

Rings in which every left ideal is idempotent and the set of ideals is linearly ordered.

Hisaya Tsutsui

Department of Mathematics
Embry-Riddle Aeronautical University
Prescott, AZ USA
E-mail address: Hisaya.Tsutsui@erau.edu

This paper constitutes a formal report on the presentation delivered by the author at the 2025 RIMS workshop entitled “*Semigroups, Algebras, Languages and Related Areas in Computer Science Rings.*” The lecture served as a preliminary exposition and introduction to the topic indicated in the title above, as well as to its closely related areas of inquiry.

The findings articulated in the presentation shed light on the intrinsic structure of fully prime rings and advance a series of novel questions concerning the primitivity of a fully prime ring under diverse frameworks. The discussion placed particular emphasis on the intricate interplay among primitivity, the idempotency of left ideals, and the linear ordering of the set of ideals. The research constitutes a collaborative endeavor with Professor Yasuyuki Hirano.

We assume throughout that all rings are associative with identity element. A few definitions are recalled below to frame the study:

- A (left) V -ring is a ring R in which every simple left R -module is injective. Equivalently, R is a left V -ring if and only if every left ideal is an intersection of maximal left ideals.
- A von Neumann regular ring (often referred to simply as a *regular ring*) is a ring R that satisfies the condition that for every element a in R , there exists an element b in R such that $a = aba$. Equivalently, R is regular if and only if every principal left ideal is generated by an idempotent element; this is further equivalent to the statement that every principal left ideal is a direct summand of the left R -module R . Another equivalent characterization is that every left R -module is flat.

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*Apart of the contents with further explanations may be submitted to elsewhere as a part of our on-going work.

- A ring R is said to be fully prime if the lattice of ideals is linearly ordered and every ideal is idempotent. Equivalently, R is fully prime if and only if every ideal of R is prime; this is also equivalent to the condition that every essential ideal of R is idempotent and the set of ideals is linearly ordered. Another equivalent characterization asserts that for any ideals P and Q of R , either $PQ = P$ or $PQ = Q$. Furthermore, R is fully prime if and only if the set of ideals of R is linearly ordered and $I \cap P \subseteq IP$ for any ideal P and left ideal I of R .
- A ring is fully left idempotent if every left ideal is idempotent.

For the commutative case, a ring R is a V -ring if and only if R is a von Neumann ring; if and only if R is fully idempotent. As noted in the definitions above, a necessary and sufficient condition for a ring R to be fully prime is that for ideals P, Q of R , either $PQ = P$ or $PQ = Q$. We remark that the product of two idempotent left ideals in a fully prime ring exhibits similar behavior: For any idempotent left ideals I and J of a fully prime ring R , either $IJ = I$, or $JI = J$. In fact, if R is fully prime, either $IR \subseteq JR$ in which case $I = I^2 = IRI \subseteq JRI = JI \subseteq I$, or $JR \subseteq IR$ in which case it follows similarly that $J = JI$.

An ideal P of a fully prime ring R is said to have the *left AR-property* if for each left ideal I of R , $I \cap P = PI$. A fully prime ring R is said to have the *left AR-property* if every ideal of R has the left AR-property.

Proposition 1. *If R is a fully prime ring satisfying the left AR-property and P is an ideal of R , then a fully prime subring $S_p = P + Z(R)$ also satisfies the left AR-property. In this case, every left ideal of S_p contained in P is a left ideal of R .*

A central theme of this study is the investigation of whether certain combinations of conditions imposed on right ideals and (two-sided) ideals lead to primitivity. We consider the following classes of rings and pose the corresponding questions:

- Is a prime regular ring necessarily primitive? (Answer: No.)
- Is a prime V -ring primitive?
- Is a fully prime regular ring primitive?

- Is a fully prime V -ring primitive?
- Is a fully prime fully left idempotent ring primitive?

One of the central questions in the theory of von Neumann regular rings was posed by Kaplansky [4]: “*Is a regular prime ring necessarily primitive?*” A counterexample to this question was first constructed by Domnao [2] in 1977. An analogous question was later raised by Fisher [3]: “*Is a prime left V -ring primitive?*” This problem has remained open ever since.

While every (two-sided) ideal of a fully prime ring is idempotent, its left ideals are not necessarily idempotent. Both von Neumann regular rings and V -rings are fully left idempotent; however, their lattices of ideals are not necessarily linearly ordered. If every left ideal of a ring R is idempotent and the set of left ideals is linearly ordered, then R must be a division ring, as shown by Blair and Tsutsui [2].

An instructive example of a fully prime ring in which every left ideal is idempotent—yet which still admits nonzero proper ideals—is the ring of linear transformations on a vector space V over a field F . It is worth noting, however, that if every left ideal of a ring R is idempotent, then R is necessarily semiprimitive.

A natural question arises: *Is a fully left idempotent, fully prime ring necessarily primitive?* If not, what about a fully prime regular ring, or a fully prime left V -ring—are these necessarily primitive?

These considerations have motivated further investigation into the structure of one-sided ideals in rings in which every ideal is prime, particularly in relation to primitivity and the nature of idempotency.

Proposition 1. *If a ring R is left Goldie, then every left ideal of R is idempotent and the set of (two-sided) ideals of R is linearly ordered if and only if R is simple.*

Proposition 2. *If a ring R is right Noetherian, then every (two-sided) ideal is idempotent and the set of left ideals is linearly ordered if and only if R is a division ring.*

We note that it is possible that the set of left ideals of a fully prime ring is linearly ordered yet the ring is not even semiprimitive as shown in [5].

Proposition 3. *A fully prime fully left idempotent ring R is primitive if R has countably many ideals.*

A prime ring R is called *right bounded* if every essential right ideal contains a nonzero ideal.

Proposition 4. *A right Goldie fully prime right bounded ring R is simple Artinian.*

A ring R is called *strongly prime* if every nonzero ideal of R contains a finitely generated left ideal whose right annihilator is zero.

Proposition 5. *Let R be fully prime right bounded ring. Then R is simple if and only if R is strongly prime.*

References

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