On Mod-separable extensions of modules

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Abstract

It is well known that separable extensions of noncommutative rings are characterized by the derivation. In this paper, using the generalized derivation, we shall introduce **Mod**-separable extensions of modules as a generalization of separable ring extensions. Moreover, we shall show that the **Mod**-separability is a hereditable property with respect to Morita equivalence in ring extensions.

1 Introduction and Preliminaries

Let A/B be a ring extension and U an A-bimodule. An additive map $\delta : A \to U$ is called a *derivation* if it satisfies $\delta(xy) = \delta(x)y + x\delta(y)$ for any $x, y \in A$. Moreover, a derivation $\delta : A \to U$ is called a *B-derivation* if $B \subset \ker \delta$. By $\mathrm{Der}(A, U)$ (resp. $\mathrm{Der}_B(A, U)$) we denote the set of all derivations (resp. B-derivations) from A to U. For a fixed element $u \in U$, it is easy to see that a map $\delta_u : A \to U$ defined by $\delta_u(x) = xu - ux$ ($x \in A$) is a derivation. Such δ_u is called an *inner derivation* by $u \in U$. We denote by $\mathrm{Inn}(A, U)$ the set of all inner derivations from A to U.

A ring extension A/B is called *separable* if the A-A-homomorphism from $A \otimes_B A$ to A defined by $\sum_i x_i \otimes y_i \mapsto \sum_i x_i y_i$ splits. The following is well known.

Proposition 1.1. *[1, Satz 4.2] A ring extension A/B is separable if and only if $\mathrm{Der}_B(A, U) \subset \mathrm{Inn}(A, U)$ for any A-bimodule U.*

Let A be a ring and S, U be A-bimodules. In [4], H. Komatsu introduced the notion of generalized derivations from S to U as follows : an additive map $\delta : S \to U$ is called a *generalized derivation* if it satisfies $\delta(asb) = \delta(as)b + a\delta(sb) - a\delta(s)b$ for any $s \in S$ and $a, b \in A$. By $\mathrm{GDer}(_A S_A, {}_A U_A)$ we denote the set of all generalized derivations of A-bimodules from S to U. It is obvious that $\mathrm{Der}(A, U) \subset \mathrm{GDer}(_A A_A, {}_A U_A)$ for any ring A and any A-bimodule U.

Let A/B be a ring extension, S an A-bimodule, and T a submodule of S as a B-bimodule. In this paper, we shall define generalized T-derivations as a generalization of B-derivations (Definition 2.1). Using generalized T-derivations, we shall introduce the notion of **Mod**-separable extensions of modules which extend the notion of separable extensions of rings (Definition 2.6). Moreover, we shall show that the **Mod**-separability is a hereditable property with respect to Morita equivalence in ring extensions (Theorem 3.3).

2 Mod-separable extension of modules

Throughout this section, let A/B be a ring extension and S/T an additive group extension (i.e., T is a subgroup of an additive group S). Noting that there always holds $\mathrm{Der}_B(A, U) \subset \mathrm{Hom}(_BA_B, {}_BU_B)$ for any A-bimodule U, we give the following definition.

Definition 2.1. (1) An additive group extension S/T is called A/B-*extension* if S is an A-bimodule and T is a submodule of S as a B-bimodule.

(2) For an A/B-extension S/T and an A-bimodule U, a map $\delta : S \to U$ is called a *generalized T-derivation* if $\delta \in \mathrm{GDer}(_AS_A, {}_AU_A) \cap \mathrm{Hom}(_BS_B, {}_BU_B)$ and $T \subset \ker \delta$. We denote by $\mathrm{GDer}_T(_AS_A, {}_AU_A)$ the set of all generalized T-derivations from S to U.

Remark 2.2. Any ring extension A/B can be considered as an A/B-extension. It is easy to see that $\mathrm{GDer}_B(_AA, {}_AU_A) = \mathrm{Der}_B(A, U)$ for any ring extension A/B and any A-bimodule U.

Example 2.3. By $\mathbb{M}_{k \times \ell}(R)$ we denote the set of all $k \times \ell$ matrices over a commutative ring R, where k and ℓ are positive integers. Let B be a commutative ring, $A = B[x]$, $D = \frac{d}{dx}$, $S = \mathbb{M}_{k \times \ell}(A)$, and $T = \mathbb{M}_{k \times \ell}(B)$. We regard S (resp. T) as an A-bimodule (resp. a B-bimodule) by the usual scalar multiplication of matrices. We define a map $\delta : S \to S$ by

$$\delta((s_{ij})) = ((Ds_{ij})_{ij}) \quad ((s_{ij}) \in S),$$

where (s_{ij}) is a $k \times \ell$ matrix whose (i, j) entry is s_{ij}. We can see that δ is a generalized T-derivation from S to S.

Let S, U be A-bimodules, $\varphi \in \mathrm{Hom}(_AS_A, {}_AA_A)$, and $u \in U$. We define a map $\delta_{\varphi,u} : S \to U$ by

$$\delta_{\varphi,u}(s) = \varphi(s)u - u\varphi(s) \quad (s \in S).$$

So it is easy to see that $\delta_{\varphi,u}$ is a generalized derivations. We give then the following definition.

Definition 2.4. For A-bimodules S and U, a map $\delta : S \to U$ is called a *generalized inner derivation* if there exist $\varphi \in \mathrm{Hom}(_AS_A, {}_AA_A)$ and $u \in U$ such that

$$\delta(s) = \varphi(s)u - u\varphi(s) \quad (s \in S).$$

By $\mathrm{GInn}(_AS_A, {}_AU_A)$ we denote the set of all generalized inner derivationsfrom S to U.

Remark 2.5. There always holds $\mathrm{Inn}(A, U) \subset \mathrm{GInn}(_AA_A, {}_AU_A)$ for any ring A and any A-bimodule U.

Using the generalized derivaions, we introduce the following definition which is a generalization of Proposition 1.1.

Definition 2.6. An A/B-extension S/T is called Mod-*separable* if, for any A-bimodule U, there holds

$$\mathrm{GDer}_T({}_AS_A, {}_AU_A) \subset \mathrm{GInn}({}_AS_A, {}_AU_A).$$

Remark 2.7. By Remark 2.2 and Remark 2.5, we see that every separable ring extension is Mod-separable.

Example 2.8. An A/B-extension S/T in Example 2.3 is not Mod-separable. Because a generalized T-derivation δ in Example 2.3 is not a generalized inner derivation.

3 Mod-separability and Morita equivalence in ring extensions

For two rings A, A', an A-A'-bimodule ${}_AM_{A'}$ is called a *Morita module* if ${}_AM$ is a progenerator and $\mathrm{End}({}_AM) \cong A'$ as a ring. Two ring extensions A/B and A'/B' are *Morita equivalent* if there exist Morita modules ${}_AM_{A'}$ and ${}_BN_{B'}$ such that ${}_AA \otimes_B N_{B'} \cong {}_AM_{B'}$. The following proposition shows that the separability is a hereditable property with respect to Morita equivalence in ring extensions.

Proposition 3.1. *[3, Theorem 5] Assume that two ring extensions A/B and A'/B' are Morita equivalent. If A/B is separable then A'/B' is also separable.*

Now we assume that two ring extensions A/B, A'/B' are Morita equivalent with a Morita module ${}_BN_{B'}$, and let $N^* = \mathrm{Hom}^r({}_BN, {}_BB)$. As was shown in [3] or [6], there holds $A' \cong N^* \otimes_B A \otimes_B N$ and $B' \cong N^* \otimes_B B \otimes_B N$ ($\cong N^* \otimes_B N$) as rings. So, Proposition 3.1 is equivalent to the following.

Proposition 3.2. *Assume that two ring extensions A/B, A'/B' are Morita equivalent with a Morita module ${}_BN_{B'}$, and let $N^* = \mathrm{Hom}^r({}_BN, {}_BB)$. If A/B is separable then $N^* \otimes_B A \otimes_B N/N^* \otimes_B B \otimes_B N$ is also separable.*

Under the conditions of the above proposition, suppose further that S/T is an A/B-extension. We can see then that $N^* \otimes_B S \otimes_B N$ (resp. $N^* \otimes_B T \otimes_B N$) is an $N^* \otimes_B A \otimes_B N$ ($\cong A'$)-bimodule (resp. a $N^* \otimes_B B \otimes_B N$ ($\cong B'$)-bimodule). Hence $N^* \otimes_B S \otimes_B N/N^* \otimes_B T \otimes_B N$ is an A'/B'-extension. The following is the main theorem of this article which is a generalization of Proposition 3.2.

Theorem 3.3. *Assume that two ring extensions A/B, A'/B' are Morita equivalent with a Morita module ${}_BN_{B'}$, and let $N^* = \mathrm{Hom}^r({}_BN, {}_BB)$. If an A/B-extesion S/T is Mod-separable then an A'/B'-extesion $N^* \otimes_B S \otimes_B N/N^* \otimes_B T \otimes_B N$ is also Mod-separable.*

To prove Theorem 3.3, we need the following lemma.

Lemma 3.4. *Assume that two ring extensions A/B, A'/B' are Morita equivalent with a Morita module ${}_B N_{B'}$, and let $N^* = \mathrm{Hom}^r({}_B N, {}_B B)$. In addition, we suppose that S/T is an A/B-extension. Then, for any A-bimodule U, there exists the following isomorphism as an additive group:*

$$\mathrm{GDer}_T({}_A S_A, {}_A U_A) \overset{\cong}{\longrightarrow} \mathrm{GDer}_{N^* \otimes_B T \otimes_B N}({}_{A'} N^* \otimes_B S \otimes_B N_{A'}, {}_{A'} N^* \otimes_B U \otimes_B N_{A'})$$
$$\delta \longmapsto 1 \otimes \delta \otimes 1$$

Outline of the proof of Theorem 3.3. Assume that two ring extensions A/B, A'/B' are Morita equivalent with a Morita module ${}_B N_{B'}$, and let $N^* = \mathrm{Hom}^r({}_B N, {}_B B)$. Moreover, we suppose that an A/B-extesion S/T is Mod-separable and U' is an A'-bimodule. Then we can see that there is an A-bimodule such that ${}_{A'} U'_{A'} \cong {}_{A'} N^* \otimes_B U \otimes_B N_{A'}$. Hence it suffices to show that

$$\mathrm{GDer}_{N^* \otimes_B T \otimes_B N}({}_{A'} N^* \otimes_B S \otimes_B N_{A'}, {}_{A'} N^* \otimes_B U \otimes_B N_{A'})$$
$$\subset \mathrm{GInn}({}_{A'} N^* \otimes_B S \otimes_B N_{A'}, {}_{A'} N^* \otimes_B U \otimes_B N_{A'}).$$

Let δ' be in $\mathrm{GDer}_{N^* \otimes_B T \otimes_B N}({}_{A'} N^* \otimes_B S \otimes_B N_{A'}, {}_{A'} N^* \otimes_B U \otimes_B N_{A'})$. By Lemma 3.4, there exists $\delta \in \mathrm{GDer}_T({}_A S_A, {}_A U_A)$ such that $\delta' = 1 \otimes \delta \otimes 1$. Since S/T is Mod-separable, there exist $\varphi \in \mathrm{Hom}({}_A S_A, {}_A A_A)$ and $u \in U$ sucht that $\delta(s) = \varphi(s)u - u\varphi(s)$ for any $s \in S$. Now we let $s' = \sum_k \rho_k \otimes s_k \otimes n_k \in N^* \otimes_B S \otimes_B N$, $\varphi' = 1 \otimes \varphi \otimes 1 \in \mathrm{Hom}({}_{A'} N^* \otimes_B S \otimes_B N_{A'}, {}_{A'} A'_{A'})$, and $u' = \sum_j f_j \otimes u \otimes m_j \in N^* \otimes_B U \otimes_B N$, where $\{f_j, m_j\}$ is a dual basis of ${}_B N$. Noting that $(bu - ub)\varphi(s) = \varphi(s)(bu - ub) = 0$ $(\forall s \in S, \forall b \in B)$, we can see that $\delta'(s') = \varphi'(s')u' - u'\varphi'(s')$. Therefore $\delta' \in \mathrm{GInn}({}_{A'} N^* \otimes_B S \otimes_B N_{A'}, {}_{A'} N^* \otimes_B U \otimes_B N_{A'})$. This complete the proof. $\square$

Finally, we state some open problems.

- What is the condition for Mod-separability that corresponds to equivalent conditions of separable ring extensions? (e.q., A/B is separable iff there exists $\sum_i x_i \otimes y_i \in A \otimes_B A$ such that $\sum_i a x_i \otimes y_i = \sum_i x_i \otimes y_i a$ $(\forall a \in A)$ and $\sum_i x_i y_i = 1$).

- Can well known extensions (e.q., G-Galois extension, Frobenius extension, Hirata separable extension) be generalized to the case of module extensions?

- Can "Morita equivalence of module extension" be introduced as a generalization of Morita equivalence of ring extension?

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