A numerical semigroup whose genus described by its minimum odd is large [1]

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Abstract

The main theorem in this article is a generalization of the result of [2]. We consider Weierstrass semigroups H attained by ramification points of double coverings of trigonal curves. Especially, we characterize H whose genus described by its minimum odd is large. We give a necessary and sufficient condition for such a numerical semigroup H to be non-Weierstrass.

1 Terminologies and introduction

Let $\mathbb{N}_0$ be the additive monoid of non-negative integers. A submonoid H of $\mathbb{N}_0$ is called a *numerical semigroup* if its complement $\mathbb{N}_0 \backslash H$ is finite. The cardinality of $\mathbb{N}_0 \backslash H$ is called the *genus* of H, denoted by $g(H)$. We denote by $m(H)$ the minimum positive integer in H, which is called the *multiplicity* of H. In this paper H always stands for a numerical semigroup. We set

$$d_2(H) = \{h \in \mathbb{N}_0 \mid 2h \in H\},$$

which is a numerical semigroup. Let n be the minimum odd integer in H. Then we get

$$g(H) \leqq 2g(d_2(H)) + \frac{n-1}{2}.$$

If the above equality holds, then we get

$$H = 2d_2(H) + \langle n \rangle,$$

where $\langle a_1, \ldots, a_t \rangle := a_1\mathbb{N}_0 + \cdots + a_t\mathbb{N}_0$ for positive integers $a_1, \ldots, a_t$. We are interested in H satisfying one of the following:

$$(1)\ \ g(H) = 2g(d_2(H)) + \frac{n-1}{2} - 1,$$

$$(2)\ \ g(H) = 2g(d_2(H)) + \frac{n-1}{2} - 2.$$

[1] This paper is an extended abstract and the details will be published (see [3])

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We set

$$c(H) = \min\{h \in H \mid h + \mathbb{N}_0 \subseteqq H\}.$$

which is called the *conductor* of H. The integer $c(H) - 1$ is called the *Frobenius number* of H. An element $\gamma \in \mathbb{N}_0 \backslash H$ is said to be a *pseudo-Frobenius number* of H if $\gamma + h \in H$ for any positive integer $h \in H$. We denote by $PF(H)$ the set of the pseudo-Frobenius numbers of H.

We are interested in whether a numerical semigroup is gained by algebraic geometry objects. A *curve* means a projective non-singular irreducible algebraic curve over an algebraically closed field k of characteristic 0. For a pointed curve (C, P) we set

$$H(P) = \{\alpha \in \mathbb{N}_0 \mid \exists f \in k(C) \text{ such that } (f)_\infty = \alpha P\},$$

where $k(C)$ is the field of rational functions on C. $H(P)$ is a numerical semigroup of genus $g(C)$ where $g(C)$ is the genus of C, which is called the *Weierstrass semigroup of* P.

Theorem 2.1 *Let n be the minimum odd integer in H. Assume that*

$$n \geqq c(d_2(H)) + m(d_2(H)) - 1 \text{ and } H = 2d_2(H) + \langle n, n+2\rangle.$$

Then the following are equivalent:

(i) $g(H) = 2g(d_2(H)) + \dfrac{n-1}{2} - 1$

(ii) $1 \in PF(d_2(H))$

(iii) $d_2(H)$ *is ordinary, that is to say,* $d_2(H) = \langle g+1, g+1+1, \ldots, g+1+g\rangle$*, which is the Weierstrass semigroup* $H(P)$ *of any ordinary point* P *of a curve of genus* g*.*

Theorem 2.2 *Let n be the minimum odd integer in H. Assume that*

$$n \geqq c(d_2(H)) + m(d_2(H)) - 1 \text{ and } H = 2d_2(H) + \langle n, n+2\rangle.$$

Then the following are equivalent:

(i) $g(H) = 2g(d_2(H)) + \dfrac{n-1}{2} - 2$

(ii) $d_2(H) = \{0, m \longrightarrow m+s-g, s+2 \longrightarrow\}$ *where* $m := m(d_2(H))$*,* $g := g(d_2(H))$ *and* $s \in \mathbb{N}_0$ *with* $g \leqq s \leqq 2g-2$ *and* $\left[\dfrac{s+1}{2}\right] + 1 \leqq m \leqq g$*. Here,* [ ] *means the Gauss symbol.*

Remark 2.3 *The following result is due to Kim [1]:*
Let C be a trigonal curve, that is to say, a three sheeted covering of the projective line $\mathbb{P}^1$, of genus $g \geqq 5$. Let $P \in C$ be a non-ramification point, but a non-ordinary point on C.

Then the Weierstrass semigroup $H(P)$ is $\{0, m \longrightarrow m+s-g, s+2 \longrightarrow\}$ for some m such that $\left[\frac{s+1}{2}\right]+1 \leqq m \leqq g$ with $g \leqq s \leqq 2g-2$. Moreover, each sequence actually occurs as the Weierstrass semigroup of some non-ramification point on a trigonal curve of genus g. This semigroup coincides with $d_2(H)$ in Theorem 2.2

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Theorem 3.1 *Assume that n is an odd integer satisfying $n \geqq c(d_2(H)) + m(d_2(H)) - 1$ and $H = 2d_2(H) + \langle n, n+4\rangle$. Assume that $m := m(d_2(H))$ is odd. Then the following are equivalent:*

(i) $g(H) = 2g(d_2(H)) + \frac{n-1}{2} - 1$.

(ii) $d_2(H) = \langle m, m+1, \cdots, m+m-1\rangle$ *or*
$\langle m, m+2, m+4, \cdots, m+m-1, m+2j+1, m+2(j+1)+1, \cdots, m+2((m-3)/2)+1$
$2m+1, 2m+3, \cdots, 2m+2(j-1)+1\rangle$ *for some j with $1 \leqq j \leqq (m-3)/2$ or*
$\langle m, m+2, m+4, \cdots, m+m-1, 2m+1, 2m+3, \cdots, 2m+m-2\rangle$.

Remark 3.2 *The following result is due to Kim [1]:*
Let C be a trigonal curve of genus $g \geqq 5$. Let $P \in C$ be a ramification point that is not total. Assume that $m(H(P))$ is odd. Then the set $\mathbb{N}_0 \backslash H(P)$ is

$$\{1, 2, 3, \cdots, 2n-1, 2n, 2n+2, 2n+4, \cdots, 2g-2n\}$$

with $\frac{g-1}{3} < n \leqq \frac{g}{2}$ where g is the genus of C. Moreover, each sequence actually occurs as the Weierstrass semigroup of some non-total ramification point on a trigonal curve of genus g.

We set $n = \frac{m(d_2(H))-1}{2}$. We set $g = m(d_2(H)) - 1$ in the first case in Theorem 3.1 (ii) and $g = m(d_2(H)) - 1 + j$ in the second case in Theorem 3.1 (ii). We set $g = \frac{3m(d_2(H))-3}{2}$ in the third case in Theorem 3.1 (ii). Then $d_2(H)$ in Theorem 3.1 (ii) coincides with $H(P)$ in the above Known result.

Theorem 3.3 *Let n be the minimum odd integer in H. Assume that $n \geqq c(d_2(H)) + m(d_2(H)) - 1$. Let $H = 2d_2(H) + \langle n, n+4\rangle$. Assume that $m := m(d_2(H))$ is even. Then the following are equivalent:*

(i) $g(H) = 2g(d_2(H)) + \frac{n-1}{2} - 1$.

(ii) $d_2(H) = \{m, m+1, \cdots\}$ *or*
$\{m, m+2, m+4, \cdots, m+2(g-m), m+2(g-m+1), m+2(g-m+1)+1, \cdots\}$
where we set $m = m(d_2(H))$ and $g = g(d_2(H))$.

Remark 3.4 *The following result is also due to Kim [1]:*
Let C be a trigonal curve of genus $g \geqq 5$. Let $P \in C$ be a ramification point that is not total. Assume that $m(H(P))$ is even. Then the set $\mathbb{N}_0 \backslash H(P)$ is $\{1, 2, 3, \cdots, 2n-1, 2n, 2n+1, 2n+3, \cdots, 2g-2n-1\}$ with $\frac{g-1}{3} \leqq n \leqq \frac{g}{2}$. Moreover, each sequence actually occurs as the Weierstrass semigroup of some non-total ramification point on a trigonal curve of genus g.

The following is the first Key Lemma in this article:

Lemma 3.5 *Assume that $m(d_2(H))$ is even. We set $n = \dfrac{m(d_2(H))-2}{2}$ and $g = g(d_2(H))$.*

(i) If $m(d_2(H)) \geqq \dfrac{2g(d_2(H))+4}{3}$, then $d_2(H)$ in Theorem 3.3 (ii) coincides with $H(P)$ in Remark 3.4.

(ii) If $m(d_2(H)) < \dfrac{2g(d_2(H))+4}{3}$, then $d_2(H)$ in Theorem 3.3 (ii) can not be attained by any non-total ramification point of a trigonal curve of genus g.

4 Non-Weierstrass numerical semigroups

A numerical semigroup H is said to be *Weierstrass* if there exists a pointed curve (C, P) with $H(P) = H$. A numerical semigroup H is said to be of *double covering type*, which is abbreviated to *DC*, if there exists a double covering of a curve with a ramification point P with $H(P) = H$. Hence, if H is DC, then it is Weierstrass.

Theorem 4.1 *Let n be the minimum odd integer in H. Assume that n is an odd integer with $n \geqq c(d_2(H)) + m(d_2(H)) - 1$. Let $H = 2d_2(H) + \langle n, n+4 \rangle$ and assume that $g(H) = 2g(d_2(H)) + \dfrac{n-1}{2} - 1$. Moreover, assume that one of the following holds:*

(i) $m := m(d_2(H))$ is odd.

(ii) $m := m(d_2(H))$ is even and $m \geqq \dfrac{2g(d_2(H))+4}{3}$.

If $n \geqq 2g(d_2(H)) + 3$, then H is DC, hence it is Weierstrass.

Outline of the proof: By the assumption which is either (i) or (ii) it follows from Remark 3.1 and (i) in Lemma 3.5 that there is a pointed trigonal curve (C, P) with a non-total ramification point P satisfying $H(P) = H$. We can construct a double covering $\pi : \tilde{C} \longrightarrow C$ with a ramification point $\tilde{P}$ over P satisfying $H(\tilde{P}) = H$. Hence the numerical semigroup H is DC

Remark 4.2 *By a way similar to the proof of Theorem 4.1 the numerical semigroups $H = 2d_2(H) + \langle n, n+2 \rangle$ in Theorem 2.1 with $n \geqq 2g(d_2(H)) + 3$ and in Theorem 2.2 with $n \geqq 2g(d_2(H)) + 5$ are Weierstrass.*

Remark 4.3 *The following result is due to Torres [4]. Assume that* $g(H) \geqq 6g(d_2(H)) + 4$*. If* H *is a Weierstrass numerical semigroup, then it is DC.*

We use the contrapositive of the above, which is the second Key Lemma in this article.

Lemma 4.4 *Assume that* $g(H) \geqq 6g(d_2(H)) + 4$*. If* H *is not DC, then it is non-Weierstrass.*

Theorem 4.5 *Let* n *be the minimum odd integer in* H*. Assume that* $n \geqq c(d_2(H)) + m(d_2(H)) - 1$*. Let* $H = 2d_2(H) + \langle n, n+4 \rangle$ *and assume that* $g(H) = 2g(d_2(H)) + \frac{n-1}{2} - 1$*. Moreover, assume that* $m := m(d_2(H))$ *is even and* $m < \frac{2g(d_2(H)) + 4}{3}$*. Then the following hold:*

(i) Then H *is not DC.*

(ii) If $n \geqq 8g(d_2(H)) + 11$*, then it is non-Weierstrass.*

Outline of the Proof: (i) Assume that H is DC. Then there exists a double covering of a pointed curve (C, P) with a ramification point $\tilde{P}$ over P satisfying that

$$H(\tilde{P}) = H = 2d_2(H) + \langle n, n+4 \rangle = 2H(P) + \langle n, n+4 \rangle,$$

which implies that we must have

$$\dim\left(\{f \in k(C) \backslash \{0\} \mid (f) \geqq -2P - Q\} \cup \{0\}\right) = 2$$

for some point Q of C distinct from P. But by the assumptions that m is even and $m < (2g(d_2(H)) + 4)/3$ it follows from Lemma 3.5 that such a pointed curve (C, P) does not exist. Hence, H is not DC.

(ii) By Lemma 4.4 the numerical semigroup H is non-Weierstrass.

The following is the main theorem in this article, which is deduced from Theorems 4.1 and 4.5.

Theorem 4.6 *Assume that* $2 \in PF(d_2(H))$ *and* $H = 2d_2(H) + \langle n, n+4 \rangle$*. Let* n *be an odd integer with* $n \geqq 8g(d_2(H)) + 11$*. Then the following are equivalent:*

(i) $m(d_2(H))$ *is even and we have* $m(d_2(H)) < \frac{2g(d_2(H)) + 4}{3}$*.*

(ii) H *is non-Weierstrass.*

The following are known examples, which are due to [1].

Remark 4.7 *Let* $l \geqq 2$ *and* n *be an odd integer with* $n \geqq 16l + 19$ *(resp.* $16l + 27$*). We set* $H = 2\langle 4, 6, 4l+1, 4l+3 \rangle + \langle n, n+4 \rangle$

(resp. $H = 2\langle 4, 6, 4l + 3, 4(l + 1) + 1\rangle + \langle n, n + 4\rangle$). Then H are non-Weierstrass numerical semigroups, which are the first examples of non-Weierstrass numerical semigroups whose multiplicities are 8, because

$$m(d_2(H)) = 4 < \frac{2g(d_2(H)) + 4}{3} = \frac{2(2l + 1) + 4}{3} \quad \text{(resp.} \frac{2(2l + 2) + 4}{3}\text{)}.$$

Hence, Theorem 4.6 is a generalization of the above important example. In fact, we get many examples of non-Weierstrass numerical semigroups.

Remark 4.8 *Let s be a non-negative integer. We set*

$$H_s = \langle 2s + 4, 2s + 4 + 2, \ldots, 2s + 4 + 2(s + 1), 2(2s + 4) + 1,$$
$$2(2s + 4) + 3, \ldots, 2(2s + 4) + 2s + 3\rangle.$$

Then we have the following:
i) $g(H_s) = 3s + 5$, $PF(H_s) \ni 2$ and $m(H_s) < \dfrac{2g(H_s) + 4}{3}$.
ii) Let n be an odd integer with $n \geqq 24s + 51$. Then $H = 2H_s + \langle n, n + 4\rangle$ is a non-Weierstrass numerical semigroup whose multiplicity is $4s + 8$.

References

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