Amagamation problems of the infinite transformation semgroups

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Abstract. In this paper we investigate the problem of whether or not the infinite full transformation semigroups are amalgamation bases for all semigroups.

1 Subsemigroups of the infinite full transformation semigroups

Let $\mathscr{T}_X^{op}$ denote the full transformation semigroup on a set X with composition being from left to right.

Let S_X be the symmetric group on the set X, which is the maximal subgroup of $\mathscr{T}_X^{op}$ containing the identity element 1_X. Let $R_X = \{e_x \mid x \in X\}$, where $(a)e_x = x$ for any $a \in X$. Then R_X is a right zero semigeroup and S_X is the symmetirc group on R_X by multiplying the elements of R_X by the elemets of S_X on the right. On the other hand, $ge_x = e_x$ hold for all $g \in S_X$ and $x \in X$. Let $U_X = S_X \cup R_X$. Then U_X is a subsemigroup of $\mathscr{T}_X^{op}$.

We define

$I_\xi = \{f \in \mathcal{T}_X^{op} \mid |_X f| < \xi\}$ and $D_\xi = \{f \in \mathcal{T}_X^{op} \mid |_X f| = \xi\}$.

Let ξ' be the smallest cardinal number exceeding ξ. Then $I_{\xi'} = I_\xi \cup D_\xi$.

Then $\mathscr{T}_X^{op} = I_{|X|} \supset \cdots \supset I_{\aleph_0} \supset \cdots \supset I_n \supset \cdots \supset I_2 \supset I_1$ is a chain of ideals of $\mathscr{T}^{op}(X_0)$ and

$$I_{\aleph_0} = \bigcup_{1 \le n < \aleph_0} I_n \text{ is a maximal ideal of } \mathscr{T}_X^{op}.$$

All the $\mathscr{J}$-classes of $\mathscr{T}_X^{op}$ are $I_{n+1} - I_n$ $(1 \le n < \aleph_0)$ or $\mathscr{T}_X^{op} - I_{\aleph_0}$, where the Green's L[resp. R, J, H]- relations are denoted by $\mathscr{L}$[resp. $\mathscr{R}$, $\mathscr{J}$, $\mathscr{H}$].

Let S be a semigroup with zero 0, and $a, b \in S$. The set $\{s \in S \mid sa = 0\}$ is called the *left annihilator* of a in S and is denoted by $Ann_l(a)$. In this case, we say that S

satisfies the condition Ann_l if $ann_l(a) = ann_l(b)$ implies $a\mathscr{R}b$. The *right annihilator* and the condition Ann_r are defined by left-right duality.

We shall prove firstly

Lemma 1. *Keep the notation as above. Then each factor semigroup I_i/I_{i-1} $(i \geq 1)$is a completely 0-simple semigroup satisfying the conditions*

(1.1) : *If $s \ngeq_{\mathscr{R}} t$ for $s, t \in I_i - I_{i-1}$, then there exist $s^* \in V(s), t^* \in V(t)$ such that $ss^*tt^*s \in I_{i-1}$*

and

(1.2) : *If $s \ngeq_{\mathscr{L}} t$ for $s, t \in I_i - I_{i-1}$, then there exist $s^* \in V(s), t^* \in V(t)$ such that $st^*ts^*s \in I_{i-1}$.*

Proof. It is easy to see that I_i/I_{i-1} a completely 0-simple semigroup.

(1.1) : Let $s, t \in I_i - I_{i-1}$. The condition that $t \ngeq_{\mathscr{R}} s$ is equivalent to that $(s,t) \notin \mathscr{R}$. In this case, $\pi_s \neq \pi_t$ where $\pi_s = \{(x,y) \in X \times X \mid (x)s = (y)s\}$. Then it does not happen that each π_s-class is contained in a π_t-class, otherwise each π_s-class is equal to a π_t-class. So, there exists at least one π_s-class which properly intersects two π_t-class. Hence there exist $x, y \in X$ such that $(x)s = (y)s$, but $(x)t \neq (y)t$. Chose $t^* \in I_i$ with $(x)tt^* = x$ and $(y)tt^* = y$ and $|(X)(tt^*s)| < i$. Hence $ss^*tt^*s \in I_{i-1}$. (Similarly, it is possible to choose $s^*, t^* \in I_i$ with $tt^*ss^*t \in I_{i-1}$)

(1.2) : Let $s, t \in I_{i+1}$ with $(s,t) \notin \mathscr{L}$. Then $(X)s \neq (X)t$ and so, there exists $x \in X$ with $(x)s \notin (X)t$. Let $y \in X$ with $(y)s \neq (X)s$. If $(y)s = (z)t$ for some $x \in X$, then $((y)s)t^* = ((x)s)t^* = z$. Thus, $|(X)st^*| < i$ and hence $st^*ts^*s \in I_{i-1}$. On the other hand, if $(y)s \notin (X)t$, then $((y)s)t^* = ((x)s)t^* = x$(it is possible to choose an arbitarary element of X). Hence $st^*ts^*s \in I_i$. (Similarly, it is possible to chose $s^*, t^* \in I_{i+1}$ with $ts^*st^*t \in I_i$) □

Remark 1. Let S be a completely 0-simple semigroup.

(1.1) [resp. (1.2)] holds if and only if S satisfies the condition Ann_l [resp. Ann_r].

Proof $(\Rightarrow)$: If $(s,t) \notin \mathscr{R}$, then by (1.1) there exist $s^* \in V(s), t^* \in V(t)$ such that $ss^*tt^*s = 0$. If $ss^*tt^* = 0$, then $ss^*s = s$, and $ann_l(s) \neq ann_l(t)$. on the other hand, if $ss^*tt^* \neq 0$, then $ann_l(s) \neq ann_l(t)$.

$(\Leftarrow)$: If $(s,t) \notin \mathcal{R}$, then by Ann_l, $ann_l(s) \neq ann_l(t)$. Suppose that there is $u \in ann_l(s)$, but $u \notin ann_l(t)$. Then $taut = t$ for some $a \in S$, since $tSut - \{0\}$ is a $\mathscr{H}$-class containing t. Hence $(au)t(au) \in V(t)$ and $ss^*t((au)t(au))s = 0$. Suppose that there is $u \in ann_l(t)$, but $u \notin ann_l(s)$. Then $sbus = s$ for some $b \in S$, since $sSus - \{0\}$ is a $\mathscr{H}$-class containing s. Hence $(au)s(au) \in V(s)$ and $s((au)s(au))tt^*s = 0$. We are done.

By left-right duality of thr above argument, the equivalence of the conditions (1.2) and Ann_r is proved. □

For undefined terms of semigroup theory, refer to [2] and [6].

Lemma 2. Let U be the semigroup $S_X \cup R_X$. Then

(1) The restriction of a right congruence on U to R_X is either the identity relation 1_{R_X} or Universal relation $R_X \times R_X$.

(2) For any right U-set M, either MR_X is disjoint union of R_X-sets which are isomorphic to the R_X-set R_X or $|MR_X| = 1$.

Proof. (1) : The result follows from a fact that for any pair of different elements (a, b) and any pair of different elements (c, d) $(a, b, c, d \in R_X)$, there exists $\alpha \in G_X$ such that $a\alpha = c$ and $b\alpha = d$.

(2) : This is an easy consequence of (1). □

We prove firstly the following.

Lemma 3. *The semigroup $U_X = S_X \cup R_X$ has the reresentation property.*

Proof. By Theorem 2.1 of [11], we suppose that there exist a right U_X-set M, a left U_X-set N containing U_X as a U_X-subsets and $m, m' \in M$ such that $m \otimes 1 = m' \otimes 1$ in $N \otimes_U N$. Then by Lemma 1.2 of [1], there exist $m_1, \cdots, m_n \in M$, $y_2, \cdots, y_n \in N$, $s_1, \cdots, s_n$, $t_1, \ldots, t_n \in U_X$ such that

$$\begin{aligned}
m &= m_1 s_1, & s_1 &= t_1 y_2 \\
m_1 t_1 &= m_2 s_2, & s_2 y_2 &= t_2 y_3 \\
&\vdots & &\vdots \\
m_{n-1} t_{n-1} &= m_n s_n, & s_n y_n &= t_n \\
x_n t_n &= x'. & &
\end{aligned} \tag{1}$$

The set of equations (1) is called a *scheme* of length n over M and N joining $(m, 1)$ to $(m', 1)$.

Then we shall show that $m = m'$ in X.

Case 1 : there exists $2 \leq i \leq n$ with $s_i \in S_X$. Then $s_{i-1}y_{i-1}(= t_{i-1}y_i) = t_{i-1}s_i^{-1}t_i y_{i+1} =$, $n_{i-1}t_{i-1}s_i^{-1}t_i(= m_i s_i s_i^{-1} t_i = m_i s_i s_i^{-1} t_i) = m_{i+1}s_{i+1}$ and so the scheme is shorten.

Case 2 : there exists $1 \leq i \leq n-1$ with $t_i \in S_X$. Then $m_{i-1}t_{i-1}(= m_i s_i = m_i t_i t_i^{-1} s_i) = m_{i+1}s_{i+1}t_i^{-1}s_i$, $s_{i+1}t_i^{-1}s_i y_i(= s_{i+1}t_i^{-1}t_i y_{i+1} = s_{i+1}y_{i+1}) = t_{i+1}y_{i+2}$ and so the scheme is shorten.

Case 3 : $s_1 \in S_X$. Then $t_1 \in S_X$. By the result of Case 2, the scheme is shorten.

Case 4 : $t_n \in S_X$. By the result of Case 1, the scheme is shorten.

Now we can assume that all of s_i, t_i $(1 \leq i \leq n)$ belong to R_X. If $t_1 = s_2$, then $m(= m_1 s_1 = m_1(t_1 s_1) = (m_2 s_2)s_1) = m_2 s_1$, $s_1(= t_1 y_2 = s_2 y_2) = t_2 y_3$, $m_2 t_2 = m_3 t_3$ and then the scheme is shorten.

So we can assume that $t_1 \neq s_2$. By Lemma 5(2), we have $|MR_X| = 1$. Consequently, $m = m_1 s_1 = m_1 t_1 = m_2 t_2 = \cdots = m_{n-1} t_{n-1} = m_n s_n = m_n t_n = m'$. We conclude that U_X has the representation extension property. □

Theorem 4. *The semigroup $U_X = S_X \cup I_X$ is an amalgamation base for semigroups.*

Proof. It is well-known that R_X is left absolutely flat. (See [1].) Also, it is easy to see that all factor semigroups $U_X/M_X, \cdots, I_{n+1}/I_n, \cdots, I_2/R_X$ satisfy the cinditions (1.1) and (1.2) in the sense of [13]. By Theorem 1 of [13], U_X is left absolutely flat. By Proposition 1.1 of [1] U_X has the free representation extension property in the sense of [3].

Next we use induction on the index k of I_k to prove that $S_X \cup I_k$ has the presentation extension prperty. Let $U_k = S_X \cup I_k$. By lemma 3, the case $p = 1$ is done. We assume that the theorem holds in the case $k = p$.

By Theorem 2.1 of [11], we suppose that there exist a right U_{p+1}-set X, a left U_{p+1}-set M containing U_{p+1} as a I_{p+1}-subsets and $x, x' \in X$ such that $x \neq x'$ in X and $x \otimes 1 = x' \otimes 1$ in $X \otimes_{U_{p+1}} Y$. Then by Lemma 1.2 of [1], there exist $x_1, \cdots, x_n \in X$, $y_2, \cdots, y_n \in Y$, $s_1, \cdots, s_n$, $t_1, \ldots, t_n \in U_{p+1}$ such that

$$\begin{aligned} x &= x_1 s_1, \quad s_1 &= t_1 y_2 \\ x_1 t_1 &= x_2 s_2, \quad s_2 y_2 &= t_2 y_3 \\ &\vdots &\vdots \\ x_{n-1} t_{n-1} &= x_n s_n, \quad s_n y_n &= t_n \\ x_n t_n &= x'. \end{aligned} \tag{2}$$

The set of equations (1) is called a *scheme* of length n over X and Y joining $(x, 1)$ to $(x', 1)$.

If $s_i \in S_X$ or $t_i \in S_X$, then as the cases (1) and (2) in the proof of Theorem 3, the scheme (2) is transformed to a scheme in whitch any element of S_X does not appear.

all of the s_i, t_i are elements of I_{p+1} and some of s_i, t_i which belong to $I_{p+1} - I_p$.

To obtain that $x = x'$ in X, we shall show to reduce the number of s_i, t_i which belong to $I_{p+1} - I_p$ to the case that all of the s_i, t_i belong to I_p.

Case 1 : $s_1 \in I_{p+1} - I_p$. Then by Ann_l we have $t_1 \in I_{p+1} - I_p$ and $s_1 \mathscr{R} t_1$.

Sucase 1.1 : $t_1, s_2 \in I_{p+1} - I_p$ and $t_1 \mathscr{L} s_2$. Then by Ann_l we have $t_2 \in I_{p+1} - I_p$ and $s_2 \mathscr{R} t_2$. Hence $x = x_1 s_1 = x_1 t_1 t_1^* s_1 = (x_2 s_2) t_1^* s_1$, $s_2 t_1^* s_1 = s_2 t_1^* t_1 y_2 = s_2 t_1^* t_1 (s_2^* s_2) y_2 = s_2 t_1^* t_1 s_2^* (t_2 y_2)$, $x_2 (s_2 t_1^* t_1 s_2^* t_2) = x_2 (s_2 s_2^* t_2) = x_2 t_2$. The scheme is shorten.

Sucase 1.2 : $t_1, s_2 \in I_{p+1} - I_p$ and $(t_1, s_2) \notin \mathscr{L}$. Then by Ann_r thre exists $u \in I_{p+1}$ such that $t_1 u = t_1$ but $s_2 u \in I_p$ (resp. $s_2 u = s_2$ but $t_1 u \in I_p$). Then $x = x_1 s_1 = x_1 t_1 t_1^* s_1$ $= x_1 t_1 u (t_1^* s_1) = x_1 t_1 ((s_2 u)^* (s_2 u))(t_1^* s_1)$, $t_1 ((s_2 u)^* (s_2 u))(t_1^* s_1) = t_1 ((s_2 u)^* (s_2 u)) t_1^* (t_1 y_2)$, $x_1 t_1 ((s_2 u)^* (s_1 2 u)) t_1^* t_1 = x_1 (t_1 u)((s_2 u)^* (s_2 u)) t_1^* t_1 = x_2 (s_2 u) t_1^* t_1 = x_1 (t_1 u)) t_1^* t_1 = x_1 (t_1 t_1^* t_1)$ $= x_1 t_1 = x_2 s_2$.

This case is reduced to the case that $s_1 \in I_p$.

By doing the same argument from the other end of the scheme (2), it suffices to deal the case that $t_n \in I_p$.

Hereater, we can assume that $s_1 \in I_p$ and $t_n \in I_p$.

Case 2: there exists $1 < i < n-1$ such that $s_i \in I_p$, $t_i \in I_{p+1} - I_p$, and $s_{i+1} \in I_p$. Then

$$s_i y_i (= t_i y_{i+1}) = (t_i t_i^* s_i s_i^* t_i) y_{i+1},$$

$$x_i (t_i t_i^* s_i s_i^* t_i)(= (x_{i+1} s_{i+1})(t_i^* s_i s_i^* t_i)) = x_{i+1}(s_{i+1} t_i^* s_i s_i^* t_i),$$

$$(s_{i+1} t_i^* s_i s_i^* t_i) y_{i+1} = (s_{i+1} t_i^* t_i) y_{i+1},$$

$$x_{i+1}(s_{i+1} t_i^* t_i)(= x_i t_i) = x_{i+1} s_{i+1}$$

Hence this case is reduced to the case that $t_i \in I_p$, although scheme gets longer. We are done.

Case 3: there exists $1 < i < n-1$ such that $t_i \in I_p$, $s_{i+1} \in I_{p+1} - I_p$ and $t_{i+1} \in I_p$. Then

$$x_i t_i (= x_{i+1} s_{i+1}) = x_{i+1}(s_{i+1} t_i^* t_i s_{i+1}^* s_{i+1}),$$

$$(s_{i+1} t_i^* t_i s_{i+1}^* s_{i+1}) y_{i+1} (= (s_{i+1} t_i^* t_i s_{i+1}^*)(s_{i+1} y_{i+1})) = (s_{i+1} t_i^* t_i s_{i+1}^* t_{i+1}) y_{i+2},$$

$$x_{i+1}(s_{i+1} t_i^* t_i s_{i+1}^* t_{i+1})(= (x_i t_i)(t_i^* t_i s_{i+1}^* t_{i+1})) = x_{i+1}(s_{i+1} s_{i+1}^* t_{i+1}),$$

$$(s_{i+1} s_{i+1}^* t_{i+1}) y_{i+2} = t_{i+1} y_{i+2}$$

We are done.

Case 4: there exists $1 < i < n-1$ such that $s_i \in I_p$, $t_i, s_{i+1} \in I_{p+1} - I_p$. If $t_i \geq_{\mathscr{L}} s_{i+1}$, then

$$s_i y_i = (t_i t_i^* s_i s_i^* t_i) y_{i+1}, \quad x_i (t_i t_i^* s_i s_i^* t_i) = x_{i+1}(s_{i+1} t_i^* s_i s_i^* t_i),$$

$$(s_{i+1} t_i^* s_i s_i^* t_i) y_{i+1} (= s_{i+1}(t_i^* t_i) y_{i+1}) = s_{i+1} y_{i+1} \text{ (since } t_i \geq_{\mathscr{L}} s_{i+1})) = t_{i+1} y_{i+2}$$

Hence this case is reduced to the case that $t_i, s_{i+1} \in I_p$.

If $t_i \not\geq_{\mathscr{L}} s_{i+1}$, then by(1.2), there exist $t_i^*, s_{i+1}^* \in I_{p+1} - I_p$ such that $s_{i+1}t_i^*t_is_{i+1}^*s_{i+1} \in I_p$. So by applying the argument of Case 2 to the equantions : $s_iy_i = t_iy_{i+1}$, $x_it_i(= x_{i+1}(s_{i+1}t_i^*t_is_{i+1}^*s_{i+1})) = x_{i+1}s'_{i+1}$, where $s'_{i+1} = s_{i+1}t_i^*t_is_{i+1}^*s_{i+1}$, we get equations :

$$s_iy_i(= t_iy_{i+1}) = (t_it_i^*s_is_i^*t_i)y_{i+1},$$

$$x_i(t_it_i^*s_is_i^*t_i)(= (x_{i+1}s'_{i+1})(t_i^*s_is_i^*t_i)) = x_{i+1}(s'_{i+1}t_i^*s_is_i^*t_i),$$

$$(s'_{i+1}t_i^*s_is_i^*t_i)y_{i+1} = (s'_{i+1}t_i^*t_i)y_{i+1},$$

$$x_{i+1}(s'_{i+1}t_i^*t_i)(= x_it_i = x_{i+1}s'_{i+1}) = x_{i+1}s_{i+1}.$$

Hence this case is reduced to the case that $t_i \in I_p$. We are done.

Case 5 : there exists $1 < i < n-1$ such that $t_i \in I_p$, $s_{i+1}, t_{i+1} \in I_{p+1} - I_p$. If $s_{i+1} \geq_{\mathscr{R}} t_{i+1}$, then

$$x_it_i = x_{i+1}(s_{i+1}t_i^*t_is_{i+1}^*s_{i+1}),$$

$$(s_{i+1}t_i^*t_is_{i+1}^*s_{i+1})y_{i+1} = (s_{i+1}t_i^*t_is_{i+1}^*t_{i+1})y_{i+2},$$

$$x_{i+1}(s_{i+1}t_i^*t_is_{i+1}^*t_{i+1})(= (x_it_i)(t_i^*t_is_{i+1}^*t_{i+1})) = x_{i+1}(s_{i+1}s_{i+1}^*t_{i+1})$$

$$= x_{i+1}t_{i+1}(\text{since } s_{i+1} \geq_{\mathscr{R}} t_{i+1}))) = x_{i+2}s_{i+2}.$$

Hence this case is reduced to the case that $s_{i+1}, t_{i+1} \in I_p$.

If $s_{i+1} \not\geq_{\mathscr{R}} t_{i+1}$, then by(1.1), there exist $s_{i+1}^*, t_{i+1}^* \in I_{p+1} - I_p$ such that $t_{i+1}t_{i+1}^*s_{i+1}s_{i+1}^*t_{i+1} \in I_p$. So by applying the argument of Case 3 to the equentions : $x_it_i = x_{i+1}s_{i+1}$, $s_{i+1}y_{i+1}(= (t_{i+1}t_{i+1}^*s_{i+1}s_{i+1}^*t_{i+1})y_{i+2}) = t'_{i+1}y_{i+2}$, where $t'_{i+1} = t_{i+1}t_{i+1}^*s_{i+1}s_{i+1}^*t_{i+1}$.

Then

$$x_it_i(= x_{i+1}s_{i+1}) = x_{i+1}(s_{i+1}t_i^*t_is_{i+1}^*s_{i+1}),$$

$$(s_{i+1}t_i^*t_is_{i+1}^*s_{i+1})y_{i+1}(= (s_{i+1}t_i^*t_is_{i+1}^*)(s_{i+1}y_{i+1})) = (s_{i+1}t_i^*t_is_{i+1}^*t'_{i+1})y_{i+2},$$

$$x_{i+1}(s_{i+1}t_i^*t_is_{i+1}^*t'_{i+1})(= (x_it_i)(t_i^*t_is_{i+1}^*t'_{i+1})) = x_{i+1}(s_{i+1}s_{i+1}^*t'_{i+1}),$$

$$(s_{i+1}s_{i+1}^*t'_{i+1})y_{i+2}(= s_{i+1}y_{i+2}) = t_{i+1}y_{i+2}$$

Hence this case is reduced to the case that $s_{i+1} \in I_p$. We are done.

Consequently, by repeatedly making use of cases 2 through 5, we can reduce all cases to the case that all of s_i, t_i belong to I_p since $s_1, t_n \in I_p$. Therefore, it follows from the induction assumption that $x = x'$. Hence U_X has the representation extension property. □

Remark 2. Also, the author found out an incorrect part in the proof of lemma 2.3 of [14]. The proof of Lemma 4 gives an correct proof of Lemma 2.4 of [14] so as to prove Theorem 1.5 (the main theorm) of [14] correctly.

Remark 3. In the Theorem 1 of the paper "Absolute Flatness of Regular Semigroups With a Finite Height Function" Semigroup Forum Vol. 52 (1996) 133-140, we found two parts to be corrected :

1. on page 135, line 5th from the bottom to the bottom :

If $t_i \not\geq_{\mathscr{L}} s_{i+1}$, then by(1.2), there exist $t_i^*, s_{i+1}^* \in I_{p+1} - I_p$ such that $s_{i+1}t_i^*t_is_{i+1}^*s_{i+1} \in I_p$. So by applying the argument of Case 2 to the equentions : $s_iy_i = t_iy_{i+1}$, $x_it_i(= x_{i+1}(s_{i+1}t_i^*t_is_{i+1}^*s_{i+1})) = x_{i+1}s'_{i+1}$, where $s'_{i+1} = s_{i+1}t_i^*t_is_{i+1}^*s_{i+1}$, we get equations :

$$s_iy_i(= t_iy_{i+1}) = (t_it_i^*s_is_i^*t_i)y_{i+1},$$

$$x_i(t_it_i^*s_is_i^*t_i)(= (x_{i+1}s'_{i+1})(t_i^*s_is_i^*t_i)) = x_{i+1}(s'_{i+1}t_i^*s_is_i^*t_i),$$

$$(s'_{i+1}t_i^*s_is_i^*t_i)y_{i+1} = (s'_{i+1}t_i^*t_i)y_{i+1},$$

$$x_{i+1}(s'_{i+1}t_i^*t_i)(= x_it_i = x_{i+1}s'_{i+1}) = x_{i+1}s_{i+1}$$

and

$$\mu(t_it_i^*s_is_i^*t_i) = \mu(s'_{i+1}t_i^*s_is_is_i^*t_i) = \mu(s'_{i+1}t_i^*t_i) = 0.$$

Hence this case is reduced to the case that $t_i \in I$. By inductive assumption, the result follows.

2. on page 136, line 8th to lin 13th :

If $s_{i+1} \not\geq_{\mathscr{R}} t_{i+1}$, then by(1.1), there exist $s_{i+1}^*, t_{i+1}^* \in I_{p+1} - I_p$ such that $t_{i+1}t_{i+1}^*s_{i+1}s_{i+1}^*t_{i+1} \in I_p$. So by applying the argument of Case 3 to the equentions : $x_it_i = x_{i+1}s_{i+1}$, $s_{i+1}y_{i+1}(= t_{i+1}t_{i+1}^*s_{i+1}s_{i+1}^*t_{i+1})y_{i+2}) = t'_{i+1}y_{i+2}$, where $t'_{i+1} = t_{i+1}t_{i+1}^*s_{i+1}s_{i+1}^*t_{i+1}$.

Then

$$x_it_i(= x_{i+1}s_{i+1}) = x_{i+1}(s_{i+1}t_i^*t_is_{i+1}^*s_{i+1}),$$

$$(s_{i+1}t_i^*t_is_{i+1}^*s_{i+1})y_{i+1}(= (s_{i+1}t_i^*t_is_{i+1}^*)(s_{i+1}y_{i+1})) = (s_{i+1}t_i^*t_is_{i+1}^*t'_{i+1})y_{i+2},$$

$$x_{i+1}(s_{i+1}t_i^*t_is_{i+1}^*t'_{i+1})(= (x_it_i)(t_i^*t_is_{i+1}^*t'_{i+1})) = x_{i+1}(s_{i+1}s_{i+1}^*t'_{i+1}),$$

$$(s_{i+1}s_{i+1}^*t'_{i+1})y_{i+2}(= t'_{i+1}y_{i+2}) = t_{i+1}y_{i+2}$$

and

$$\mu(s_{i+1}t_i^*t_is_{i+1}^*s_{i+1}) = \mu(s_{i+1}t_i^*t_is_{i+1}^*t'_{i+1}) = \mu(s_{i+1}s_{i+1}^*t'_{i+1}) = 0.$$

Hence this case is reduced to the case that $s_{i+1} \in I$. By inductive assumption, the result follows.

2 Amalgamation bases and the infinite full transformation semigroups

A semigroup U is called an *amalgamation base for finite semigroups* if every amalgam $[S, T; U]$ of semigroups S, T with U as a core is embedded in a semigroup. In [11], it was proved that that the finite full transformatuon semigroups are amalgamation bases for semigroups.

In order to prove that the infinite full transformation semigroups are amalgamation bases, we will extend the lemmas for the proof that the finite full transformation semigroupsis are amalgamation bases in [14] from finite semigroups to infinite semigroups.

Actually, we can use a infinite version of Result in [14] as a criterion of amalgamation bases for semigroups.

Let $\mathscr{T}_X$ denote the full transformation semigroup on a set X with composition being from right to left.

Let S be a semigroup. Then a left [resp. right] S-set is a set with an associative operation of S on the left [resp. right]. A left [resp. right] S-set X is *faithful* if for distinct $s, t \in S$, there exists $x \in X$ with $sx \neq tx$. Thus, given a faithful left [resp. right] S-set X, we obtain a canonical embedding of S into $\mathscr{T}_X$ and vice-versa.

The following result can give characterizations of amalgamation bases for semigroups

Criterion(Lemma 1 and its corollary of [6]). *Let U be a semigroup. Then the following are equivalent :*

(1) *U is an amalgamation base for semigroups ;*

(2) *For any two embeddings ϕ_1, ϕ_2 of U into the full transformation semigroup $\mathscr{T}_X$, there exist a set Y and two embeddings $\delta_1, \delta_2 : \mathscr{T}_X \to \mathscr{T}(Y)$ such that Y contains X as a subset and $\delta_1\phi_1$ and $\delta_2\phi_2$ coincide on U;*

(3) *For any semigroups S, T, any faithful left [right] S-set X and any faithful left [right] T-set Y, there exist a faithful left [right] S-set $X' \supseteq X$ and a faithful left [right] T-set $Y' \supseteq Y$ such that the U-sets X', Y' are U-isomorphic to each other.*

Let $\mathscr{T}_X$ denote the full transformation semigroup on a set X with composition being from right to left.

The following Lemma 5 and Lemma 6 are infinite case versions of Lemma 2 and Lemma 4 of [15], respectively.

Lemma 5. (Compare Lemma 2 of of [15]) *Let U be a regular semigroup whose $\mathscr{J}$-classes form a chain. Suppose that there is a chain of ideals $U = I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n$ such that I_n is a right zero semigroup and each I_i/I_{i+1} is a completely 0-simple semigroup. Asssume that each I_i/I_{i+1} satisfies the condition Ann_l $(1 \leq i \leq n-1)$. Let ϕ_1, ϕ_2 be embeddings of U into the full transformation semigroup $\mathscr{T}_X$ such that $|Y^{(1)}| = |Y^{(2)}|$, where $Y^{(1)} = (\bigcup_{u\in U} \phi_1(u)(X))$ and $Y^{(2)} = (\bigcup_{u\in U} \phi_2(u)(X))$. Then any U-isomorphism between the right U-set $\mathscr{T}_X\phi_1(U)$ and the right U-set $\mathscr{T}_X\phi_2(U)$ extends a U-isomorphism from the right $\phi_1(U)$-set $\mathscr{T}_X$ to the right $\phi_2(U)$-set $\mathscr{T}_X$.*

Proof. Suppose that there exists a U-isomorphism θ from the right U-set $\mathscr{T}_X\phi_1(U)$ to the right U-set $\mathscr{T}_X\phi_2(U)$. Let $f \in Map(Y^{(1)}, X)$. Then there exists uniquely $f' \in Map(Y^{(2)}, X)$ such that $\theta(f\phi_1(e)) = f'\phi_2(e)$ for all $e \in E_U$. In fact, we define a mapping $f' \in Map(Y^{(2)}, X)$ by $f'(x) = \theta(f\phi_1(e))(x)$ if $x \in \phi_2(e)(X)$ for some $e \in E_U$, where E_U denotes the set of all idempotents of U.

For $x \in \bigcup_{e\in E_U} \phi_2(e)(X))$, there exists an idempotent $e_x \in E_U$ such that $\phi_2(e_x)(x) = x$, and $h\mathcal{R}_U e_x$ for any $h \in E_U$ with $\phi_2(h)x = x$ and $e_x h = h$, since every descending chain of $\mathscr{R}_U$-classes of U is finitely stops. Then we shall prove that $he_x = e_x he_x$ for all $h \in E_U$ with $\phi_2(h)(x) = x$. For, firstly by the property of e_x we have $e_x he_x(e_x he_x)^* e_x \mathcal{R}_U e_x$, since $\phi_2(e_x he_x)(x) = (x)$. So $e_x he_x \mathcal{R}_U e_x$. Hence $he_x \mathcal{J}_U e_x$. Since I_n is a right zero semigroup, if $(he_x, e_x) \notin \mathcal{R}_U$, then $e_x, he_x \in I_i - I_{i-1}$ for some $2 \leq i \leq n$. By the condition Ann_l on the factor semigroup I_i/I_{i-1}, it follows that there exist an element $u \in I_i$ such that either (1) $uhe_x = he_x$, but $(ue_x, e_x) \notin \mathcal{J}_U$ or (2) $ue_x = e_x$, but $(uhe_x, he_x) \notin \mathcal{J}_U$. In the case (1), $\phi_2(u)(x)(x) = \phi_2(u)(\phi_2(he_x)(x)) = \phi_2(uhe_x)(x) = \phi_2(he_x)(x) = x$. So $\phi_2(e_x ue_x)(x)= x$ and hence $\phi_2((e_x ue_x)(e_x ue_x)^*)(x)= x$. Since $(e_x ue_x)(e_x ue_x)^*$ is an idempotent, it follows from the property of e_x that $(e_x ue_x)(e_x ue_x)^* \mathcal{R}_U e_x$, which contradicts that $(ue_x, e_x) \notin \mathcal{J}_U$. In the case (2), $\phi_2(u)(x) = \phi_2(u)(\phi_2(e_x)(x) = \phi_2(ue_x)(x) = \phi_2(e_x)(x) = x$. So $\phi_2(e_x uhe_x)(x)= x$ and hence $\phi_2((e_x uhe_x)(e_x uhe_x)^*)(x)= x$. By the property of e_x that $(e_x uhe_x)(e_x uhe_x)^* \mathcal{R}_U e_x$, which contradicts that $(uhe_x, e_x) \notin \mathcal{J}_U$.

Thus it must hold that $he_x = e_x he_x$ for all $h \in E_U$ with $\phi_2(h)(x) = x$.

If $x \in \phi_2(h)(X)$, where $h \in E_U$, then $\phi_2(h)(x) = x$ and hence, $\theta(f\phi_1(h))(x) = \theta(f\phi_1(h))(\phi_2(e_x)(x)) = \theta(f\phi_1(h)\phi_1(e_x))(x) = \theta(f\phi_1(he_x))(x) = \theta(f\phi_1(e_x he_x))(x) = \theta(f\phi_1(e_x))(\phi_2(he_x)(x)) = \theta(f\phi_1(e_x))(x)$.

Moreover, if $x \in \phi_2(h)_X \cap \phi_2(h')(x)$, where $h, h' \in E_U$, then $\theta(f\phi_1(h))(x) = \theta(f\phi_1(e_x))(x) =$

$\theta(f\phi_1(h'))(x)$.

Thus f' is well-defined and unique. So we obtain a bijection $\xi : Map(Y^{(1)}, X) \to Map(Y^{(2)}, X)$ with $\xi(f) = f'$.

If $t = f\phi_1(u)$, where $u \in U$, $f \in \mathscr{T}_X$, then $t = t|_{Y^{(1)}}\phi_1(u^*u)$. Conversely, for $g \in Map(Y^{(1)}, X)$ and $e \in E_U$, we have $g\phi_1(e) \in \mathscr{T}_X\phi_1(U)$. Then

$$\mathscr{T}_X\phi_1(U) = \{g\phi_1(e) \mid g \in Map(Y^{(1)}, X),\ e \in E_U\} \tag{3}$$

.

Also,

$$\mathscr{T}_X\phi_2(U) = \{g'\phi_2(e) \mid g' \in Map(Y^{(2)}, X),\ e \in E_U\} \tag{4}$$

.

Now we have

$$\theta(f\phi_1(u)) = \xi(f)\phi_2(u) \tag{5}$$

.

Actually, $(\xi(f)\phi_2(u))(x) = \xi(f)(\phi_2(uu^*)(\phi_2(u)(x)) = \theta(f\phi_1(uu^*))(\phi_2(u)(x)) = \theta(f\phi_1(uu^*)\phi_1(u))(x) = \theta(f\phi_1(u))(x)$.

For any $f \in Map(Y^{(1)}, X)$, let $V^{(1)}(f)$ be the set

$$\{h|_{X-Y^{(1)}} \in Map(X - Y^{(1)}, X) \mid h \in \mathscr{T}_X - \mathscr{T}_X\phi_1(U)\text{and } h|_{Y^{(1)}} = f\}.$$

Also, for any $f' \in Map(Y^{(2)}, X)$, let $V^{(2)}(f')$ be the set

$$\{h'|_{X-Y^{(2)}} \in Map(X - Y^{(2)}, X) \mid h' \in \mathscr{T}_X - \mathscr{T}_X\phi_2(U) \text{ and } h'|_{Y^{(2)}} = f'\}.$$

Hence $V^{(1)}(f)$= $Map(X - Y^{(1)}, X) - \{h|_{X-Y^{(1)}} \mid h \in \mathscr{T}_X\phi_1(U)$ and $h|_{Y^{(1)}} = f\}$ and $V^{(2)}(f')$= $Map(X - Y^{(2)}, X) - \{h'|_{X-Y^{(2)}} \mid h' \in \mathscr{T}_X\phi_2(U)$ and $h'|_{Y^{(2)}} = f'\}$

We shall show that $|V^{(1)}(f) = |V^{(2)}(f)|$ for each $f \in Map(Y^{(1)}, X)$.

For $f \in Map(Y^{(1)}, X)$ and $f' = \xi(f)$, we obtain that

for any $e \in E_U$, $f = f\phi_1(e)|_{Y^{(1)}}$ if only if $f' = \theta(f\phi_1(e))|_{Y^{(2)}}$. Actually, if $f = f\phi_1(e)|_{Y^{(1)}}$ for some $e \in E_U$ then for any $x \in Y^{(2)}$, $f'(x) = \theta(f\phi_1(e'))(x)$ $(x \in \phi_2(e')(X),\ e' \in E_U) = \theta((f\phi_1(e))\phi_1(e'))(x) = \theta((f\phi_1(e))(\phi_2(e')(x)) = \theta(f\phi_1(e))(x)$. Hence $f' = \theta(f\phi_1(e))|_{Y^{(2)}}$. The converse is true.

Next it follows from (5) that

$$\text{for any } e, e' \in E_U,\ f\phi_1(e) = f\phi_1(e') \text{ if and only if } f'\phi_2(e) = f'\phi_2(e'). \tag{6}$$

Consequently, by (3), (4), (6) and θ, we have

$|\{h|_{X-Y^{(1)}} \mid h \in \mathcal{T}_X\phi_1(U) \text{ and } h|_{Y^{(1)}} = f\}| = |\{h'|_{X-Y^{(2)}} \mid h' \in \mathscr{T}_X\phi_2(U) \text{ and } h'|_{Y^{(2)}} = f'\}|$.

Hence $|V^{(1)}(f) = |V^{(2)}(f)|$ for each $f \in Map(Y^{(1)}, X)$.

So, for each $f \in Map(Y^{(1)}, X)$, there esixts a bijection $\Xi_f : V^{(1)}(f) \to V^{(2)}(\xi(f))$.

Next, when each $h \in \mathscr{T}_X$ is written as the form $(h|_{Y^{(1)}}, h|_{X-Y^{(1)}})$, we have

$$\mathscr{T}_X - \mathscr{T}_X\phi_1(U) \text{ if and only if } h|_{X-Y^{(1)}} \in V^{(1)}(h|_{Y^{(1)}}).$$

also, when each $h \in \mathscr{T}_X$ is written as the form $(h|_{Y^{(2)}}, h|_{X-Y^{(2)}})$, we have

$$\mathscr{T}_X - \mathscr{T}_X\phi_2(U) \text{ if and only if } h|_{X-Y^{(2)}} \in V^{(2)}(h|_{Y^{(2)}}).$$

Therefore,

$$\mathscr{T}_X - \mathscr{T}_X\phi_1(U) = \bigcup_{f\in Map(Y^{(1)},X)} \{f\} \times V^{(1)}(f)$$

,

and so,

$$\mathscr{T}_X - \mathscr{T}_X\phi_1(U) = \bigcup_{f\in Map(Y^{(1)},X)} \{\xi(f)\} \times V^{(2)}(\xi(f))$$

Thus we define a bijection $\Theta : \mathscr{T}_X \to \mathscr{T}_X$ by θ and gluing ξ and Ξ_f as follows : .

For $h \in \mathscr{T}_X$,

$$\Theta(h) = \begin{cases} \theta(h) & \text{if } h \in \mathscr{T}_X\phi_1(U) \\ (\xi(h|_{Y_1}), \Xi_{h|_{Y_1}}(h|_{X-Y_1})) & \text{if } h \in \mathscr{T}_X - \mathscr{T}_X\phi_1(U) \end{cases}$$

Then if $h \in \mathscr{T}_X\phi_1(U)$ and $u \in U$, then $\Theta(h\phi_1(u)) = \theta(h\phi_1(u)) = \theta(h)\phi_2(u) = \Theta(h)\phi_2(u)$.

For $x \in X$, $u \in U$, $(h\phi_1(u))_X = ((h|_{Y^{(1)}}, h|_{X-Y^{(1)}})\phi_1(u))_X$ and hence $h\phi_1(u) = h|_{Y^{(1)}}\phi_1(u)$.

If $h \in \mathscr{T}_X - \mathscr{T}_X\phi_1(U)$, then $\Theta(h)\phi_2(u) = (\xi(h|_{Y_1}), \Xi_{h|_{Y_1}}(h|_{X-Y_1}))\phi_2(u) = \xi(h|_{Y_1})\phi_2(u)$ $= \theta(h|_{Y_1}\phi_1(u)) = \theta(h\phi_1(u)) = \Theta(h\phi_1(u))$.

Therefore, Θ is a U-isomorphism of the right $\phi_1(U)$-set $\mathscr{T}_X$ to the right $\phi_2(U)$-set $\mathscr{T}_X$.

The lemma is proved. □

Let G be a group and X a G-set. Then X is a disjoint union of cyclic left G-set Gx_i $(i \in I)$. Each left G-set Gx_i is isomorhic to a left G-set $G/H = \{gH|g \in G\}$, where H is a subgroup of G.

For subgroups H, F of G, left G-sets G/H and G/F are G-isomorphic to each other if and only if thre exists $g \in G$ with $F = g^{-1}Hg$.

If there exist a G-isomorphism ϕ of left G-sets G/H to G/F, then letting $\phi(H)= aF$ $(a \in G)$, we have $h(aF) = haF$ for all $h \in H$ since $h\phi(H) = \phi(hH) = \phi(H)$, So $HaF = aF$ and hence $a^{-1}Ha = F$.

Conversely, the mapping ϕ of a left G-set G/H to a left G-set $G/a^{-1}Ha$ with $\phi(gH) = ga(a^{-1}Ha) = gHa$ is a G-isomorphism between them. For a subgroup H of G, if a left G-set A is isomorhc to the left G-set G/H, then A is called to be *of type H*. Then the type is uniquely determined up to conjugacy.

Lemma 6. (Compare Lemma 4 of of [15]) *Let G be a group with an identity element e and I a regular semigroup. Let U be a regular semigroup semigroup which is a disjoint union of G and I and I is an ideal of U.*

.

Suppose that there are embeddings ϕ_1, ϕ_2 of U into a semigroup S such that the restrictions to $I \cup \{e\}$ of ϕ_1 and ϕ_2 are equal. Then there exist a set Y and embeddings $\xi_1 : \mathscr{T}_X \to \mathscr{T}(Y)$ and $\xi_2 : \mathscr{T}_X \to \mathscr{T}(Y)$ such that $\xi_1\phi_1|_U = \xi_2\phi_2|_U$.

Proof Since groups are amalgamation bases, it follows from Theorem 1 of [8] that there exist a set X' and an embedding ϕ of $\mathscr{T}_X$ into $\mathscr{T}(X')$ such that there exists $c \in \mathscr{T}(X')$ such that $c(\phi\phi_1(g))c^{-1} = \phi\phi_2(g)$ for all $g \in G$. Let $\phi'_i = \phi\phi_i$ $(i = 1, 2)$. So we may assume that $cc^{-1} = c^{-1}c = \phi'_1(e)(= \phi'_2(e))$ and $c = \phi'_1(e)c = c\phi'_1(e)$ and $c^{-1} = \phi'_1(e)c^{-1} = c^{-1}\phi'_1(e)$.

Firstly, we observe that $\phi'_1(e)(X')$ is both a $\phi'_1(G)$-set and a $\phi'_2(G)$-set. Also it holds that $c\phi'_1(g) = \phi'_2(g)c$. So c induce a homomorphism of a $\phi'e(X')$. Specially, $c(\phi'_1(G)z) = \phi'_2(G)c(z)$ for all $z \in \phi'_1(e)(X')$.

Let $Z_1 = \phi'_1(e)(X') - \left(\bigcup_{u\in I} \phi'_1(u)(X')\right)$ and $Z_2 = \phi'_1(e)(X') \cap \left(\bigcup_{u\in I} \phi'_1(u)(X')\right)$.

Note that $\phi'_1(g)(\bigcup_{u\in I} \phi'_1(u)(X')) \subseteq \bigcup_{u\in I} \phi'_1(u)(X')$ for all $g \in G$, since $gI \subseteq I$. Hence $\phi'_1(g)(Z_2) = Z_2$ and $\phi'_1(g)(Z_1) = Z_1$ for all $g \in G$. Similarly, we have $\phi'_2(g)(Z_2) = Z_2$ and $\phi'_2(g)(Z_1) = Z_1$ for all $g \in G$.

Set $Z_{11} = \{x \in Z_1 \mid c(x) \in Z_1\}$, $Z_{12} = \{x \in Z_1 \mid c(x) \in Z_2\}$, $Z_{21} = \{x \in Z_2 \mid c(x) \in Z_1\}$, $Z_{22} = \{x \in Z_2 \mid c(x) \in Z_2\}$. Then $Z_1 = Z_{11} + Z_{12} = c(Z_{11}) + c(Z_{21})$ and $Z_2 = Z_{21} + Z_{22}$=$c(Z_{12}) + c(Z_{22})$, where + denotes disjoint union. Note that $|Z_{12}| = |Z_{21}|$.

Further, we have $\phi'_1(G)(Z_{12}) = Z_{12}$ and $\phi'_2(G)(c(Z_{12})) = c(Z_{12})$. Actually, for $z \in Z_{12}$

and $g \in G$, $c(\phi'_1(g)(z)) = c\phi'_1(g)((c^{-1}c)(z)) = (c\phi'_1(g)c^{-1})(c(z)) = \phi'_2(g)(c(z)) \in Z_2$. Then $\phi'_1(g)(z) \in Z_{12}$, since $\phi'_1(g)(z) \in Z_1$. Hence $\phi'_1(G)(Z_{12}) = Z_{12}$. Consequently, $\phi'_2(G)(c(Z_{12})) = c(\phi'_1(G)Z_{12}) = c(Z_{12})$.

In the same way as above, we obtain that $\phi'_1(G)(Z_{21}) = Z_{21}$ and $\phi'_2(G)(c(Z_{21})) = c(Z_{21})$.

' Here we shall prove that there exists a bijection $\alpha \in \mathscr{T}(X')$ such that $\phi'_1(e)\alpha = \alpha\phi'_1(e)$, $\alpha\phi'_1(g) = \phi'_2(g)\alpha$ for all $g \in G$ and α is an identity mapping of $X' - Z_1$.

To do so, let $z \in Z_{12}$. Then if $\phi'_1(G)z$ is isomorphic to $\phi'_1(G)/\phi'_1(H)$ as $\phi'_1(G)$-set, then $\phi'_2(G)c(z)$ is isomorphic to $\phi'_2(G)/\phi'_2(H)$ as as $\phi'_2(G)$-sets. On the other hand, let $z \in Z_{21}$. Then if $\phi'_1(G)z$ is isomorphic to $\phi'_1(G)/\phi'_1(H)$ as $\phi'_1(G)$-set, then $\phi'_2(G)c(z)$ is isomorphic to $\phi'_2(G)/\phi'_2(H)$ as $\phi'_2(G)$-sets.

For a subgroup H of G and $z \in \phi'_1(e)(X)$, if $\phi'_1(G)z$ is of $\phi'_1(H)$ type, then $c(\phi'_1(G)z) = \phi'_2(G)c(z)$ is of $\phi'_2(H)$ type.

Let κ be the cardinal number of the set of $\phi'_1(G)$-sets of tye of $\phi'_1(H)$ in Z_{11} and π the cardinal number of the set of $\phi'_2(G)$-sets of type of $\phi'_2(H)$ in $c(Z_{22})$. Then the cardinal number of the set of $\phi'_1(G)$-sets of type of $\phi'_1(H)$ in $C(Z_{12})$ is equal to κ. Also, the cardinal number of the set of $\phi'_1(G)$-sets of type of $\phi'_1(H)$ in Z_{12} is equal to π. Since the operations of $\phi'_1(G)$ and $\phi'_2(G)$ on Z_2 are the same and $Z_2 = Z_{21}+Z_{22}=c(Z_{12})+c(Z_{22})$, it follows that $\kappa = \pi$. Therefore it follows that there exists a bijective homomorhism t of the $\phi'_1(G)$-set Z_{12} to the $\phi'_2(G)$-set $c(Z_{21})$ with $t\phi'_1(g) = \phi'_2(g)t$ for all $g \in G$. On the other hand, c is a homomorphism of the $\phi'_1(G)$-set Z_{11} to $\phi'_2(G)$-set $c(Z_{11})$.

We define a bijection α of X' by defining as follows :

For any $x \in X'$,

$$\alpha(x) = \begin{cases} c(x) & \text{if } x \in Z_{11} \\ t(x) & \text{if } x \in Z_{12} \\ x & \text{if } x \in X' - Z_1 \end{cases}$$

Then it is easy to see that $\phi'_1(e)\alpha = \alpha\phi'_1(e)$ and

$$\phi'_1(u) = \alpha\phi'_1(u) = \alpha^{-1}\phi'_1(u) \ \text{ for all } \ u \in I \quad \cdots\cdots \quad (*)$$

.

Since for all $g \in G$, $c\phi'_1(g) = \phi'_2(g)c$ and $t\phi'_1(g) = \phi'_2(g)t$ on Z_{12}, it holds that

$$\phi'_2(g)\alpha = \alpha\phi'_1(g) \ \text{ for all } g \in G \qquad \cdots\cdots \quad (**)$$

Now we define a map $\phi_3' : U \to \mathscr{T}(X')$ by $\phi_3'(u) = \alpha^{-1}\phi_2'(u)\alpha$. Then for any $u, u' \in U$, $\phi_3'(uu') = \alpha^{-1}\phi_2'(uu')\alpha = \alpha^{-1}(\phi_2'(u)\phi_2'(u'))\alpha = \alpha^{-1}(\phi_2'(u)(\alpha\alpha^{-1}\phi_2'(u'))\alpha = \phi_3'(u)\phi_3'(u')$. Then ϕ_3' is a homomorphism. Obviously, ϕ_3' is injective. Also, by $(*)$ we have

$$\phi_3'(u) = \phi_1'(u)\alpha \quad \text{for all} \quad u \in I \quad \cdots\cdots \quad (***)$$

.

and by $(**)$ we have

$$\phi_3'(g) = \phi_1'(g) \quad \text{for all} \quad u \in G \quad \cdots\cdots \quad (****)$$

.

Now we define a map $\theta : \mathscr{T}(X')\phi_1'(U) \to \mathcal{T}(X')\phi_3'(U)$ as follows :

For any $f \in \mathcal{T}(X')\phi_1'(U)$,

$$\theta(f) = \begin{cases} f & \text{if } f \in \mathscr{T}(X')\phi_1'(U) - \mathcal{T}(X')\phi_1'(I) \\ f\alpha & \text{if } f \in \mathscr{T}(X')\phi_1'(I) \end{cases}$$

Then it is obvious that θ is an bijection.

We shall prove that θ is a U-homomorphism from the right $\phi_1'(U)$-set $\mathscr{T}(X')\phi_1'(U)$ to the right $\phi_3'(U)$-set $\mathscr{T}(X')\phi_3(U)$.

Case 1 : $u \in I$ and $f \in \mathcal{T}(X')\phi_1'(I)$. Then $\theta(f\phi_1'(u)) = (f\phi_1'(u))\alpha = (f\phi_1'(u))\alpha = f(\alpha\phi_1'(u))\alpha$ (by $(*)$) $= (f\alpha)(\phi_1'(u)\alpha) = \theta(f)\phi_3'(u)$ (by $(***)$).

Case 2 : $u \in I$ and $f \in \mathscr{T}(X')\phi_1'(U) - \mathscr{T}(X')\phi_1'(I)$.

Then $\theta(f\phi_1'(u)) = (f\phi_1'(u))\alpha = f(\phi_1'(u)\alpha) = \theta(f)\phi_3'(u)$.

Case 3 : $u \in G$ and $f \in \mathscr{T}(X')\phi_1'(I)$. Then $f = f'\phi_1'(a)$ where $f' \in \mathscr{T}(X')$, $a \in I$. Hence $\theta(f\phi_1'(u)) = (f\phi_1'(u))\alpha = ((f'\phi_1'(a))\phi_1'(u))\alpha = (f'\phi_1'(au))\alpha = (f'\phi_2'(au))\alpha$ $= (f'\phi_2'(a)\phi_2'(u))\alpha = ((f'\phi_1'(a))\phi_2'(u))\alpha = f\phi_2'(u))\alpha = (f(1_{X'}\phi_2'(u))\alpha = (f(\alpha\alpha^{-1})\phi_2'(u))\alpha$ $= (f\alpha)(\alpha^{-1}\phi_2'(u)\alpha) = (f\alpha)\phi_1'(u) = = \theta(f)\phi_3'(u)$ (by $(****)$).

Case 4 : $u \in G$ and $f \in \mathscr{T}(X')\phi_1'(U) - \mathscr{T}(X')\phi_1'(I)$. Then $\theta(f\phi_1'(u)) = f\phi_1'(u) = \theta(f)\phi_3'(u)$.

By Lemma 5, there exist a set Y' and two embeddings $\delta_1, \delta_2 : \mathscr{T}(X') \to \mathscr{T}(Y')$ such that Y' contains X' as a subset and $\xi_1\phi_1$ and $\xi_2\phi_3$ coincide on U. $\square$

Remark 4. The proof of Lemma 6 is a simplified form of Lemma 4 of [15].

Remark 5. Lemma 3 of of [15] from finite cases to infinite cases is not obtained yet. Neither an extension of Lemma 5 of [15] is obtained.

Actually, let U be a finite regular semigroup which is a disjoint union of 1-idempotent semigroup $\{e\}$ and I such that I is an ideal of U and $I = IeI$. Suppose that there exist two embeddings ϕ_1, ϕ_2 of a finite semigroup $U = e \cup I$ into the finite full transformation semigroup $\mathscr{T}(X)$ with $\phi_1|_I = \phi_2|_I$. Then we shall show that $|\phi_1(e)(X)| > |\phi_1(u)(X)|$ for all $u \in U$. Suppose that there is some $u \in U$ with $|\phi_1(e)(X)| = |\phi_1(u)(X)|$. By $I = IeI$, we have $u = aeb$, wherer $a, b \in I$. Then $|\phi_1(e)(X)| = |\phi_1(ae)(X)|$ or $|\phi_1(e)(X)| = |\phi_1(eb)(X)|$. We know that $es \in V(ae)$ for $s \in V(ae)$ [resp. $se \in V(eb)$ for $s \in V(eb)$]. Then $esae$ [resp. $ebse$] is an idempotent and $e > esae$ [resp. $e > ebse$]. It is a contradiction, since any $\mathscr{J}$-class of $\mathscr{T}(X)$ with an adjoined zero element becomes a completely 0-simple semigroup. Hence there exists an ideal J of $\mathscr{T}(X)$ which contains $\phi_1(I)$, but neither $\phi_1(e)$ nor $\phi_2(e)$. Consequently as shown in Lemma 9 of [6], we may assume that $\phi_1(e)$ and $\phi_2(e)$ belong to a $\mathscr{J}$-class of $\mathscr{T}(X)$. However, $\mathscr{L}$-class of the infinite full transformation semigroups contain infinite chain of idempotents is not completely simple.

References

[1] S. Bulman-Fleming and K. McDowell. *Absolutely flat semigroups.* Pacific J. Math. **107**(1983), 319-333.

[2] A. H. Clifford and G. B. Preston, *Algebraic theory of semigroups*, Amer. Math. Soc., Math. Survey, No.7, Providence, R.I., Vol.I(1961); Vol.II(1967).

[3] T. E. Hall. *Representation extension and amalgamation for semigroups.* Quart. J. Math. Oxford (2) **29**(1978), 309-334.

[4] T. E. Hall. *Finite inverse semigroups and amalgamations.* Semigroups and their applications, Reidel, 1987, pp. 51-56.

[5] T. E. Hall and M.S. Putcha, *The potential $\mathscr{J}$-relation and amalgamation bases for finite semigroups*, Proc. Amer. Math. Soc. **95**(1985), 309-334.

[6] T. E. Hall and K. Shoji, *Finite bands and amalgamation bases for finite semigroups*, Communications in Algebra **30**(2)(2002), 911-933.

[7] B. H. Neumann, *An essay on free products of groups with amalgamations* Trans. Roy. Soc. London Ser. A **246**(1954), 503-554.

[8] G.Higman, B. H. Neumann and H. Neumann, *Embedding theorems for groups*, J, London Math. S0c. **24**(4)(1949), 247-254.

[9] B. H. Neumann, *An essay on free products of groups with amalgamations* Trans. Roy. Soc. London Ser. A **246**(1954), 503-554.

[10] J. Okniński and M.S. Putcha, *Embedding finite semigroup amalgams*, J. Austral. Math. Soc. (Series *A*) **51**(1991), 489-496.

[11] T. K. Shoji, *Absolute flatness of the full transformation semigroups*, J. Algebra **118**(1988), 477-486.

[12] K. Shoji, *Absolute flatness of regular semigroups with a finite height function*, Semigroup Forum **52**(1996), 133-140.

[13] K. Shoji,*A proof of Okniński and Putcha's theorem*, Proc. of the Third Inter. Colloq. on Words, Languages and Combinatorics. Ed. by M. Ito &T Imaoka World Scientific, 2003, 420-427.

[14] K. Shoji, *Regular semigroups which are amalgamation bases for finite semigroups*, Algebra Colloquium **14**(2007) 245-254.

[15] K. Shoji, *The full transformation semigroups of finite rank are amalgamation bases for finite semigroups*, Communications in Algebra **44**(2016) 3278-3289.