$ *is associative.*

実際 para Hurwitz algebra $\mathbf{O}$ の conjugation algebra は Hurwitz algebra であり, Hurwitz algebra の一種の Cayley algebra は alternative algebra であることより, Proposition 3.3 を用いて type(a) の conjugation algebra が, つまり 8 次元の Cayley algebra が この性質を満足します. しかしながら, pseudo algebra $\mathbf{O}_p$ の場合は乗積表より $(e_1e_1)e_2 = e_2 \neq e_1(e_1e_2) = -e_2$ and $(e_4e_5)e_5 = -e_4 \neq e_4(e_5e_5) = e_4$, 等から Artin's theorem を満たさない代数系です. 更に para octonion は para unit を持ちますが, pseudo octonion は unit(単位元) も para unit も持ちません. つまり, symmetric composition algebra において (a) type (para case) and (b) type (pseudo case) の代数系では 性質が異なります. しかし同じ性質を所有する場合も存在します. 次節で説明します.

§3.3. Miscellaneous results

この節でいくつかの symmetric composition algebras ([K-O.2], [K-O.3], [K.3]) についての結果を述べます. That is, para and Okubo algebra cases です.

Let $\Sigma = \{a = (a_1, a_2, a_3) \in A^3 | a_j a_{j+1} = a_{j+2},\ < a_j, a_j >= 1, \forall j = 0, 1, 2\}$, and put $\sigma_j(a) = R(a_{j+1})R(a_{j+2})$, $\theta_j(a) = L(a_{j+2})L(a_{j+1})$. Also we set $G :=< \sigma_j(a)\theta_j(b)\ and\ \theta_j(a)\sigma_j(b) | \forall\ \ a,\ b \in \Sigma >_{gen}$. Then we have the following.

Proposition 3.4([K-O.3]). *Let A be a symmetric composition algebra. Then*

$$\sigma_j(a)(xy) = (\sigma_{j+1}(a)x)(\sigma_{j+2}(a)(y)),\ \theta_j(a)(xy) = (\theta_{j+1}(a)x)(\theta_{j+2}(a)y),$$

$$< \sigma_j(a)x, y >=< x, \theta_j(a)y >,$$

and furthermore, G is a invariant subgroup of $Trig\ A$.

A が symmetric composition algebra with $dim\ A = 2$ の場合, $Aut\ A \cong S_3$ if $\sqrt{3} \in F$, $Trig\ A$ is a infinite group if F is infinite.

A be a symmetric composition algebra. The two cases of para and pseudo have a subalgebra $B =< e, f >_{span}$ of A eqipped with $e, f \in A$ satisfying

$$ee = e, ff = -e, ef = fe = -f, < e, e >=< f, f >= 1, < e, f >=< f, e >= 0.$$

The case of para is $e = e_0$ and $f = one\ of\ e_j\ (1 \leq j \leq 7)$, the case of pseudo is $e = -e_8$ and $f =\ one\ of\ e_1, e_2, e_3$. Any $x, y \in B$ satisfies the quadratic relation; $xy = yx = - < e, x > y - < e, y > x + (4 < e, x >< e, y > - < x, y >)e$. Hence this subalgebra B is a quadratic algebra with the para-unit element.

§4. Magic square tables and bisymmetric spaces

§4.1. Magic square table of Lie algebras

Following ([K-O.2],[K.4],[K.5] and references of therein), triple systems or normal triality algebras A に附随した five graded exceptional Lie algebras $L(A)$ を考察する. 以後, 基礎体 F は標数 0 の代数閉体とする.

I) $A = \mathbf{O} \otimes \mathbf{O}$ の場合 (tensor product case , and $dim\ g_{-2} = dim\ g_2 = 14$) $A = A_1 \otimes A_2,\ dim\ A_1, dim\ A_2$ なる記号を用いると, それぞれ construct される Lie algebra は以下の様になり, そして $A = \mathbf{O} \otimes \mathbf{O}$ の subalgebras も考察する.

$L(A) = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2 \cong E_8,\ g_{-2} \oplus g_0 \oplus g_2 \cong D_8, g_0 \cong D_7 \oplus gl(1), A = g_{-1}$

$dimA_2 \backslash dimA_1$	1	2	4	8
1	A_1	A_2	C_3	F_4
2	A_2	$A_2 \oplus A_2$	A_5	E_6
4	C_3	A_5	D_6	E_7
8	F_4	E_6	E_7	E_8

E_8 を extended Dynkin diagram で表すと $g_{-1} \oplus g_1 \cong L(A)/(g_{-2} \oplus g_0 \oplus g_2) = E_8/D_8$ は 128 次元の対称空間;

◎ − ○ − ○ − ○ − ○ − ○ − ○−$\boxed{\circ}$, $\boxed{\circ}$ omitted ≅ D_8, and ◎ is highest root.
|
○

II) $A = \begin{pmatrix} \alpha & \mathbf{a} \\ \mathbf{b} & \beta \end{pmatrix}$ の場合 (balanced case, and $dim\ g_{-2} = dim\ g_2 = 1$)

$L(A) \cong E_8,\ g_{-2} \oplus g_0 \oplus g_2 \cong E_7 \oplus A_1,\ g_0 \cong E_7 \oplus gl(1), A = g_{-1}$.
ここで $H_3(\mathbf{O}) \rightarrow H_3(\mathfrak{A})(= B)$ なる記号に置き換える. ただし $\mathfrak{A}$ は Hurwitz algebras over F. そして $\forall \begin{pmatrix} \alpha & \mathbf{a} \\ \mathbf{b} & \beta \end{pmatrix} \in \begin{pmatrix} F & B \\ B & F \end{pmatrix} = A$ とすると $dimB$ に対応して, その Lie algebra は以下の様になります.

$dimB$	1	6	9	15	27
$dimA$	4	14	20	32	56
$dimL(A)$	14	52	78	133	248
$L(A)$	G_2	F_4	E_6	E_7	E_8

この場合の construct された E_8 を extended Dynkin diagram で表すと $g_{-1} \oplus g_1 \cong L(A)/(g_{-2} \oplus g_0 \oplus g_2) = E_8/(A_1 \oplus E_7)$ は 112 次元の対称空間;

◎− $\boxed{\circ}$− ○ − ○ − ○ − ○ − ○ −○, $\boxed{\circ}$ omitted ≅ $A_1 \oplus E_7$, ◎ is highest root.
|
○

Remark. 56 次元の balanced な空間 A は最初に F.Hreudenthal による metasymmetric geometry の研究に現れ, normal triality algebra or triple system により特徴ずけが可能な代数的な structure を持ち, generalized Zorn's vector matrix と呼ばれています ([K-O.2],[K-O.3]).

§4.2. Bisymmetric spaces associated with exceptional Lie algebras

リー群の接空間がリー代数と対応する関係と同様に対称空間の接空間がリー三項系代数であるのでここでは三項系で考察します. つまり Bisymmetric space の定義は次のような $(B_\alpha, B_\beta, B_\gamma, B_\delta)$ の Lie triple system の pair です:

(I) *dim* B_δ/*dim* B_γ = *dim* B_γ/*dim* B_β = *dim* B_β/*dim* $B_\alpha = 2$, *and*

$$B_\alpha < B_\beta < B_\gamma < B_\delta$$

as Lie triple subsystem's series of the Lie triple system $g_{-1} \oplus g_1$ of 5-graded Lie algebra $g = L(A) = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2$ associated the normal triality algebra $A = g_{-1}$ and $Der\ (g_{-1} \oplus\ g_1) \cong g_{-2} \oplus g_0 \oplus g_2$.

従って, § 4.1 で述べた結果を用いて (I) type の場合, 次のような bisymmetric spaces の series が得られます. これを Type (I) の Bisymmetric spaces と呼ぶ.

$$(\heartsuit)\quad F_4/B_4 < E_6/(D_5 \oplus gl(1)) < E_7/(D_6 \oplus A_1) < E_8/D_8$$

and the dimensin of bisymmetric spaces of type (I);

$$16, \quad 32, \quad 64, \quad 128 \qquad respectively.$$

extended Dynkin diagram (対称空間) の E_7, E_6, F_4 は以下の様になります;

○–[○]– ○ – ○ – ○ – ○ –◎, [○] omitted $\cong D_6 \oplus A_1$, and ◎ is highest root.
|
○

[○] – ○ – ○ – ○ – [○], [○] omitted $\cong D_5 \oplus gl(1)$, and ◎ is highest root.
|
○
|
◎

◎ – ○ –○ => ○– [○], [○] omitted $\cong B_4$, and ◎ is highest root.

(II) type の場合 (balanced case) も同様の結果が構成可能です. しかし

$$(II) \quad (dim\ B_\delta + 16)/dim\ B_\gamma = (dim\ B_\gamma + 16)/dim\ B_\beta =$$

$$(dim\ B_\beta + 16)/dim\ B_\alpha = 2,\ and\ B_\alpha < B_\beta < B_\gamma < B_\delta$$

と定義する. そして Type(II) の Bisymmetric space と呼ぶことにします.

$$(\heartsuit\heartsuit) \quad F_4/(C_3 \oplus A_1) < E_6/(A_5 \oplus A_1) < E_7/(D_6 \oplus A_1) < E_8/(E_7 \oplus A_1)$$

and the dimension of bisymmetric spaces of type (II) ;

$$28, \quad 40, \quad 64, \quad 112 \qquad respectively.$$

type (II) の extended Dynkin diagram（対称空間）の E_7, E_6, F_4 の場合;

○–○– ○ – ○ – ○ – [○] –◎, [○] omitted $\cong D_6 \oplus A_1$, and ◎ is highest root.
|
○

○ – ○ – ○ – ○ – ○, [○] omitted $\cong A_5 \oplus A_1$, and ◎ is highest root.
|
[○]
|
◎

◎ – [○]–○ => ○– ○, [○] omitted $\cong C_3 \oplus A_1$. and ◎ is highest root.

Here $A_1, A_5, B_4, C_3, D_5,$ and D_6 mean classical simple Lie algebras.

Remark. Type (II) においての $L(A)/g_0$ として現れる vector spaces の series;

$$F_4/(C_3 \oplus gl(1)) < E_6/(A_5 \oplus gl(1)) < E_7/(D_6 \oplus gl(1)) < E_8/(E_7 \oplus gl(1))$$

$(dim\ B_\delta + 18)/dim\ B_\gamma = (dim\ B_\gamma + 18)/dim\ B_\beta = (dim\ B_\beta + 18)/dim\ B_\alpha = 2,$ $and\ B_\alpha < B_\beta < B_\gamma < B_\delta$ も bivector space としてある種の代数構造が存在 (bireductive homogeneous spaces) and their dimensions are 30, 42, 66, 114, respectively, normal triality の拡張概念としての algebraic structure が存在していると考えています. —-今後の研究課題の一つです.

Concluding Remark. One of our fundamntal philosophy is to study the construction of 5-graded Lie (super)algebras $g = g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2$, satisfying $[g_i, g_j] \subseteq g_{i+j}$without using root systems and Cartan matrix.

筆者の最近の論文 ([K.5]) が収録されています Springer より出版されたこの本 (国際会議報告集) は非結合的代数の分野の研究動向を知るためには役に立つと考えますので興味のある方はご参照下さい. また筆者の力量不足と時間的な制約のために英語と日本語の混在した論文になりました, ご寛容ください.

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