

NOTES ON THE ABELIANIZATION OF THE HYPERELLIPTIC TORELLI GROUP

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ABSTRACT. In this survey paper, we introduce the hyperelliptic mapping class group of an oriented topological surface and its subgroup, the hyperelliptic Torelli group. The hyperelliptic mapping class group is defined as the subgroup of the mapping class group that centralizes a fixed hyperelliptic involution, capturing the symmetries induced by this involution. We begin by reviewing foundational results on the Torelli group, focusing on Johnson's computation of its abelianization for genus $g \geq 3$ via a homomorphism now known as the Johnson homomorphism. We then present the hyperelliptic Johnson homomorphism, which plays an analogous role in the study of the hyperelliptic Torelli group. Finally, we discuss known results and techniques for bounding below the dimension of its abelianization. This survey is based on joint work with Ma Luo.

1. INTRODUCTION

In this survey notes we will introduce the hyperelliptic Johnson homomorphism and the study of the abelianization of the hyperelliptic Torelli group. We will also review a key tool called the relative completion of a discrete group. The readers can find the detail of related results in [7].

Let $S_{g,n}$ be an oriented topological surface of genus g with n marked points. The mapping class group $\Gamma_{g,n}$ of $S_{g,n}$ is the group of isotopy classes of orientation-preserving diffeomorphisms of $S_{g,n}$ fixing the marked points pointwise. Denote the Torelli group of $\Gamma_{g,n}$ by $T_{g,n}$. Johnson proved in [5] that the Torelli group T_g is finitely generated for $g \geq 3$. When $g = 2$, it was proved by Mess in [8] that T_2 is a countably generated free group. Let σ be a hyperelliptic involution of S_g . The hyperelliptic mapping class group Δ_g is the centralizer of the isotopy class $[\sigma]$ in Γ_g . The hyperelliptic Torelli group, denoted by $T\Delta_g$, is the intersection $T_g \cap \Delta_g$. For $g \geq 3$, it is not known whether it is finitely generated or not. We will discuss a lower bound for the dimension of the abelianization of $T\Delta_g$ over \mathbb{Q} .

2. TOPOLOGY OF $S_{g,n}$

For a path connected topological space X , the fundamental group $\pi_1(X, x)$ of X with base point x is defined as the group of homotopy classes of loops in X based at x . For any points x

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and y in X , there is a natural isomorphism

$$\pi_1(X, x) \cong \pi_1(X, y),$$

which is unique up to a conjugation action by an element of $\pi_1(X, x)$. Therefore, we will omit the base point from the notation.

Let S_g be a compact smooth oriented topological surface of genus g . Let P be a subset of S_g consisting of n distinct points. Define $S_{g,n}$ as the marked surface (S_g, P) . It is an oriented topological surface of genus g with n marked points. Let $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ be the standard generators for $\pi_1(S_g)$. It has a minimal presentation given by

$$\pi_1(S_g) \cong \langle \alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g \mid [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] \rangle.$$

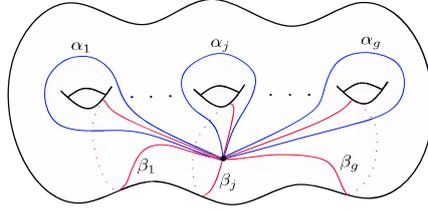


FIGURE 1. Standard Generators

By the Hurewicz theorem, the natural map $\pi_1(S_g) \rightarrow H_1(S_g; \mathbb{Z})$ induces an isomorphism from the abelianization of $\pi_1(S_g)$ to the homology group $H_1(S_g; \mathbb{Z})$. Denote the images of α_j and β_j in $H_1(S_g; \mathbb{Z})$ by a_j and b_j for $j = 1, \dots, g$. The abelianization $H_1(S_g; \mathbb{Z})$ is a free abelian group of rank $2g$.

2.1. Symplectic group. The symplectic group with integer entries is defined as

$$\mathrm{Sp}(2g; \mathbb{Z}) = \{M \in \mathrm{GL}(2g; \mathbb{Z}) \mid M^T J M = J\},$$

where $J = \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix}$ and I_g is the g -by- g identity matrix. The group $H_1(S_g; \mathbb{Z})$ is equipped with the algebraic intersection pairing $\langle \cdot, \cdot \rangle$. The pairing $\langle \cdot, \cdot \rangle$ is a non-degenerate bilinear alternating form, and H becomes a symplectic space of rank $2g$ with $\langle \cdot, \cdot \rangle$. Set $H = H_1(S_g; \mathbb{Z})$. Fix a symplectic basis given by $a_1, \dots, a_g, b_1, \dots, b_g$ for H . Then there is an isomorphism of the automorphism of H preserving $\langle \cdot, \cdot \rangle$ with $\mathrm{Sp}(2g; \mathbb{Z})$:

$$\mathrm{Aut}(H, \langle \cdot, \cdot \rangle) \cong \mathrm{Sp}(2g; \mathbb{Z}).$$

3. MAPPING CLASS GROUPS

Assume that $2g - 2 + n > 0$. The mapping class group of $S_{g,n}$, denoted by $\Gamma_{g,n}$, is defined as the group of isotopy classes of orientation-preserving diffeomorphisms of $S_{g,n}$ fixing the subset P pointwise:

$$\Gamma_{g,n} := \mathrm{Diff}^+(S_g, P) / \sim,$$

where \sim denotes the isotopy relation. The group $\Gamma_{g,n}$ is independent of the choice of the subset P by the classification of surfaces. When $n = 0$, we simply denote $\Gamma_{g,0}$ by Γ_g . By forgetting a marked point, we obtain a surjection $\mathcal{F}\mathrm{orget} : \Gamma_{g,n+1} \rightarrow \Gamma_{g,n}$, called the forgetful map. Hence, by composing n forgetful maps, we obtain the surjection $\Gamma_{g,n} \rightarrow \Gamma_g$.

3.1. Dehn twists. The group Γ_g is finitely generated by the mapping class elements called the Dehn twists. Let γ be a simple closed curve in S_g . Consider a tubular neighborhood N of γ shown in Figure 2. A Dehn twist T_γ about γ is a left-twist map about γ , fixing the boundary of N . The isotopy class of T_γ does not depend on the choice of N , and furthermore

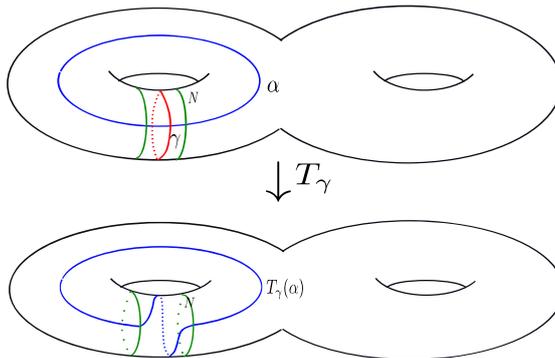


FIGURE 2. A View of a Dehn Twist

it is independent of the choice of the simple curve γ within its isotopy class. Thus, by abuse of notation, T_γ also denotes its isotopy class in Γ_g .

A simple closed curve γ in S_g is said to be separating if the surface obtained by cutting S_g along γ is disconnected. Otherwise, it is said to be nonseparating. When $g = 1$, Γ_g is generated by the Dehn twists about α_1 and β_1 in the torus. For $g \geq 2$, the mapping class group Γ_g is finitely generated by the isotopy classes of Dehn twists about $2g + 1$ nonseparating simple closed curves in S_g (see [4, Thm. 4.14]). Furthermore, it is also finitely presented [4, Thm. 5.3].

3.2. Symplectic representation of $\Gamma_{g,n}$. Each mapping class element $[\phi]$ in Γ_g induces an automorphism $\phi_* : H \rightarrow H$, which is independent of the choice of the representative of the class. The automorphism ϕ_* preserves the intersection pairing $\langle \cdot, \cdot \rangle$. Hence there is a representation

$$\rho_g : \Gamma_g \rightarrow \mathrm{Sp}(2g; \mathbb{Z}).$$

The homomorphism ρ_g is surjective for $g \geq 1$ [4, Thm. 6.4]. By composing with the forgetful map $\Gamma_{g,n} \rightarrow \Gamma_g$, we obtain a representation

$$\rho_{g,n} : \Gamma_{g,n} \rightarrow \mathrm{Sp}(2g; \mathbb{Z}).$$

This is called the symplectic representation of $\Gamma_{g,n}$.

3.3. Torelli groups. The Torelli group T_g is defined as the kernel of the symplectic representation $\rho_{g,n}$:

$$T_{g,n} := \ker \rho_{g,n}$$

and there is an exact sequence

$$1 \rightarrow T_{g,n} \rightarrow \Gamma_{g,n} \xrightarrow{\rho_{g,n}} \mathrm{Sp}(2g; \mathbb{Z}) \rightarrow 1.$$

It has an infinite-index in $\Gamma_{g,n}$.

3.4. Finite generation of T_g . To describe generators of the Torelli group T_g , we consider two types of elements. The first is the *separating twist*, which is a Dehn twist about a separating simple closed curve in the surface S_g . The second is the *bounding pair map*, defined as follows: a *bounding pair* in S_g consists of two disjoint, homologous, nonseparating simple closed curves d_1 and d_2 , and the associated mapping class

$$T_{d_1}T_{d_2}^{-1}$$

is called a *bounding pair map*. Birman and Powell [1, 9] proved that T_g is generated by all separating twists and bounding pair maps.

In 1980, Johnson [5] proved that the Torelli group T_g is finitely generated by bounding pair maps alone for $g \geq 3$. However, his generating set is extremely large, with its size growing exponentially in g . This result was later significantly improved by Putman [10], who constructed a much smaller generating set consisting of both separating twists and bounding pair maps, whose size grows only cubically with g .

For genus $g = 2$, the situation is notably different: Mess [8] showed that T_2 is a countably generated free group, generated entirely by separating twists.

3.5. Open questions. There are several major open problems concerning the Torelli group. One fundamental question is whether T_g is finitely presented for $g \geq 3$. Another closely related problem involves the subgroup $\mathcal{K}(S) \subset T_g$, generated by all separating twists: is $\mathcal{K}(S)$ finitely generated for $g \geq 3$?

4. HYPERELLIPTIC MAPPING CLASS GROUPS

We consider a subgroup of the mapping class group consisting of elements that preserve a certain symmetry on S_g . This symmetry arises from an orientation-preserving involution σ of order 2, which fixes exactly $2g + 2$ points on S_g . We refer to σ as a *hyperelliptic involution*, and a visualization of σ is provided in Figure 3.

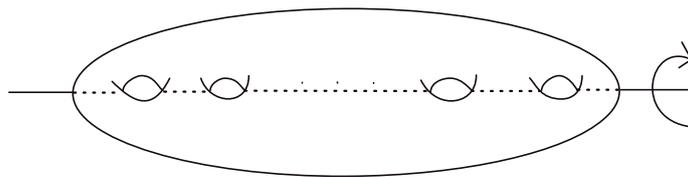


FIGURE 3. A hyperelliptic involution of S_g , rotation by π

Fix a hyperelliptic involution σ of S_g . We define the hyperelliptic mapping class group Δ_g of S_g as

$$\Delta_g := \text{the centralizer of the isotopy class of } \sigma \text{ in } \Gamma_g.$$

4.1. Generators. A simple closed curve γ is called *symmetric* if $[\sigma(\gamma)] = [\gamma]$, where σ denotes the hyperelliptic involution. The hyperelliptic mapping class group Δ_g is generated by Dehn twists about the $2g + 1$ symmetric nonseparating simple closed curves shown in Figure 4.

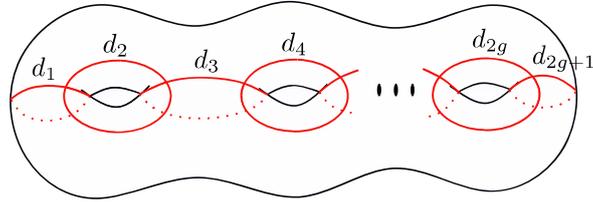


FIGURE 4. Dehn twists about symmetric nonseparating curves generating Δ_g

4.2. Hyperelliptic Torelli group. The hyperelliptic Torelli group $T\Delta_g$ is defined as the intersection of the hyperelliptic mapping class group and the Torelli group:

$$T\Delta_g := \Delta_g \cap T_g.$$

Although it remains unknown whether $T\Delta_g$ is finitely generated, a remarkable result by Brendle, Margalit, and Putman provides significant insight.

Theorem 4.1 ([3]). *If $g \geq 2$, then $T\Delta_g$ is generated by Dehn twists about symmetric separating curves.*

Remark 4.2. When $g = 2$, any two simple separating curves intersect at least four times (see Figure 5). In contrast, for $g \geq 3$, there exist disjoint symmetric separating curves, as illustrated in Figure 6, which give rise to commuting Dehn twists in $T\Delta_g$.

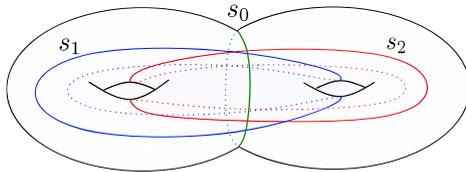


FIGURE 5. Symmetric separating curves in S_2

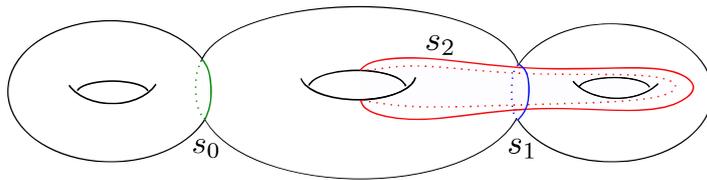


FIGURE 6. Symmetric separating curves in S_3

4.3. Level structure. For an integer $m \geq 0$, the level m subgroup of Δ_g is defined as the kernel of the mod m reduction map:

$$\Delta_g[m] := \ker \left(\Delta_g \xrightarrow{\rho} \mathrm{Sp}_{2g}(\mathbb{Z}) \rightarrow \mathrm{Sp}_{2g}(\mathbb{Z}/m) \right).$$

Similarly, the level m subgroup of $\Gamma_{g,n}$ is given by

$$\Gamma_{g,n}[m] := \ker \left(\Gamma_{g,n} \rightarrow \mathrm{Sp}_{2g}(\mathbb{Z}/m) \right).$$

Let W denote the set of Weierstrass points. Then there is a short exact sequence:

$$1 \rightarrow \Delta_g[2] \rightarrow \Delta_g \rightarrow \text{Aut}(W) \rightarrow 1.$$

5. THE JOHNSON HOMOMORPHISM

Let $\Pi = \pi_1(S_g)$, and let $L^\bullet \Pi$ denote the lower central series of Π , defined by

$$L^1 \Pi = \Pi, \quad L^2 \Pi = [\Pi, \Pi], \quad L^3 \Pi = [\Pi, L^2 \Pi], \dots$$

Then there is a short exact sequence

$$1 \rightarrow L^2 \Pi / L^3 \Pi \rightarrow \Pi / L^3 \Pi \rightarrow \Pi / L^2 \Pi \rightarrow 1.$$

Set $H = \Pi^{\text{ab}}$. We then have $\Pi / L^2 \Pi \cong H$ and $L^2 \Pi / L^3 \Pi \cong \Lambda^2 H$.

The *Johnson homomorphism* is a map

$$\tau : T_{g,1} \rightarrow \text{Hom}(H, \Lambda^2 H)$$

defined by

$$\tau(\phi)(x) = \phi(\tilde{x})\tilde{x}^{-1},$$

where \tilde{x} is any lift of $x \in H$ to $\Pi / L^3 \Pi$.

Using this homomorphism, Johnson computed the abelianization of T_g . Over \mathbb{Q} , the result is stated as follows:

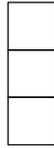
Theorem 5.1 ([6]). *Let $g \geq 3$. Then the Johnson homomorphism τ induces $\text{Sp}_{2g}(\mathbb{Q})$ -equivariant isomorphisms*

$$\tau_{\mathbb{Q}}^{\text{ab}} : H_1(T_{g,1}; \mathbb{Q}) \cong \Lambda^3 H_{\mathbb{Q}} \subset \text{Hom}(H_{\mathbb{Q}}, \Lambda^2 H_{\mathbb{Q}}),$$

and

$$H_1(T_g; \mathbb{Q}) \cong \Lambda^3 H_{\mathbb{Q}} / H_{\mathbb{Q}}.$$

The $\text{Sp}_{2g}(\mathbb{Q})$ -representation $\Lambda^3 H_{\mathbb{Q}} / H_{\mathbb{Q}}$ is irreducible and corresponds to the partition $1+1+1$ as an $\text{Sp}_{2g}(\mathbb{Q})$ -module. This representation is indexed by the following Young diagram:



5.1. The Hyperelliptic Johnson homomorphism. Since the class $[\sigma]$ acts as $-\text{id}$ on H , and every element of $T\Delta_g$ commutes with $[\sigma]$, it follows that the Johnson homomorphism restricts trivially to $T\Delta_g$. Instead, we consider the following exact sequence:

$$1 \rightarrow L^3 \Pi / L^4 \Pi \rightarrow \Pi / L^4 \Pi \rightarrow \Pi / L^3 \Pi \rightarrow 1.$$

The group $L^3 \Pi / L^4 \Pi$ carries an $\text{Sp}_{2g}(\mathbb{Z})$ -representation isomorphic to $(\Lambda^2 H \otimes H) / \Lambda^3 H$.

Fix a Weierstrass point q . The *hyperelliptic Johnson homomorphism* is defined as

$$\tau_q^{\text{hyp}} : T\Delta_g \rightarrow \text{Hom}(H, (\Lambda^2 H \otimes H) / \Lambda^3 H)$$

by the formula

$$\tau_q^{\text{hyp}}(\phi)(x) = \phi(\tilde{x})\tilde{x}^{-1},$$

where \tilde{x} is any lift of $x \in H$ to $\Pi / L^4 \Pi$.

The image of τ_q^{hyp} lies in the $\text{Sp}_{2g}(\mathbb{Q})$ -representation given by the sum

$$\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array},$$

where $\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}$ corresponds to $\Lambda^2 H / \langle \theta \rangle$, and $\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}$ denotes the highest weight component of $S^2 \Lambda^2 H$.

6. OPEN PROBLEMS ON $T\Delta_g$

There are several open problems concerning the group $T\Delta_g$. Among these, we highlight the following fundamental questions:

- For $g \geq 3$, determine $H_1(T\Delta_g; \mathbb{Z})$ and $H_1(T\Delta_g; \mathbb{Q})$.
- For $g \geq 3$, is $T\Delta_g$ finitely generated?
- For $g \geq 4$, is $T\Delta_g$ finitely presented?

Brendle, Childers, and Margalit showed in [2] that $T\Delta_3$ is not finitely presentable.

7. RELATIVE COMPLETION OF $\Delta_g[m]$

Our approach to studying $T\Delta_g$ involves a tool known as the *relative completion* of a discrete group. Relative completion provides a linearization of the group, governed by its cohomology, making it partially computable.

Definition 7.1. The relative completion of $\Delta_g[m]$ with respect to $\rho : \Delta_g[m] \rightarrow \text{Sp}_{2g}(\mathbb{Q})$ is an extension of $\text{Sp}_{2g}(\mathbb{Q})$ by a pronipotent \mathbb{Q} -group $\mathcal{V}_{g,n}$:

$$\begin{array}{ccccccc} T\Delta_g & \longrightarrow & \Delta_g[m] & & & & \\ & & \downarrow & \searrow \rho & & & \\ & & \mathcal{V}_g[m] & & \mathcal{D}_g[m] & \longrightarrow & \text{Sp}_{2g}(\mathbb{Q}) \longrightarrow 1, \\ & & \downarrow & & \downarrow \tilde{\rho} & & \\ 1 & \longrightarrow & \mathcal{V}_g[m] & \longrightarrow & \mathcal{D}_g[m] & \longrightarrow & \text{Sp}_{2g}(\mathbb{Q}) \longrightarrow 1, \end{array}$$

satisfying the following universal property. If G is a proalgebraic \mathbb{Q} -group that is also an extension of $\text{Sp}_{2g}(\mathbb{Q})$ by a pronipotent \mathbb{Q} -group U such that ρ factors through $G \rightarrow \text{Sp}_{2g}(\mathbb{Q})$ with Zariski-dense image in G , then there exists a unique morphism $\phi : \mathcal{D}_g[m] \rightarrow G$ of proalgebraic groups over \mathbb{Q} such that the diagram

$$\begin{array}{ccc} \Delta_g[m] & \xrightarrow{\tilde{\rho}} & \mathcal{D}_g[m] \\ \downarrow & \searrow \phi & \downarrow \\ G & \longrightarrow & \text{Sp}_{2g}(\mathbb{Q}) \end{array}$$

commutes.

By the Levi's theorem, the exact sequence

$$1 \rightarrow \mathcal{V}_g[m] \rightarrow \mathcal{D}_g[m] \rightarrow \text{Sp}_{2g}(\mathbb{Q}) \rightarrow 1$$

splits, and hence there is an isomorphism $\mathcal{D}_g[m] \cong \mathcal{V}_g[m] \rtimes \text{Sp}_{2g}(\mathbb{Q})$.

7.1. **A lower bound for $\dim H_1(T\Delta_g; \mathbb{Q})$.** Denote the Lie algebra of $\mathcal{V}_g[m]$ by $\mathfrak{v}_g[m]$. It is a pronilpotent Lie algebra over \mathbb{Q} . The following result follows from a result in [7, Thm. 2].

Theorem 7.2. *If $g \geq 3$, there is a surjection*

$$\tilde{\zeta} : H_1(\mathfrak{v}_g[2]; \mathbb{Q}) \rightarrow \begin{array}{|c|} \hline \square^{2g+1} \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}.$$

Combining with the surjection $H_1(T\Delta_g; \mathbb{Q}) \rightarrow H_1(\mathfrak{v}_g[2])$, we obtain a surjection

$$\zeta : H_1(T\Delta_g; \mathbb{Q}) \rightarrow \begin{array}{|c|} \hline \square^{2g+1} \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}.$$

In fact, we conjecture that the map ζ is an isomorphism.

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