Algebraic Theory of Emergence & Reduction of Information

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1. Introduction

This is a report on the continuation of my research on the topic of emergence, currently one of the central concepts in philosophy, physics, biology, cognitive science, and other domains of scientific inquiry. The methodological and conceptual background of this concept was extensively explained elsewhere in my earlier publication [1]. Emergence has become fashionable, almost a buzzword frequently exploited to spice research reports and philosophical reflections, but it was never properly, formally defined to become accessible to a precise methodological or theoretical study. Ironically, probably its best characterization, but not a definition, can be found in the famous article by Philp W. Anderson *More is Different* published in 1972 that initiated intensive interest in its role in scientific inquiries, where it was associated with symmetry-breaking in complex systems [2].

A half-century later, emergence has become a celebrated idea [3], but its meaning is as obscured as it was before, or even more due to its frequent occurrence in multiple and diverse contexts and with a striking diversity of mutually contradictory characteristics. Moreover, the interest in pair emergence–reduction hidden in the diverse terminology is not new, even if the term emergence is quite recent. We can find its antecedents, if not in the answers, then in the questions, as far back as in Plato's dialog *Cratylus*, where it is hidden behind the antiquated vocabulary of the study of names [4]. Then we can trace similar motifs of the inquiry of the emergence of new, possibly irreducible meaning of what emerges through a combination of lower-level units in the Scholastic and modern philosophy. Even in *Cratylus*, there are indirect questions about the distinction between what now we call the epistemic and ontic aspect of emergence, the difference between what emerges to the inquiring mind (to someone) and what emerges as a new or different entity irreducible to components, independently from any inquiry.

The inquiries were not limited to metaphysics or semantics. Through the centuries, the question about whether the transformation into a new quality requires just an appropriate recipe of proportions of substrates (reductionist view) or whether it is necessary to engage a philosophers' stone or secret ritual (emergentist view), preoccupied alchemists, including the most famous of them, Isaac Newton. The confusion caused by the lack or lack of clarity on the distinction between epistemic and ontic emergence delayed the (epistemic) emergence of the study of the (ontic) emergence until recently.

In this study, as well as in the previous one mentioned above, only ontic emergence is considered [1]. Epistemic emergence is too dependent on the context and diachrony to be considered in a synchronic, systematic, abstract way. One of the main methodological tools here is the transition from studying the emergence of properties to the emergence of information. The use of the concept of a property based on this word in vernacular can be blamed for the confusion present in almost all works from Plato's *Cratylus* to recent philosophical articles devoted explicitly to emergence. It took the revolution in physics brought by quantum mechanics to realize how misleading it is [1]. The concept of property loses its universal meaning with the distinction between the state of a system (system as it is) and observables (system characteristics as they appear in the process of experimental inquiry, in the traditional terminology, the appearance of the system) which is critical for quantum theory. Of course, this currently universally accepted distinction in physics is irrelevant in common sense everyday communication, but it gives us a new and very different perspective on metaphysical issues, including those with a centuries-long philosophical tradition. One of them is the status of the concept of supervenience of sets of properties (i.e. a binary relation between sets of properties), closely related to the emergence of properties, but at the higher level of set-theoretical hierarchy. Supervenience will be addressed in this paper, although its importance is more for methodological matters than the subject of ontological inquiry.

While the deficiency of the unqualified concept of property is commonly recognized, my solution to the problem of the erroneous conflation of the epistemic and ontic understanding of properties by replacing the concept of property with the concept of information is much more idiosyncratic. In the absence of consensus on the definition of information, it requires a specific choice of such a definition and following it choice of a theoretical conceptual framework, which I based on the mathematical theory of closure spaces [1,5]. Whether someone agrees with my choices or not, this paper has independent, purely mathematical, algebraic content of potential interest for a reader completely disengaged from the theoretical study of information or philosophical considerations of emergence. The subject of this content is a question about the reducibility or irreducibility of closure spaces resulting from the extension/contraction of the set on which they are defined or the combining of component closure spaces.

The main step ahead in the presented here research from its earlier reported stage [1], is, on the one hand, a more specific characterization of the dual process of extension/restriction of the set on which closure space is defined and its relation to the concepts such as supervenience, and on the other hand in taking into account a different form of the process which can be better-called irreducibility/reducibility to the component spaces. In this case, irreducibility into components indicates that the structure of the closure space, and in consequence of the information system, is emergent, i.e. cannot be derived from simpler spaces.

2. Conceptual and Mathematical Preliminaries

To make this paper self-sufficient, a brief review of the concepts involved in this study will precede the report of new developments. The central concepts are of a closure space representing an information system, its logic generalizing the specific paradigmatic Boolean logic of information systems based on language, and encoding of information (i.e. an instance of information). They have been introduced [6], associated with information [6], and extensively studied in several of my earlier publications [5,6,8]. Here, there is only their brief overview or just a reminder. The mathematical concepts used in the following text have foundations in the literature addressing lattice theory and closure spaces [7].

Def. *A closure space* $\langle S,f \rangle$ *is a set S with a function* $f: 2^S \rightarrow 2^S$ *on the power set of S called a (transitive) closure operator that satisfies three conditions: (i)* $\forall A \subseteq S: A \subseteq f(A)$, *(ii)* $\forall A, B \subseteq S$: *If* $A \subseteq B$, *then* $f(A) \subseteq f(B)$, *(iii)* $\forall A \subseteq S: f(f(A)) = f(A)$.

Every closure space $\langle S,f \rangle$ can be defined in an equivalent (cryptomorphic) way by a Moore family of subsets of S, i.e. family closed with respect to arbitrary intersections and including the set S. Every Moore family $\mathcal{M}$ defines a transitive operator: $f(A) = \cap \{M \subseteq \mathcal{M}: A \subseteq M\}$ and in turn, the family f-Cl = $\{M \subseteq S: f(M) = M\}$ is a Moore family. The family f-Cl is a complete lattice $\mathcal{L}_f$ with respect to the set inclusion $\subseteq$. However, this lattice does not have to be considered in its set-theoretical representation defined by inclusion and can be, and frequently is associated with its general partial order.

If needed, the concept of a closure space $\langle S,f \rangle$ and its lattice of closed elements $\mathcal{L}_f$ can be defined on an arbitrary bounded complete lattice $\mathcal{L}$ instead of the power set 2^S by replacing every occurrence of the set inclusion $\subseteq$ with the symbol of the partial order $\leq$ of $\mathcal{L}$.

The family $\Im$ of elements of a complete lattice $\mathcal{L}$ is called a *filter*, if it satisfies two conditions: (1) $\forall A, B \in \mathcal{L}$: If $A \in \Im$ and $A \leq B$, then $B \in \Im$. (2) $\forall A,B \in \mathcal{L}$: If $A \in \Im$ and $B \in \Im$, then $A \wedge B \in \Im$.
A (proper) filter does not have the least element of $\mathcal{L}$ as its element. The maximal (proper) filter on $\mathcal{L}$ is called an *ultrafilter*. All these concepts are defined in the same way as in the more familiar special case of the complete lattice introduced in the power set 2^S of set S by the set inclusion $\subseteq$. However, in this more general case, the properties of filters may ramify. For instance, in general, ultrafilters do not have to be prime filters and ultrafilters may exist in the absence of prime filters.

With these mathematical preliminaries, we can introduce the basics of the mathematical theory of information used here. We will consider a closure space $\langle S,f \rangle$ with its corresponding Moore family $\mathcal{M}$ of closed subsets as an **information system**. The specific choice of closure space depends on the choice of the type of information. For instance, we can consider geometric, topological, logical (linguistic) information, etc. with corresponding lattices of closed subsets defined by geometric, topological, or logical consequence closure operators.

The family of closed subsets $\mathcal{M}$ = f-Cl is equipped with the structure of a complete lattice $\mathcal{L}_f$ which we can consider to be the **logic of an information system**. It plays a similar role in the generalizations to various information systems as in the algebraic formulation of traditional logic, although it does not have to be a Boolean algebra, but only a complete lattice with or without orthocomplementation.

Encoding of information **(or instance of information)** is a distinction of a subfamily $\Im$ of $\mathcal{M}$, which is a filter in the lattice $\mathcal{L}_f$. In a more abstract, algebraic approach, it is simply a filter. The reasons for the association of information with filters are related to the need for semantical analysis of information and to the fact that traditionally understood filters serve as algebraic tools for localization or identification that is of special importance for the definition of information used here and in my earlier publications. It turns out that this abstract form of information, although very different and much more general, can be easily related to Shannonian information theory within the latter's restricted context [5,8]. Encoding of information is an invariant of the group of symmetry acting on S defined by the transformations of the set S that preserve the filter structure of the information logic $\mathscr{L}_f$. This provides a bridge for the study of information in terms of symmetry or its breaking, both of critical importance for emergence in physics, natural sciences, or the study of complexity [5].

In the general case, the logic of a closure space $\mathscr{L}(S,f)$ is not necessarily a Boolean ortholattice of traditional logic, nor the ortholattice $\mathscr{L}_f$(H) of closed subspaces of a Hilbert space as in quantum logic. Therefore, we have to be cautious not to import without close inspection into this general theory the facts about the structures (e.g. filters, ultrafilters) from more familiar cases of Boolean or quantum logics which may not be true in general.

3. Emergence and Reduction in Terms of Extensions or Reductions of Closure Spaces

Anderson's metaphoric "More is different" representing emergence was at the already reported earlier stage of this study interpreted in two ways [1]:

- Extension of the closure space $<S,f>$ to a closure space $<T,g>$ such that $S \subseteq T$ & $S \neq T$
- Extension of the closure space $<S,f>$ to a closure space $<T,g>$ such that $T = 2^S$.

Both extensions were understood as dual to corresponding reductions but in their natural occurrences from which a more general description was derived, the first had its source in reduction and the other in extension. In the first case, in the previous report, the extension was derived from the process of restriction from a set T to its subset S called a frame, such that the closure in $<T,g>$ is determined by the closure in $<S,f>$ as follows: Let $<S,f>$ be a closure space and B its subset (not necessarily proper). By the definition, subset B is a *frame for* $<S,f>$*, if* $\forall A \subseteq S\ \exists B_A \subseteq B: f(A) = f(B_A)$. This defining condition is equivalent to: $\forall A \subseteq S: f(A) = f(B \cap f(A))$ [9,10]. At this point, it should be noted that the term "frame" introduced independently in several different and inequivalent ways in other contexts does not have any application here [11].

One of the main results of the earlier study was the conclusion that the reversed process of extension based on the restriction of the first type to a frame may be not unique. This means, that different closure spaces on set T may be restricted to a given frame. Or the other way around, that from a given closure space $<S,f>$ there are possible different extensions to closure spaces $<T,g>$ on the same set T, but with different closures *g*. This means, that each of these different closure spaces $<T,g>$ is consistent with the underlying closure space $<S,f>$ on the frame S, but is not determined by it. This is the first type of emergence in which the extension from $<S,f>$ to a particular closure space $<T,g>$ requires additional conditions for the construction of the latter.

There was a second type of extension listed above and studied in the previous paper [1]. In this case, a closure space <*S,f*> is extended to a closure space <*T,g*> such that $T = 2^S$, and the term "extension" has a very different meaning of an increased cardinality or a higher position in the set-theoretical hierarchy, and S is not a subset of T ($S \not\subset T$). Moreover, T in the first case was an arbitrary set extending S (i.e. the only condition was that $S \subseteq T$ & $S \neq T$), here $T = 2^S$ is uniquely determined and the process of extension was carried out in a very specific way. In the earlier paper, this specific choice of the process was neither discussed nor justified, but this issue is addressed in this paper. To do this, we have to present the process.

Let <S, *f*> be a closure space and $T= 2^S$. Define a binary relation R between S and $T= 2^S$ by: $\forall x \in S\ \forall A \subseteq S$: xRA *iff* $x \in f(A)$. Then the relation R defines a Galois connection (polarity) between the Boolean algebra of subsets of S and Boolean algebra of subsets of T and it turns out that the first Galois closure is the original closure *f*(A) on the subsets of S (well-known fact used to demonstrate the early result of the study of closure spaces that every closure operator is a Galois operator [7]). The other Galois closure operator *g*(β) on T was never studied. It was ignored, as not important for the purpose of the study of closure operators on S:

- $\forall A \subseteq S: f(A) = R^{*a}R^a(A)$, where $R^a(A) = \{y \in S: \forall x \in A: xRy\}$and R* is the inverse of R.
- $\forall \beta \subseteq T = 2^S: g(\beta) = R^aR^{*a}(\beta) = \{A \subseteq S: \cap\{f(B): B \in \beta\} \subseteq f(A)\}$.

Then in the latter case, we have $\forall \beta \subseteq 2^S: \beta \in \mathscr{L}_g$ *iff* $[\forall A \subseteq S: \cap\{f(B): B \in \beta\} \subseteq f(A) \Rightarrow A \in \beta\}$ and the lattices of closed subsets $\mathscr{L}_g$ and $\mathscr{L}_f$ are dually isomorphic. Therefore, we have here a very clear instance of the lifting (dually, as the isomorphism is dual, i.e. order inverting) of the logic from <S, *f*> to <*T,g*>. The relation R uniquely associated with closure operator *f* on S, working here as a lifting tool, has in the case of infinite set S interesting associations with the subject of the Generalized Continuum hypothesis, but it cannot be discussed here due to the limited volume of this paper.

The main result of the earlier paper demonstrating the existence of emergent closure spaces in sets T of the type $T = 2^S$, where S is an infinite set, was based on the demonstration that the following theorem holds:

Let (as above, g be the closure operator on 2^S generated by a closure operator f on S)
$\forall \beta \subseteq T = 2^S: g(\beta) = \{A \subseteq S: \cap \{f(B): B \in \beta\} \subseteq f(A)\}$. *Then:*

(i) *(quite obviously)* $g(\emptyset) \neq \emptyset$. *Always* $\{S\} \in g(\emptyset) \neq \emptyset$. [Note the typo in [1]!] *and moreover* $\forall A \subseteq S: f(A)=S \Rightarrow A \in g(\emptyset)$.

(ii) *The subset of closed subsets f-Cl*$\subseteq 2^S$ *is always not only a frame for* $<2^S,g>$ *but also a minimal frame.*

Since there are many simple examples of infinite closure spaces that do not have minimal frames and, quite obviously there are many closure spaces with the empty closure of the empty set (e.g. topological spaces) [10], we can see that the emergence of closure spaces, i.e. emergence of information systems is ubiquitous.

To show that emergent closure spaces are not just marginal, pathological cases, in the earlier paper, a highly nontrivial example was given of the meta-closure space [12].

We can consider a closure space $<2^S, g>$ defined by the Moore family of all Moore families of closed subsets, each for uniquely corresponding closure operator on S. Then the meta-closure space $<2^S, g>$ is defined by: $\forall \mathscr{B} \subseteq 2^S$: $g(\mathscr{B}) = \{B \subseteq S: \exists \mathfrak{C} \subseteq \mathscr{B}: B = \cap \mathfrak{C}\}$. If S is infinite, then its meta-closure space does not have minimal frames [1,12].

4. Minors in Closure Spaces: Contractions and Restrictions

The first type of emergence studied in the previous publication [1], was considered in the context of the extension understood as a reversal of a simple reduction of a closure space $<S, f>$ by restriction to a closure space $<T,g>$ such that $T \subseteq S$ & $S \neq T$, defined by the simple rule $\forall A \subseteq T$: $g(A) = f(A) \cap T$. This form of reduction is very natural, in particular when T is an f–closed subset of $<S,f>$, i.e. $f(T) = T$, but the general study of extensions based on the reversal in this special case of such reductions, may be considered oversimplification. Moreover, we will see that the consideration of an extensive class of alternative forms of reduction provides stronger arguments for diverse occurrences of emergence. Please note the exchange of the symbols for closure spaces $<S,f>$ and $<T,g>$ such that $T \subseteq S$, as it will simplify some notation in the following.

The simple case of restrictions considered before will be extended to a much more general concept of a minor of the given closure space.

The first type of extension defined as the opposite of the reduction to frames requires in its definition of reduction $\forall A \subseteq B$: $f|_B(A) = f(A) \cap B$ the assumption that the closure space $<S,f>$ which is being reduced has a nontrivial frame B. This makes the emergence quite limited as we get a ramification to different extended closure spaces but with isomorphic logics $\mathscr{L}_f$. Since the reduction considered before was limited to (non-trivial) frames, the extension is also limited and cannot be considered canonical. For this reason, we have to consider a more general concept of reduction. A much more general concept of a reduction is that of a minor of closure space.

The concept of a minor was originally introduced by Tutte [13] in matroid theory and later was extended by me to arbitrary closure spaces [14]. In the following, the proofs are omitted due to the limitations of the size of the paper.

Def. 4.1: *Let $<S,f>$ be a closure space and $R \subseteq T \subseteq S$. A minor is a closure space on $T \setminus R$ with the closure operator $f^R{}_T$ defined by: $\forall A \subseteq T \setminus R$: $f^R{}_T(A) = f(A \cup R) \cap (T \setminus R)$.*
For $U = S \setminus T = T^c$, the minor $f^T = f^U{}_S$ is called contraction and $\forall A \subseteq T$: $f^T(A) = f(A \cup T^c) \cap T$.
For $R = \varnothing$, the minor $f_T = f^R{}_T = f^{\varnothing}{}_T$ is called restriction and $\forall A \subseteq T$: $f_T(A) = f(A) \cap T$.

Prop. 4.2: *Let $<S,f>$ be a closure space and $R \subseteq T \subseteq S$ and $M \subseteq N \subseteq T \setminus R$. Then:*

- *$(f^R{}_T)^M{}_N = f^{R \cup M}{}_{T \cap N}$ is a closure operator on the set $(T \cap N) \setminus (R \cup M)$*
- *if $P = R^c$, then $(f^P)_T = (f^R{}_S)^{\varnothing}{}_T = f^R{}_{T \cup R} = f^R{}_T$ and similarly $(f_T)^P{}_N = f^R{}_T$.*

This means that every minor is a composition of contraction and restriction.

- Corollary.: $(f^R{}_T)^R{}_T = f^R{}_T$ on T\R, i.e. $f \to f^R{}_T$ is a projection from S to T\R.

Now we can consider a reversal of the projection into a minor parametrized by a set that does not intersect with the original set:

Extension of the closure space <T,g> to a closure space <S,f> where $T \subseteq S$ & $S \neq T$, with the distinguished, fixed subset R of S\T,i.e. $R \subseteq S\backslash T$, the closure space <T,g> is a minor of <S,f>, i.e. $g = f^R{}_T$ defined by the condition: $\forall A \subseteq T: g(A)\ f^R{}_T(A) = f(A \cup R) \cap (T \backslash R)$.

This type of extension is a generalization of the extension from a frame. Moreover, it can be easily seen that this very general form of extension can produce not only different closure spaces but also that because we have the freedom to choose the fixed set R out of the many sets not intersecting with T, these spaces may have non-isomorphic logics. Therefore, the first type of extension as a reversal of reduction to the minor includes non-trivial emergent extensions.

5. Supervenience

The concept of supervenience between sets of properties has been in philosophical literature closely related to that of emergence, but while the latter was most frequently a characteristic of a property, the former was a relationship between sets of properties. As in the case of emergence, supervenience was understood in many different ways but it had the advantage of being rarely used in vernacular, and the dictionary description of the word "supervene" as something coming or occurring as something additional, extraneous, or unexpected does not make it very attractive outside of academic discourse. This made it easier to define in a general way without excluding its use in some contexts. Moreover, because it has a relational form it can be expressed in an easily comprehensible way in terms of the concept of the difference between objects that to a layperson may seem familiar and obvious.

Def. 5.1: *Suppose we have two sets of properties $\mathscr{B}$ and $\mathfrak{C}$ of objects in some set of objects S. Then $\mathscr{B}$ supervenes upon $\mathfrak{C}$ if no two objects differ with respect to properties in $\mathscr{B}$ without differing with respect to properties in $\mathfrak{C}$.*

In philosophical discussions of supervenience, in particular, in analytic philosophy, the preferable form of the definition involves the expression "*can differ*" instead of "*differ*" [15]. However, in this case, it is necessary to involve modal logic, and the study with its enriched logic loses its context of natural sciences, in which the modality "can" does not have any meaning, at least with the present scientific methodology. Thus, in the following, supervenience of the set of properties $\mathscr{B}$ upon set $\mathfrak{C}$ will have its short explanation as "*there are no objects in the set of objects S with $\mathscr{B}$-difference without $\mathfrak{C}$-difference*" instead of "*there cannot be an $\mathscr{B}$-difference without a $\mathfrak{C}$-difference*".

The above definition of supervenience in terms of difference can be reformulated equivalently with the use of the concept of indiscernibility with respect to a set of properties $\mathscr{B}$ understood as:

Two objects x and y from S (i.e. $x,y \in S$) are $\mathscr{B}$-indiscernible if ($\forall A \in \mathscr{B}$: x has property A iff y has property A).

Now we can reformulate the definition as:

Def. 5.2: *Suppose we have two sets of properties $\mathscr{B}$ and $\mathfrak{C}$ of objects in some set of objects S. Then $\mathscr{B}$ supervenes upon $\mathfrak{C}$ if for all objects x, y in S if x and y are $\mathfrak{C}$-indiscernible, then they are $\mathscr{B}$-indiscernible (i.e. $\forall x,y \in S$: x and y are $\mathfrak{C}$-indiscernible $\Rightarrow$ x and y are $\mathscr{B}$-indiscernible).*

With supervenience defined this way, we can start placing it in the context of information systems and representing them as closure spaces by placing it in the conceptual frame of the special case of a closure space $<S,f>$ representing an information system in which the closure operator is trivial. Thus, let's consider the familiar case, the simple case of a trivial closure space in which every subset is closed and its logic is the Boolean algebra 2^S of all subsets of the set S. This means $\forall A \subseteq S$: $A = f(A)$ and we have $\forall x \in S \forall A \subseteq S$: xRA *iff* $x \in A$. Then the Galois closure operator on 2^S defined in general by $g(\mathscr{B}) = \{A \subseteq S: \cap\{f(B): B \in \mathscr{B}\} \subseteq f(A)\}$ simplifies to the following: $g(\mathscr{B}) = \{A \subseteq S: \cap\{B: B \in \mathscr{B}\} \subseteq A\}$.

In this very special case, we can use the traditional concept of property and consider that xRA describes the relationship "x has all properties defining set A" and $g(\mathscr{B})$ is the principal filter in 2^S generated by the intersection of all subsets in $\mathscr{B}$, i.e. it is the family of all subsets which include this intersection. The key point is in simplification of "xRA *iff* $x \in f(A)$" to "xRA *iff* $x \in A$". In the former, general case, we have an instance of general information (or encoding of information in an information system), and in the latter, a set-theoretical interpretation of the instance of information about x that x has all properties defining set A.

Then we get a simple explanation of the meaning of supervenience.

Prop. 5.3: *Suppose we have two sets of properties $\mathscr{B}$ and $\mathfrak{C}$ of objects in some set of objects S. Then $\mathscr{B}$ supervenes upon $\mathfrak{C}$ if* $g(\mathscr{B}) \subseteq g(\mathfrak{C})$.

The next step is towards the generalization to the general case of information systems not restricted to language, such as geometric or topological information systems. When we consider a unique but arbitrary information system described by a closure space $<S,f>$, we can use the preceding proposition for the special case of trivial closure space as a heuristic for the formulation of the general definition of supervenience.

Def. 5.4: *Suppose we have two arbitrary families $\mathscr{B}$ and $\mathfrak{C}$ of subsets of set S in a closure space $<S,f>$ and g is the closure operator in the Galois closure space $<2^S,g>$ dual to $<S,f>$. Then $\mathscr{B}$ supervenes upon $\mathfrak{C}$ if $g(\mathscr{B}) \subseteq g(\mathfrak{C})$.*

This generalization of supervenience opens to us an extensive formal toolbox for its study. For instance, we can ask about a family $\mathfrak{C}$ (and by extension minimal family $\mathfrak{C}$) that is the subject of supervenience by a given family $\mathscr{B}$. Naturally, the answers can be given in terms of frames (minimal frames, respectively) in the closure space $<2^S,g>$.

There are some additional advantages of such a generalization of supervenience. The source of the increased interest in supervenience was the issue of the need for relating properties of very different types. For instance, cognitive–mental properties of phenomenal consciousness have been considered to supervene over the physiological properties of the brain. However, in typical attempts to describe this supervenience, there was always an assumption that both types of properties apply to the same objects or their combinations. This hidden assumption did not address the issue of the possible and likely difference between the objects to which properties are ascribed. This difference can be naturally addressed by considering the differences between information systems and instances of information in them. Obviously, mental characteristics are very different from physiological ones, and putting them together is highly questionable.

We can prevent it by considering two or more different information systems defined on the same set S, i.e. two or more closure operations $f_1, f_2, \ldots$ defined on the same set S. Then we have dual Galois closure operators $g_1, g_2, \ldots$ on 2^S. Now we can redefine supervenience for this case.

Def. 5.5: *Suppose we have two arbitrary families* $\mathscr{B}$ *and* $\mathfrak{C}$ *of subsets of set S with a class of closure spaces* $<S, f_1>, <S, f_2>, \ldots$ *defined on S and* $g_1, g_2, \ldots$ *are the dual Galois closure operators in the Galois closure spaces* $<2^S, g_1>, <2^S, g_2>, \ldots$ *dual to* $<S, f_1>, <S, f_2>, \ldots$*, respectively. Then* $\mathscr{B}$ *supervenes upon* $\mathfrak{C}$ *if there exist i, j (not necessarily different) such that* $g_i(\mathscr{B}) \subseteq g_j(\mathfrak{C})$.

This again opens access to formal methods of the study of supervenience. The study of supervenience is more about the methodology of the study of emergence than about its ontological aspects, but the tools acquired in it are of great value for the latter.

There is a possible direction of further generalization linking closure spaces defined on different sets, but even its shortest presentation would have required quite an extensive additional conceptual framework of closure space morphisms going too far beyond the scope of this paper.

6. Disjoint Sums of Closure Spaces and Products of Their Logics

There is another way how closure spaces can be extended, or rather constructed. Since this construction requires an even more extensive conceptual framework than mentioned above, due to the involvement of closure spaces defined on different sets, it will be addressed very briefly in a very general outline. The concept of a direct sum of closure spaces defined on several different sets has been defined and studied before [16].

The first step is the definition of a disjoint sum of closure spaces which requires an explanation of the concepts involved in the definition [16].

Def.6.1: *Let f be an operator on a set S, g an operator on set T, and* φ *be a function from S to T. The function* φ *is (f,g)-continuous if* $\forall A \subseteq S$: $\varphi(f(A)) \subseteq g(\varphi(A))$*. They can be called continuous instead of (f,g)-continuous if no confusion is likely. This general definition of continuity becomes the usual concept of continuity when the closure space is topological.*

Then we can proceed to the definition of the direct sum of closure spaces.

Def. 6.2: *A disjoint sum* $\oplus\boldsymbol{A_I}$ *of the indexed family* $\boldsymbol{A_I}$ *is defined as*
$\oplus\boldsymbol{A_I} = \{(a,i): a \in \cup\{\gamma(i): i \in I\},\ a \in \gamma(i), \text{ and } i \in I\}$.
With every disjoint sum $\oplus\boldsymbol{A_I}$ *there is associated a family of injective functions, called canonical injections of the family* $\boldsymbol{A_I}$ *into its disjoint sum* $\oplus\boldsymbol{A_I}$, $\{\theta_i: \gamma(i) \rightarrow \oplus\boldsymbol{A_I},\ i \in I\}$ *defined for every* $i \in I$ *and for every* $a \in \gamma(i)$ *by* $\theta_i(a) = \varphi(i)$.

Def. 6.3: *Disjoint sum* $\langle\oplus\boldsymbol{S_I}, \boldsymbol{g}\rangle$ *of the indexed family* $\boldsymbol{S_I}$ *of sets equipped with closure operators is the disjoint sum of sets* $\oplus\boldsymbol{S_I}$ *with its family of canonical injections* $\{\theta_i: S_i \rightarrow \oplus\boldsymbol{S_I},\ i \in I\}$ *equipped with the operator g defined by* $\forall A \subseteq \oplus\boldsymbol{S_I}$: $g(A) = \cup\{\theta_i f_i \theta_i^{-1}(A): i \in I\}$. *If no confusion is likely the simpler symbol* $\oplus\boldsymbol{S_I}$ *replaces the more precise symbol* $\langle\oplus\boldsymbol{S_I}, \boldsymbol{g}\rangle$.
With the operator g defined as above, all canonical injections become (f_i, g)*-continuous.*

The main result in the study of the disjoint sums of closure spaces is the following.

Prop. 6.4: *The lattice of closed subsets of the disjoint sum of an arbitrary family of closure spaces is isomorphic to the direct product of lattices of closed subsets for the component closure spaces, i.e.* $\langle \boldsymbol{L_g}, \leq\rangle \approx \langle\otimes\boldsymbol{L_I}, \leq\rangle$, *where* $\langle\boldsymbol{L_g}, \leq\rangle$ *is a lattice of g-closed sets in the disjoint union of closure spaces* $\langle\oplus\boldsymbol{S_I}, \boldsymbol{g}\rangle$ *whose components are closure spaces from the family* $\boldsymbol{S_I} = \{\langle S_i, f_i\rangle: i \in I\}$, $\langle\otimes\boldsymbol{L_I}, \leq\rangle$ *is the cardinal product of lattices from the family* $\{\langle L_i, \leq_i\rangle: i \in I\}$, *where each lattice* $\langle L_i, \leq_i\rangle$ *is a lattice of closed subsets in the closure space* $\langle S_i, f_i\rangle$, *and* $\approx$ *is the order isomorphism*

The proposition is important for the study of the emergence of a closure space (information system) understood as the impossibility of its decomposition to a disjoint sum of component closure spaces. From Proposition 6.4 we know that the question about such reducibility or irreducibility of a closure space can be answered through the inspection of its logic. In turn, there is an extensive theory of the reducibility of lattices (or partially ordered sets) to direct products. For instance, we can analyze the center of the lattice. In completely irreducible complete bound lattices, the center is trivial and consists of the pair of the least and greatest elements [7].

There is a very important example of such a completely irreducible lattice in the study of quantum logics for quantum mechanical systems without superobservables [9]. This corresponds to the fact that Hilbert spaces of dimension three or higher have irreducible lattices of closed subspaces. The transition from information systems of classical mechanics to quantum mechanics is an example of emergence understood as irreducibility to components.

7. Conclusion

We have an extended list of extensions in which emergence occurs than those studied before [1]:

- *Extension of the closure space* $<T,g>$ *to a closure space* $<S,f>$ *where* $T \subseteq S$ & $S \neq T$, *with the distinguished, fixed subset R of* $S \backslash T$, *i.e.* $R \subseteq S \backslash T$, *the closure space* $<T,g>$ *is a minor of* $<S,f>$, *i.e.* $g = f^R{}_T$ *defined by the condition:* $\forall A \subseteq T: g(A)\ f^R{}_T(A) = f(A \cup R) \cap (T \backslash R)$.
- *Extension of the closure space* $<S,f>$ *to a closure space* $<T,g>$ *such that* $T = 2^S$. *The closure space of meta-closure for infinite set S provides an example of emergence, i.e.irreduciility.*

- *Extension from the component closure spaces to their disjoint sums. There are multiple examples of closure spaces that are irreducible into a disjoint sum of component closure spaces.*

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