Geometry of weighted Finsler spacetimes: A review

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Abstract

This is an overview for recent development concerning differential geometric investigations of weighted Finsler spacetimes with weighted Ricci curvature bounded below. We discuss comparison and splitting theorems (being reminiscent of positive-definite Finsler geometry), singularity theorems related to general relativity, and the timelike curvature-dimension condition motivated by recent progress of synthetic Lorentzian geometry.

1 Introduction

Lorentz–Finsler manifolds generalize Lorentzian manifolds in the same manner as Finsler manifolds generalize Riemannian manifolds, and a time-oriented Lorentz–Finsler manifold will be called a *Finsler spacetime*. This class of spacetimes has been attracting growing interest from the viewpoint of synthetic Lorentzian geometry and for widening room for physical interpretations. In this review, we summarize recent development, mainly motivated by the former viewpoint, for a further generalized class of *weighted* (or measured) Finsler spacetimes. Recently, we have witnessed strong interplay between Riemannian and Lorentzian geometries via synthetic theories by means of optimal transport and various comparison-type theorems concerning the sectional (flag) or Ricci curvature. This review could be read as an introduction of Lorentzian geometry to Finsler geometers, as well as a survey of the Finsler framework to Lorentzian geometers. Because of the limitation on length, we will not include detailed background; interested readers can consult references for further information.

Let us briefly explain the contents of the review. After preliminaries for weighted Finsler spacetimes in Sections 2, 3, we see in Section 4 some comparison theorems shown in essentially the same manner as the positive-definite case of Riemannian or Finsler manifolds. Precisely, we discuss the Bonnet–Myers diameter bound and d'Alembertian (spacetime Laplacian) comparison theorem along [33, 34]. We in fact consider those results in terms of ϵ-range, which provides a unification of unweighted and weighted comparison theorems.

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Then, Section 5 is devoted to a genuinely Lorentzian subject of singularity theorems along [33]. Still, they can be regarded as comparison theorems under nonnegative Ricci curvature. Very roughly speaking, singularity theorems assert that, under nonnegative curvature and some condition violating product (splitting) structure, we encounter incompleteness of timelike or lightlike geodesics.

In Section 6, we consider timelike splitting theorems established in [13] along the lines of a recent breakthrough [9]. This is a Lorentzian counterpart to the Cheeger–Gromoll splitting theorem in Riemannian geometry, and could be regarded as complementing singularity theorems to present the product structure under the existence of a timelike straight line.

Last but not least, Section 7 is concerned with the timelike curvature-dimension condition $\mathrm{TCD}_q(K, N)$ along [11], which was introduced by McCann [36] as a Lorentzian counterpart to Sturm and Lott–Villani's curvature-dimension condition $\mathrm{CD}(K, N)$ in terms of optimal transport theory. Establishing $\mathrm{TCD}_q(K, N)$ for Finsler spacetimes is meaningful from the synthetic viewpoint, as it tells that $\mathrm{TCD}_q(K, N)$ does not rule out Finsler spacetimes.

We do not include comparison theorems for the flag curvature. In that direction, it was recently shown in [5] that a Berwald spacetime has nonnegative flag curvature in timelike directions if and only if the time separation function satisfies a local concavity condition. This result can regarded as a Lorentzian analog to Busemann's classical characterization of nonpositively curved Riemannian manifolds [12]. We refer to [6] for the concavity in the nonsmooth setting of Lorentzian pre-length spaces [27], and to [18] for a local-to-global result.

2 Finsler spacetimes

We begin with preliminaries for Lorentz–Finsler geometry along [33, 34]. Lorentz–Finsler geometry can be regarded as a Lorentzian counterpart to Finsler geometry as well as a Finsler counterpart to Lorentzian geometry. Because of the author's expertise, the standpoint of this review is closer to the former one. In fact, as is seen in this section, to a large extent, one can introduce differential geometric quantities regardless of the signature of metrics. We refer to [1, 50, 54] for the basics of Finsler geometry, and fundamental references of Lorentzian geometry are [3, 51]. For further reading on Lorentz–Finsler geometry and more general structures, see, e.g., [37, 41, 42, 53].

Throughout the article, let M be a connected C^∞-manifold without boundary of dimension n (≥ 2). Given a local coordinate system $(x^i)_{i=1}^n$ on an open set $U \subset M$, we will use the fiber-wise linear coordinates $v = \sum_{i=1}^n v^i (\partial/\partial x^i)|_x \in T_xM$, $x \in U$. (We remark that $\dim M = n + 1$ in [33, 34].)

2.1 Lorentz–Finsler manifolds

We follow Beem's definition [2] of a Finsler version of Lorentzian metrics as follows.

Definition 2.1 (Lorentz–Finsler structures) A *Lorentz–Finsler structure* of M is a function $L\colon TM \longrightarrow \mathbb{R}$ satisfying the following conditions:

(1) $L \in C^\infty(TM \setminus \{0\})$;

(2) $L(cv) = c^2 L(v)$ for all $v \in TM$ and $c > 0$;

(3) For any $v \in TM \setminus \{0\}$, the symmetric matrix

$$\big(g_{ij}(v)\big)_{i,j=1}^n := \left(\frac{\partial^2 L}{\partial v^i \partial v^j}(v)\right)_{i,j=1}^n \tag{2.1}$$

is non-degenerate with signature $(-,+,\dots,+)$.

Then, we call (M, L) a (C^∞-)*Lorentz–Finsler manifold*.

We say that (M, L) is *reversible* if $L(-v) = L(v)$ for all $v \in TM$. For each $v \in T_xM \setminus \{0\}$, define a Lorentzian metric g_v of T_xM by using (2.1) as

$$g_v\left(\sum_{i=1}^n a_i \frac{\partial}{\partial x^i}\bigg|_x, \sum_{j=1}^n b_j \frac{\partial}{\partial x^j}\bigg|_x\right) := \sum_{i,j=1}^n g_{ij}(v) a_i b_j.$$

We have $g_v(v, v) = 2L(v)$ by Euler's homogeneous function theorem.

A tangent vector $v \in TM$ is said to be *timelike* (resp. *null*) if $L(v) < 0$ (resp. $L(v) = 0$). We say that v is *lightlike* if it is null and nonzero, and *causal* if it is timelike or lightlike (i.e., $L(v) \le 0$ and $v \neq 0$). *Spacelike* vectors are v satisfying $L(v) > 0$ or $v = 0$. For causal vectors v, we set

$$F(v) := \sqrt{-2L(v)} = \sqrt{-g_v(v, v)}. \tag{2.2}$$

Denote by $\Omega'_x \subset T_xM$ the set of timelike vectors and put $\Omega' := \bigcup_{x \in M} \Omega'_x$. Note that $\Omega'_x \neq \emptyset$, every connected component of Ω'_x is a convex cone (see [2], [33, Lemma 2.3]), and that the closures of different components intersect only at 0 (see [38, Proposition 1]). In general, the number of connected components of Ω'_x may be larger than 2 (see [2], [33, Example 2.4]). This fact will not affect our discussion because we shall deal with only future-directed (timelike or causal) vectors. We also remark that Ω'_x has exactly two connected components in reversible Lorentz–Finsler manifolds of dimension ≥ 3 (thanks to [38, Theorem 7]).

2.2 Finsler spacetimes

Definition 2.2 (Finsler spacetimes) If a Lorentz–Finsler manifold (M, L) admits a smooth timelike vector field X, then (M, L) is said to be *time-oriented* (by X). A time oriented Lorentz–Finsler manifold is called a *Finsler spacetime*.

In a Finsler spacetime time-oriented by X, a causal vector $v \in T_xM$ is said to be *future-directed* if it lies in the same connected component of $\overline{\Omega'_x} \setminus \{0\}$ as $X(x)$. We denote by $\Omega_x \subset \Omega'_x$ the set of future-directed timelike vectors, and define

$$\Omega := \bigcup_{x \in M} \Omega_x, \qquad \overline{\Omega} := \bigcup_{x \in M} \overline{\Omega}_x.$$

A C^1-curve in (M, L) is said to be *timelike* (resp. *causal*) if its tangent vector is always timelike (resp. causal). As usual, without otherwise indicated, all causal curves will be future-directed.

Given $x, y \in M$, we write $x \ll y$ (resp. $x < y$) if there is a future-directed timelike (resp. causal) curve from x to y, and $x \le y$ means that $x = y$ or $x < y$. Define the *chronological past* and *future* of $x \in M$ by

$$I^-(x) := \{y \in M \mid y \ll x\}, \qquad I^+(x) := \{y \in M \mid x \ll y\},$$

and the *causal past* and *future* of x by

$$J^-(x) := \{y \in M \mid y \le x\}, \qquad J^+(x) := \{y \in M \mid x \le y\}.$$

Let us introduce several causality conditions.

Definition 2.3 (Causality conditions) Let (M, L) be a Finsler spacetime.

(1) (M, L) is said to be *chronological* if $x \notin I^+(x)$ for all $x \in M$.

(2) We say that (M, L) is *causal* if there is no closed causal curve.

(3) (M, L) is said to be *strongly causal* if, for all $x \in M$, every neighborhood U of x contains a neighborhood V of x such that no causal curve intersects V more than once.

(4) We say that (M, L) is *globally hyperbolic* if it is strongly causal and, for any $x, y \in M$, $J^+(x) \cap J^-(y)$ is compact (or empty).

Clearly strong causality implies causality, and a causal spacetime is chronological. The chronological condition implies that the spacetime is non-compact. Global hyperbolicity is regarded as a kind of completeness. Another fundamental completeness condition is the following.

Definition 2.4 (Timelike geodesic completeness) We say that a Finsler spacetime (M, L) is *future timelike geodesically complete* if any timelike geodesic $\eta\colon [0, 1] \longrightarrow M$ can be extended to a geodesic $\tilde{\eta}\colon [0, \infty) \longrightarrow M$. If η can be extended to whole $\mathbb{R}$ (i.e., also *past timelike geodesically complete*), then (M, L) is said to be *timelike geodesically complete*.

Next, we define the *time separation* (also called the *Lorentz–Finsler distance*) $\tau(x,y)$ for $x,y \in M$ by

$$\tau(x,y) := \sup_{\eta} \mathbf{L}(\eta), \qquad \mathbf{L}(\eta) := \int_0^1 F\big(\dot{\eta}(t)\big)\,\mathrm{d}t,$$

where $\eta\colon [0,1] \longrightarrow M$ runs over all causal curves from x to y (recall (2.2) for F). We set $\tau(x,y) := 0$ if there is no causal curve from x to y (namely $x \not\leq y$). By definition, τ enjoys the *reverse triangle inequality*

$$\tau(x,z) \geq \tau(x,y) + \tau(y,z) \quad \text{for all } x \leq y \leq z.$$

In a similar manner to metric geometry, a timelike curve $\eta\colon [0,1] \longrightarrow M$ is called a *geodesic* if it is locally maximizing of constant speed, i.e., there is $c > 0$ such that, for any $t \in [0,1]$, we can take $\varepsilon > 0$ for which $\tau(\eta(a),\eta(b)) = c(b-a)$ holds for all $a,b \in (t-\varepsilon, t+\varepsilon) \cap [0,1]$ with $a < b$. If $\tau(\eta(0),\eta(1)) = \mathbf{L}(\eta)$ also holds, then we call η a (globally) *maximizing* geodesic.

In general, τ is only lower semi-continuous and can be infinite (see, e.g., [39, Proposition 6.7]). In globally hyperbolic Finsler spacetimes, τ is finite and continuous, and any pair of points $x,y \in M$ with $x < y$ admits a maximizing geodesic from x to y, where a lightlike geodesic is understood as a solution to the geodesic equation (2.4) below (*Avez–Seifert-type theorem*; see [39, Propositions 6.8, 6.9]).

2.3 Covariant derivative and Ricci curvature

Let us introduce several differential geometric quantities in a local coordinate system $(x^i)_{i=1}^n$, in the same way as the positive-definite case (see, e.g., [50]). Define

$$\gamma^i_{jk}(v) := \frac{1}{2}\sum_{l=1}^n g^{il}(v)\left(\frac{\partial g_{lk}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right)(v)$$

for $i,j,k = 1,\dots,n$ and $v \in TM \setminus \{0\}$, where $(g^{ij}(v))$ is the inverse matrix of $(g_{ij}(v))$,

$$G^i(v) := \sum_{j,k=1}^n \gamma^i_{jk}(v) v^j v^k, \qquad N^i_j(v) := \frac{1}{2}\frac{\partial G^i}{\partial v^j}(v)$$

for $v \in TM \setminus \{0\}$ (and $G^i(0) = N^i_j(0) := 0$), and

$$\Gamma^i_{jk}(v) := \gamma^i_{jk}(v) - \frac{1}{2}\sum_{l,m=1}^n g^{il}(v)\left(\frac{\partial g_{lk}}{\partial v^m}N^m_j + \frac{\partial g_{jl}}{\partial v^m}N^m_k - \frac{\partial g_{jk}}{\partial v^m}N^m_l\right)(v) \tag{2.3}$$

on $TM \setminus \{0\}$. (We remark that the above definition of $G^i(v)$ is same as [50] and corresponds to $2G^\alpha(v)$ in [33, 34].) Then, the *covariant derivative* of a vector field $Y = \sum_{i=1}^n Y^i(\partial/\partial x^i)$ is defined as

$$D^w_v Y := \sum_{i,j=1}^n \left\{ v^j \frac{\partial Y^i}{\partial x^j}(x) + \sum_{k=1}^n \Gamma^i_{jk}(w) v^j Y^k(x) \right\} \frac{\partial}{\partial x^i}\bigg|_x$$

for $v \in T_xM$ with reference vector $w \in T_xM \setminus \{0\}$. We remark that the functions Γ^i_{jk} in (2.3) are the coefficients of the *Chern(–Rund) connection*.

In the Lorentzian case, g_{ij} is constant in each slit tangent space $T_xM \setminus \{0\}$ (thus, $\Gamma^i_{jk} = \gamma^i_{jk}$) and the covariant derivative does not depend on the choice of a reference vector. In the Lorentz–Finsler setting, the following class is worth considering.

Definition 2.5 (Berwald spacetimes) A Finsler spacetime (M, L) is said to be of *Berwald type* (or called a *Berwald spacetime*) if Γ^i_{jk} is constant on the slit tangent space $T_xM \setminus \{0\}$ for any x in the domain of every local coordinate system.

The Berwald condition is strong but provides a reasonable (nontrivial) class where we can mimic Lorentzian techniques. By the very definition, the covariant derivative on a Berwald spacetime is defined independently from the choice of a reference vector. Examples of Berwald spacetimes include Lorentzian manifolds and flat Lorentz–Finsler structures L of $\mathbb{R}^n$ (every tangent space $(T_x\mathbb{R}^n, L)$ is canonically isometric to $(T_0\mathbb{R}^n, L)$ and we have $\Gamma^i_{jk} = \gamma^i_{jk} = 0$). We refer to [20, 21, 22] for some investigations on Berwald spacetimes, and to [1, Chapter 10], [50, §6.3] for the positive-definite case.

Remark 2.6 (Metrizability) In the positive-definite case, Szabó [57] showed that a Berwald space (M, F) admits a Riemannian metric g whose Levi-Civita connection coincides with the Chern connection of F, i.e., the Christoffel symbols of g coincide with Γ^i_{jk} of F (see also [1, Exercise 10.1.4]). This is called the (Riemannian) *metrizability theorem*. It is not known whether the metrizability can be generalized to the Lorentz–Finsler setting. In [20], some counter-examples were constructed for Lorentz–Finsler structures defined on a subset of TM. Their discussion is not applicable to Lorentz–Finsler structures defined on whole TM as in Definition 2.1.

The *geodesic equation* for a C^2-curve $\eta\colon [0,1] \longrightarrow M$ is written as

$$D^{\dot\eta}_{\dot\eta}\dot\eta = 0, \tag{2.4}$$

which makes sense also for lightlike and spacelike curves. The *exponential map* is defined in the same way as the Riemannian case, which is C^∞ on a neighborhood of the zero section only in Berwald spacetimes (see, e.g., [40] as well as [1, Exercise 5.3.5] in the positive-definite case).

Next, we introduce the Ricci curvature. A C^∞-vector field J along a geodesic η is called a *Jacobi field* if it satisfies the Jacobi equation

$$D^{\dot\eta}_{\dot\eta}D^{\dot\eta}_{\dot\eta}J + R_{\dot\eta}(J) = 0,$$

where

$$R_v(w) := \sum_{i,j=1}^n R^i_j(v)w^j \frac{\partial}{\partial x^i}\bigg|_x \in T_xM \tag{2.5}$$

for $v, w \in T_xM$ and

$$R^i_j(v) := \frac{\partial G^i}{\partial x^j}(v) - \sum_{k=1}^n \left\{ \frac{\partial N^i_j}{\partial x^k}(v)v^k - \frac{\partial N^i_j}{\partial v^k}(v)G^k(v) + N^i_k(v)N^k_j(v) \right\}$$

is the *curvature tensor*. Similarly to the positive-definite case, a Jacobi field is characterized as the variational vector field of a geodesic variation.

Note that $R_v(w)$ is linear in w, thereby $R_v \colon T_xM \longrightarrow T_xM$ is an endomorphism for each $v \in T_xM$.

Definition 2.7 (Ricci curvature) For $v \in \overline{\Omega}_x$, we define the *Ricci curvature* (or *Ricci scalar*) of v as the trace of R_v: $\mathrm{Ric}(v) := \mathrm{trace}(R_v)$.

We remark that $\mathrm{Ric}(cv) = c^2 \mathrm{Ric}(v)$ for $c > 0$.

2.4 Legendre transforms and differential operators

In order to introduce a Lorentzian analog of Laplacian, we consider the dual structure to L and the Legendre transform (see [34, §4.4], [38], [41, §3.1] for further discussions). Define the *polar cone* to Ω_x by

$$\Omega_x^* := \bigl\{\omega \in T_x^*M \mid \omega(v) < 0 \text{ for all } v \in \overline{\Omega}_x \setminus \{0\}\bigr\}.$$

This is an open convex cone in T_x^*M. For $\omega \in \Omega_x^*$, we define

$$L^*(\omega) := -\frac{1}{2}\Bigl(\sup_{v \in \Omega_x \cap F^{-1}(1)} \omega(v)\Bigr)^2 = -\frac{1}{2}\inf_{v \in \Omega_x \cap F^{-1}(1)} \bigl(\omega(v)\bigr)^2.$$

Then, by definition, we have the *reverse Cauchy–Schwarz inequality*

$$4L^*(\omega)L(v) \le \bigl(\omega(v)\bigr)^2$$

for $v \in \Omega_x$ and $\omega \in \Omega_x^*$, and arrive at the following variational definition of the Legendre transform.

Definition 2.8 (Legendre transform) Define the *Legendre transform* $\mathscr{L}^* \colon \Omega_x^* \longrightarrow \Omega_x$ as the map sending $\omega \in \Omega_x^*$ to the unique element $v \in \Omega_x$ satisfying $L(v) = L^*(\omega) = \omega(v)/2$. We also define $\mathscr{L}^*(0) := 0$.

A coordinate expression of the Legendre transform is given by

$$\mathscr{L}^*(\omega) = \sum_{i=1}^n \frac{\partial L^*}{\partial \omega_i}(\omega)\frac{\partial}{\partial x^i}, \qquad \text{where } \omega = \sum_{i=1}^n \omega_i \, \mathrm{d}x^i. \tag{2.6}$$

We refer to [34, 38, 41] for some basic properties of the Legendre transform.

A continuous function $f \colon M \longrightarrow \mathbb{R}$ is called a *time function* if $f(x) < f(y)$ for all $x, y \in M$ with $x < y$. A C^1-function $f \colon M \longrightarrow \mathbb{R}$ is said to be *temporal* if $-\mathrm{d}f(x) \in \Omega_x^*$ for all $x \in M$. Observe that temporal functions are time functions.

For a temporal function $f \colon M \longrightarrow \mathbb{R}$, define the *gradient vector* of $-f$ at $x \in M$ by

$$\nabla(-f)(x) := \mathscr{L}^*\bigl(-\mathrm{d}f(x)\bigr) \in \Omega_x.$$

We deduce from (2.6) that $g_{\nabla(-f)}(\nabla(-f)(x), v) = -\mathrm{d}f(v)$ for any $v \in T_xM$. For a C^2-temporal function $f\colon M \longrightarrow \mathbb{R}$ and $x \in M$, define the *Hessian* $\nabla^2(-f)\colon T_xM \longrightarrow T_xM$ by

$$\nabla^2(-f)(v) := D_v^{\nabla(-f)}\big(\nabla(-f)\big).$$

This spacetime Hessian has the following symmetry (see [34, Lemma 4.13]):

$$g_{\nabla(-f)}\big(\nabla^2(-f)(v), w\big) = g_{\nabla(-f)}\big(v, \nabla^2(-f)(w)\big)$$

for all $v, w \in T_xM$. Then we define the *d'Alembertian* (or *spacetime Laplacian*) as the trace of the Hessian:

$$\Box(-f) := \mathrm{trace}\big(\nabla^2(-f)\big)$$

(denoted by $\boldsymbol{\Delta}(-f)$ in [34]). We remark that the d'Alembertian is not elliptic but hyperbolic, and is nonlinear (since the Legendre transform is nonlinear).

Remark 2.9 (Berwald case) For $v \in T_xM$ and the geodesic $\eta\colon (-\varepsilon, \varepsilon) \longrightarrow M$ with $\dot\eta(0) = v$, the second order derivative $(-f \circ \eta)''(0)$ does not necessarily coincide with $g_{\nabla(-f)}(\nabla^2(-f)(v), v)$. They coincide in Berwald spacetimes thanks to the fiber-wise constancy of the connection coefficients Γ^i_{jk} (see [50, §12.1] for the positive-definite case).

3 Weighted Finsler spacetimes

In what follows, we fix a smooth positive measure $\mathfrak{m}$ on M, and call $(M, L, \mathfrak{m})$ a *weighted* (or *measured*) Finsler spacetime. Depending on the choice of a measure, we naturally need to modify the d'Alembertian and Ricci curvature.

Given $v \in \overline{\Omega} \setminus \{0\}$, let $\eta\colon (-\varepsilon, \varepsilon) \longrightarrow M$ be the causal geodesic with $\dot\eta(0) = v$ and decompose $\mathfrak{m}$ in local coordinates along η as

$$\mathfrak{m}(\mathrm{d}x) = \mathrm{e}^{-\psi_\eta}\sqrt{-\det\big[\big(g_{ij}(\dot\eta)\big)\big]}\,\mathrm{d}x^1 \cdots \mathrm{d}x^n, \tag{3.1}$$

where $\psi_\eta\colon (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$. For a C^2-temporal function $f\colon M \longrightarrow \mathbb{R}$, define the *weighted d'Alembertian* of $-f$ associated with $\mathfrak{m}$ by

$$\Box_{\mathfrak{m}}(-f)(x) := \Box(-f)(x) - \psi'_\eta(0), \tag{3.2}$$

where ψ_η is given by (3.1) with $v = \nabla(-f)(x)$. Equivalently, one can define $\Box_{\mathfrak{m}}(-f) := \mathrm{div}_{\mathfrak{m}}[\nabla(-f)]$ using the divergence with respect to $\mathfrak{m}$; for any test function $\phi \in C^\infty(M)$ of compact support, we have

$$\int_M \phi \cdot \Box_{\mathfrak{m}}(-f)\,\mathrm{d}\mathfrak{m} = -\int_M \mathrm{d}\phi\big(\nabla(-f)\big)\,\mathrm{d}\mathfrak{m}.$$

The following definition generalizes the weighted Ricci curvature (also called the *Bakry–Émery Ricci curvature*) for measured Finsler manifolds introduced in [45] (see also [48, 49] for the case of $N \leq 0$).

Definition 3.1 (Weighted Ricci curvature) Given $v \in \overline{\Omega} \setminus 0$ and ψ_η as in (3.1), we define the *weighted Ricci curvature* by

$$\mathrm{Ric}_N(v) := \mathrm{Ric}(v) + \psi_\eta''(0) - \frac{\psi_\eta'(0)^2}{N-n}$$

for $N \in \mathbb{R} \setminus \{n\}$. We also define $\mathrm{Ric}_\infty(v) := \mathrm{Ric}(v) + \psi_\eta''(0)$, $\mathrm{Ric}_n(v) := \lim_{N \downarrow n} \mathrm{Ric}_N(v)$, and $\mathrm{Ric}_N(0) := 0$ for all N.

Remark 3.2 (Weight function) Note that, for a Lorentzian manifold (M, g), the reference measure $\sqrt{-\det[(g_{ij})]}\,\mathrm{d}x^1 \cdots \mathrm{d}x^n$ in (3.1) coincides with the canonical volume measure vol_g. Hence, choosing a measure $\mathfrak{m}$ on M is equivalent to choosing a function Ψ on M via the relation $\mathfrak{m} = \mathrm{e}^{-\Psi}\,\mathrm{vol}_g$. In the Lorentz–Finsler setting, however, they are not equivalent. On the one hand, one way to unify these two approaches is to consider 0-homogeneous functions $\psi \colon \overline{\Omega} \setminus \{0\} \longrightarrow \mathbb{R}$ as in [33, 34] (we remark that, in general, such a function may not be associated with any measure; see [46, 50] for the positive-definite case). On the other hand, a weighted Laplacian corresponding to general ψ is yet to be developed, even for Riemannian manifolds.

We will say that $\mathrm{Ric}_N \geq K$ holds *in timelike directions* for some $K \in \mathbb{R}$ if we have $\mathrm{Ric}_N(v) \geq KF^2(v) = -2KL(v)$ for all $v \in \Omega$ (recall (2.2)). The case of $K = 0$ is also called the *timelike N-convergence condition* (see [33]). Observe that, by definition,

$$\mathrm{Ric}_n(v) \leq \mathrm{Ric}_N(v) \leq \mathrm{Ric}_\infty(v) \leq \mathrm{Ric}_{N'}(v)$$

holds for $n < N < \infty$ and $-\infty < N' < n$.

Associated with the real parameter N, the following notion was introduced in [33].

Definition 3.3 (ϵ-range) Given $N \in (-\infty, 1] \cup [n, \infty]$, we will consider $\epsilon \in \mathbb{R}$ in the following (timelike) *ϵ-range*:

$$\epsilon = 0 \ \text{ for } N = 1, \qquad |\epsilon| < \sqrt{\frac{N-1}{N-n}} \ \text{ for } N \neq 1, n, \qquad \epsilon \in \mathbb{R} \ \text{ for } N = n. \tag{3.3}$$

The associated constant $c = c(N, \epsilon) > 0$ is defined as

$$c(N, \epsilon) := \frac{1}{n-1}\left(1 - \epsilon^2 \frac{N-n}{N-1}\right) \ \text{ for } N \neq 1, \qquad c(1, 0) := \frac{1}{n-1}. \tag{3.4}$$

Note that $\epsilon = 1$ is admissible only for $N \in [n, \infty)$ and $\epsilon = 0$ is always admissible. In terms of the ϵ-range, we can unify results for usual constant curvature bounds and those for the variable curvature bound studied by Wylie–Yeroshkin [61] (see also [28]) into a single framework, by taking $\epsilon = 1$ and $\epsilon = 0$, respectively. We refer to [34] for more details, and to [29, 30, 31] for some further applications in the Riemannian setting.

It will be useful to consider the *reverse Lorentz–Finsler structure* of L defined as $\overleftarrow{L}(v) := L(-v)$. We will put an arrow $\leftarrow$ on a quantity associated with $\overleftarrow{L}$. The Lorentz–Finsler manifold $(M, \overleftarrow{L})$ is time-oriented by $-X$, so that $\overleftarrow{\Omega} = -\Omega$. The corresponding weighted d'Alembertian is given by $\overleftarrow{\Box}_{\mathfrak{m}} f = -\Box_{\mathfrak{m}}(-f)$ for temporal functions f with respect to L. We remark that, for the weighted Ricci curvature $\overleftarrow{\mathrm{Ric}}$ of $(M, \overleftarrow{L}, \mathfrak{m})$, we have $\overleftarrow{\mathrm{Ric}}_N(v) = \mathrm{Ric}_N(-v)$ for $v \in \overleftarrow{\Omega}$, and hence $\mathrm{Ric}_N \geq K$ in timelike directions (in L) if and only if $\overleftarrow{\mathrm{Ric}}_N \geq K$ in timelike directions (in $\overleftarrow{L}$).

The following condition, introduced in [33], will play an essential role. A future directed causal curve $\eta\colon (a,b) \longrightarrow M$ is said to be *future inextendible* if $\eta(t)$ does not converge as $t \to b$.

Definition 3.4 (ϵ-completeness) A weighted Finsler spacetime $(M, L, \mathfrak{m})$ is said to be *future timelike ϵ-complete* if, for any future inextendible timelike geodesic $\eta\colon [0,T) \longrightarrow M$,

$$\int_0^T \mathrm{e}^{\frac{2(\epsilon-1)}{n-1}\psi_\eta(s)}\,\mathrm{d}s = \infty$$

holds. We say that $(M, L, \mathfrak{m})$ is *timelike ϵ-complete* if both $(M, L, \mathfrak{m})$ and $(M, \overleftarrow{L}, \mathfrak{m})$ are future timelike ϵ-complete.

The 1-completeness corresponds to the timelike geodesic completeness (Definition 2.4) and the 0-completeness recovers the ψ-completeness introduced in [60] in the Riemannian case (see [58, 59] for the Lorentzian case). A typical situation is that $\epsilon < 1$, ψ_η is bounded above and (M, L) is (future) timelike geodesically complete. For later use, we define

$$\varphi_\eta(t) := \int_0^t \mathrm{e}^{\frac{2(\epsilon-1)}{n-1}\psi_\eta(s)}\,\mathrm{d}s, \tag{3.5}$$

called the (timelike) *ϵ-proper time* in [33, (5.6)].

4 Comparison theorems

In the setting of weighted Finsler spacetimes as in the previous section, we first discuss direct analogs to two comparison theorems on measured Finsler manifolds as applications of the Raychaudhuri inequality suitably generalized to the current setting. For describing the Raychaudhuri inequality, we introduce some notations. We refer to [33, 34] for more details.

Definition 4.1 (Jacobi & Lagrange tensor fields) Let $\eta\colon [0,T) \longrightarrow M$ be a timelike geodesic of unit speed (i.e., $F(\dot\eta) \equiv 1$), and denote by $N_\eta(t) \subset T_{\eta(t)}M$ the space of vectors orthogonal to $\dot\eta(t)$ with respect to $g_{\dot\eta}$. For brevity, the covariant derivative $D_{\dot\eta}^{\dot\eta} Y$ of a vector field Y along η will be written as Y'.

(1) A smooth tensor field J, giving an endomorphism $\mathsf{J}(t)\colon N_\eta(t) \longrightarrow N_\eta(t)$ for each $t \in [0,T)$, is called a *Jacobi tensor field* along η if we have

$$\mathsf{J}'' + \mathsf{R}\mathsf{J} = 0 \tag{4.1}$$

and $\ker(\mathsf{J}(t)) \cap \ker(\mathsf{J}'(t)) = \{0\}$ for all t, where $\mathsf{R}(t) := R_{\dot\eta(t)}\colon N_\eta(t) \longrightarrow N_\eta(t)$ is the curvature endomorphism defined in (2.5).

(2) A Jacobi tensor field J is called a *Lagrange tensor field* if

$$(\mathsf{J}')^{\mathsf{T}}\mathsf{J} - \mathsf{J}^{\mathsf{T}}\mathsf{J}' = 0 \tag{4.2}$$

holds on $[0, T)$, where the transpose T is taken with respect to $g_{\dot\eta}$.

Remark 4.2 (a) The equation (4.1) means that, for any $g_{\dot\eta}$-parallel vector field P along η (i.e., $P' \equiv 0$), $Y(t) := \mathsf{J}(t)(P(t))$ is a Jacobi field along η. Then, the condition $\ker(\mathsf{J}(t)) \cap \ker(\mathsf{J}'(t)) = \{0\}$ implies that $Y = \mathsf{J}(P)$ is not identically zero for every nonzero P. Note also that $R_{\dot\eta(t)}(w) \in N_\eta(t)$ for all $w \in T_{\eta(t)}M$.

(b) The equation (4.2) means that $\mathsf{J}^{\mathsf{T}}\mathsf{J}'$ is $g_{\dot\eta}$-symmetric. Precisely, given two $g_{\dot\eta}$-parallel vector fields P_1, P_2 along η, the Jacobi fields $Y_i := \mathsf{J}(P_i)$ satisfy

$$g_{\dot\eta}(Y_1', Y_2) - g_{\dot\eta}(Y_1, Y_2') = 0. \tag{4.3}$$

Since $[g_{\dot\eta}(Y_1', Y_2) - g_{\dot\eta}(Y_1, Y_2')]' = 0$, we have (4.3) for all t if it holds at some t.

Given a Lagrange tensor field J along η, define $\mathsf{B} := \mathsf{J}'\mathsf{J}^{-1}$, which is symmetric by (4.2). Multiplying (4.1) by J^{-1} from right provides the *Riccati equation*

$$\mathsf{B}' + \mathsf{B}^2 + \mathsf{R} = 0$$

(see [33, (5.3)]). We also define the *expansion scalar*

$$\theta(t) := \operatorname{trace}\big(\mathsf{B}(t)\big) \tag{4.4}$$

and the *shear tensor* (the traceless part of B)

$$\sigma(t) := \mathsf{B}(t) - \frac{\theta(t)}{n-1}\mathsf{I}_{n-1}(t),$$

where $\mathsf{I}_{n-1}(t)$ denotes the identity of $N_\eta(t)$.

The weighted counterparts will make use of the parametrization φ_η as in (3.5). Note that

$$(\eta \circ \varphi_\eta^{-1})'(s) = \mathrm{e}^{-\frac{2(\epsilon-1)}{n-1}\psi_\eta(\varphi_\eta^{-1}(s))}\dot\eta\big(\varphi_\eta^{-1}(s)\big) \tag{4.5}$$

for $s \in [0, \varphi_\eta(T))$. We define, for $\epsilon \in \mathbb{R}$ and $t \in [0, T)$,

$$\begin{aligned}
\mathsf{J}_{\mathfrak{m}}(t) &:= \mathrm{e}^{-\psi_\eta(t)/(n-1)}\mathsf{J}(t),\\
\mathsf{B}_\epsilon(t) &:= (\mathsf{J}_{\mathfrak{m}} \circ \varphi_\eta^{-1})'\big(\varphi_\eta(t)\big) \cdot \mathsf{J}_{\mathfrak{m}}(t)^{-1} = \mathrm{e}^{-\frac{2(\epsilon-1)}{n-1}\psi_\eta(t)}\bigg(\mathsf{B}(t) - \frac{\psi_\eta'(t)}{n-1}\mathsf{I}_{n-1}(t)\bigg),\\
\theta_\epsilon(t) &:= \operatorname{trace}\big(\mathsf{B}_\epsilon(t)\big) = \mathrm{e}^{-\frac{2(\epsilon-1)}{n-1}\psi_\eta(t)}\big(\theta(t) - \psi_\eta'(t)\big),\\
\sigma_\epsilon(t) &:= \mathsf{B}_\epsilon(t) - \frac{\theta_\epsilon(t)}{n-1}\mathsf{I}_{n-1}(t) = \mathrm{e}^{-\frac{2(\epsilon-1)}{n-1}\psi_\eta(t)}\sigma(t).
\end{aligned} \tag{4.6}$$

Then, the *weighted Riccati equation* is given by

$$(\mathsf{B}_\epsilon \circ \varphi_\eta^{-1})' + \frac{2\epsilon}{n-1}(\psi_\eta \circ \varphi_\eta^{-1})' \cdot \mathsf{B}_\epsilon(\varphi_\eta^{-1}) + \mathsf{B}_\epsilon^2(\varphi_\eta^{-1}) + \mathsf{R}_{(1,\epsilon)}(\varphi_\eta^{-1}) = 0$$

on $(0, \varphi_\eta(T))$, where

$$\mathsf{R}_{(N,\epsilon)}(t) := \mathrm{e}^{-\frac{4(\epsilon-1)}{n-1}\psi_\eta(t)}\left\{\mathsf{R}(t) + \frac{1}{n-1}\left(\psi_\eta''(t) - \frac{\psi_\eta'(t)^2}{N-n}\right)\mathsf{I}_{n-1}(t)\right\}$$

(see [33, Lemma 5.5], [34, §5.2]). Observe that, by (4.5),

$$\mathrm{trace}\big(\mathsf{R}_{(N,\epsilon)}(t)\big) = \mathrm{e}^{-\frac{4(\epsilon-1)}{n-1}\psi_\eta(t)}\,\mathrm{Ric}_N\big(\dot\eta(t)\big) = \mathrm{Ric}_N\Big((\eta\circ\varphi_\eta^{-1})'\big(\varphi_\eta(t)\big)\Big).$$

The *timelike weighted Raychaudhuri inequality* is a consequence of the above weighted Riccati equation (see [33, Proposition 5.7], [34, Theorem 5.6]).

Theorem 4.3 (Raychaudhuri inequality) *Let J be a nonsingular Lagrange tensor field along a timelike geodesic $\eta\colon [0,T) \longrightarrow M$ of unit speed. Then, for every $\epsilon \in \mathbb{R}$ and $N \in (-\infty,1] \cup [n,\infty]$, we have*

$$(\theta_\epsilon\circ\varphi_\eta^{-1})' \le -\mathrm{Ric}_N\big((\eta\circ\varphi_\eta^{-1})'\big) - \mathrm{trace}\big(\sigma_\epsilon^2(\varphi_\eta^{-1})\big) - c\theta_\epsilon^2(\varphi_\eta^{-1})$$

on $(0,\varphi_\eta(T))$ with $c = c(N,\epsilon)$ in (3.4).

When ϵ is taken from the ϵ-range (3.3), we have $c > 0$ and the above Raychaudhuri inequality yields the *Bishop-type inequality* (see [34, (5.9)]):

$$h_1''(s) \le -ch_1(s)\,\mathrm{Ric}_N\big((\eta\circ\varphi_\eta^{-1})'(s)\big), \qquad h_1(s) := \Big(\det\big[\mathsf{J}_{\mathfrak{m}}\big(\varphi_\eta^{-1}(s)\big)\big]\Big)^c,$$

from which we can readily deduce the Bonnet–Myers and d'Alembertian comparison theorems.

Remark 4.4 The Raychaudhuri inequality is closely related to the Bochner inequality, whereas there is a limitation due to the requirement that J is an endomorphism of $N_\eta(t)$ (note that every $v \in N_\eta(t)$ is $g_{\dot\eta}$-spacelike, i.e., $g_{\dot\eta}(v,v) > 0$ or $v = 0$). Nonetheless, the Raychaudhuri inequality is sufficient to derive the Bishop inequality, which is concerned with the behavior of concentric "spheres" (with respect to τ). In timelike splitting theorems in Section 6, we will need a Bochner-type inequality (see [13, Remark 5.4] for more details).

The *timelike diameter* of (M,L) is defined as $\mathrm{diam}(M) := \sup_{x,y\in M}\tau(x,y)$ (recall that $\tau(x,y) = 0$ if $x \not< y$); we refer to [3, §11.1] for some accounts on $\mathrm{diam}(M)$. We remark that the finite diameter does not imply the compactness in the Lorentzian setting. A *Bonnet–Myers theorem* with ϵ-range was given in [34, Theorem 5.7] as follows.

Theorem 4.5 (Bonnet–Myers theorem) *Let $(M,L,\mathfrak{m})$ be a globally hyperbolic, weighted Finsler spacetime. Suppose that, for some $N \in (-\infty,1] \cup [n,\infty]$, ϵ in the ϵ-range (3.3), $K > 0$ and $b > 0$, we have*

$$\mathrm{Ric}_N(v) \ge KF^2(v)\mathrm{e}^{\frac{4(\epsilon-1)}{n-1}\psi_\eta(0)},$$

$$\mathrm{e}^{-\frac{2(\epsilon-1)}{n-1}\psi_\eta(0)} \le b \tag{4.7}$$

for all $v \in \Omega$ and η given as in (3.1). Then, we have

$$\mathrm{diam}(M) \le \frac{b\pi}{\sqrt{cK}}$$

for $c = c(N, \epsilon) > 0$ as in (3.4).

As an intermediate estimate, we also obtain

$$\varphi_\eta(t_0) = \int_0^{t_0} \mathrm{e}^{\frac{2(\epsilon-1)}{n-1}\psi_\eta(s)}\,\mathrm{d}s \le \frac{\pi}{\sqrt{cK}}$$

without assuming (4.7), where $\eta(t_0)$ is the first conjugate point to $\eta(0)$. For $N \in [n, \infty)$, we can take $\epsilon = 1$ for which $c = 1/(N-1)$, and (4.7) automatically holds with $b = 1$. Hence, we have $\mathrm{diam}(M) \le \pi\sqrt{(N-1)/K}$ as in the standard (weighted) Bonnet–Myers theorem (see, e.g., [50, §9.4]).

Given $z \in M$, we say that $x \in I^+(z)$ is a (future) *timelike cut point* to z if there is a maximizing timelike geodesic $\eta\colon [0,1] \longrightarrow M$ from z to x such that its extension $\bar{\eta}\colon [0, 1+\varepsilon] \longrightarrow M$ is not maximal for any $\varepsilon > 0$. The *timelike cut locus* $\mathrm{Cut}(z)$ is the set of all timelike cut points to z. Notice that the function $f(x) := \tau(z, x)$ satisfies $-\mathrm{d}f(x) \in \Omega_x^*$ for $x \in I^+(z) \setminus \mathrm{Cut}(z)$, thereby the weighted d'Alembertian $\Box_{\mathfrak{m}}(-f)$ as in (3.2) is well-defined on $I^+(z) \setminus \mathrm{Cut}(z)$.

Now, the *d'Alembertian comparison theorem* [34, Theorem 5.8] asserts the following. We refer to [14, 59] for the weighted Lorentzian case. The function $\mathbf{s}_\kappa$ with $\kappa \in \mathbb{R}$ is defined by

$$\mathbf{s}_\kappa(r) := \begin{cases} \frac{1}{\sqrt{\kappa}}\sin(\sqrt{\kappa}r) & \kappa > 0, \\ r & \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}}\sinh(\sqrt{-\kappa}r) & \kappa < 0, \end{cases} \tag{4.8}$$

where $r \in [0, \pi/\sqrt{\kappa}]$ for $\kappa > 0$ and $r \ge 0$ for $\kappa \le 0$.

Theorem 4.6 (d'Alembertian comparison theorem) *Let $(M, L, \mathfrak{m})$ be a globally hyperbolic, weighted Finsler spacetime and $N \in (-\infty, 1] \cup [n, \infty]$, $\epsilon \in \mathbb{R}$ in the ϵ-range (3.3), $K \in \mathbb{R}$ and $b \ge a > 0$. Suppose that*

$$\mathrm{Ric}_N(v) \ge KF^2(v)\mathrm{e}^{\frac{4(\epsilon-1)}{n-1}\psi_\eta(0)},$$

$$a \le \mathrm{e}^{-\frac{2(\epsilon-1)}{n-1}\psi_\eta(0)} \le b \tag{4.9}$$

for all $v \in \Omega$ and η given as in (3.1). Then, for any $z \in M$, $f(x) := \tau(z, x)$ satisfies

$$\Box_{\mathfrak{m}}(-f)(x) \le \frac{1}{c\rho}\frac{\mathbf{s}'_{cK}(f(x)/b)}{\mathbf{s}_{cK}(f(x)/b)}$$

on $I^+(z) \setminus \mathrm{Cut}(z)$, where $\rho := a$ if $\mathbf{s}'_{cK}(f(x)/b) \ge 0$ and $\rho := b$ if $\mathbf{s}'_{cK}(f(x)/b) < 0$.

Similarly to Theorem 4.5, we also have an intermediate estimate

$$\square_{\mathfrak{m}}(-f)\big(\eta(t)\big) \le \mathrm{e}^{\frac{2(\epsilon-1)}{n-1}\psi_\eta(t)} \frac{\mathbf{s}'_{cK}(\varphi_\eta(t))}{c\mathbf{s}_{cK}(\varphi_\eta(t))}$$

without assuming (4.9). When $N \in [n,\infty)$ and $\epsilon = 1$, we can take $a = b = 1$ and obtain

$$\square_{\mathfrak{m}}(-f)(x) \le (N-1)\frac{\mathbf{s}'_{K/(N-1)}(f(x))}{\mathbf{s}_{K/(N-1)}(f(x))}$$

as in the standard (weighted) Laplacian comparison theorem (see, e.g., [50, §11.3]).

5 Singularity theorems

Singularity theorems are a set of results in general relativity asserting that, under suitable energy and genericity conditions, we encounter some singularity in future or past (see, e.g., [26]). On the one hand, the energy condition is read as the nonnegative Ricci curvature in timelike or lightlike directions, thus singularity theorems could also be regarded as comparison theorems. On the other hand, genericity conditions are imposed to exclude static spacetimes having the product structure (cf. timelike splitting theorems in the next section).

For the sake of simplicity and keeping this review compact, we shall consider only the Hawking singularity theorem concerning timelike incompleteness along [33]. We refer to [33] for the Penrose and Hawking–Penrose singularity theorems dealing with lightlike incompleteness, as well as for further discussions on the structure of singularity theorems.

Let $S \subset M$ be a C^2-spacelike hypersurface and V be its future-directed normal vector field, in the sense that $V(x) \in \Omega_x$ and $\ker[g_V(V(x),\cdot)] = T_xS$ for all $x \in S$. Consider the *geodesic congruence* generated by V, namely the family of timelike geodesics $\eta\colon [0,T) \longrightarrow M$ such that $\eta(0) \in S$ and $\dot\eta(0) = V(\eta(0))$. Associated with such a family of geodesics, we have a Jacobi tensor field J along η, which is in fact a Lagrange tensor field. Then, in the same way as (4.4) and (4.6), we define the *expansion* $\theta\colon S \longrightarrow \mathbb{R}$ of the geodesic congruence by

$$\theta(x) := \mathrm{trace}(\mathsf{J}'\mathsf{J}^{-1})(0)$$

for $x = \eta(0) \in S$ (the right-hand side can be interpreted as the trace of the shape operator of S), and the *ϵ-expansion* $\theta_\epsilon\colon S \longrightarrow \mathbb{R}$ by

$$\theta_\epsilon(x) := \mathrm{e}^{\frac{-2(\epsilon-1)}{n-1}\psi_\eta(0)}\big(\theta(x) - \psi'_\eta(0)\big).$$

As a kind of genericity condition, we employ the following.

Definition 5.1 (Contraction) We say that S is *contracting* (resp. $\mathfrak{m}$-*contracting*) if $\theta < 0$ (resp. $\theta_1 < 0$) on S.

We are ready to state the *Hawking singularity theorem* in this context (see [33, Theorem 8.13]). Recall Definition 3.4 for the definition of timelike ϵ-completeness.

Theorem 5.2 (Hawking singularity theorem) *Let $(M, L, \mathfrak{m})$ be a weighted Finsler spacetime satisfying $\mathrm{Ric}_N \geq 0$ in timelike directions for some $N \in (-\infty, 1] \cup [n, \infty]$. If M contains a compact C^2-spacelike hypersurface S which is $\mathfrak{m}$-contracting, then there exists a timelike geodesic issued normally from S which is future ϵ-incomplete for every $\epsilon \in \mathbb{R}$ in the ϵ-range (3.3).*

A time reversed version says that, if $\theta_1 > 0$ on S (i.e., S is $\mathfrak{m}$-*expanding*), then there exists a timelike geodesic issued normally from S which is past ϵ-incomplete for every $\epsilon \in \mathbb{R}$ in (3.3) (see [33, Remark 8.14]). This is also understood as Theorem 5.2 for the reverse structure $\overleftarrow{L}(v) = L(-v)$ (recall Section 3).

6 Timelike splitting theorems

Timelike splitting theorem is a Lorentzian analog to the celebrated *Cheeger–Gromoll splitting theorem* [16] in Riemannian geometry; the latter asserts that, if a Riemannian manifold of nonnegative Ricci curvature contains a straight line, then it necessarily splits off a real line $\mathbb{R}$. The Lorentzian version could be regarded as a complement to singularity theorems discussed in the previous section, as the existence of a *timelike straight line* (i.e., $\eta\colon \mathbb{R} \longrightarrow M$ satisfying $\tau(\eta(s), \eta(t)) = t - s$ for all $s, t \in \mathbb{R}$ with $s < t$) will prevent the occurrence of singularity. The precise statement is as follows.

Theorem 6.1 (Lorentzian splitting theorem) *Let (M, g) be a Lorentzian spacetime with $\mathrm{Ric} \geq 0$ in timelike directions, and suppose that it is timelike geodesically complete or globally hyperbolic. If (M, g) contains a timelike straight line, then it is isometric to the product spacetime $(\mathbb{R} \times \Sigma, -\mathrm{d}t^2 + h)$, where (Σ, h) is a complete Riemannian manifold of nonnegative Ricci curvature.*

This splitting theorem was conjectured by Yau [62], and established by Galloway [23] under the global hyperbolicity and by Newman [44] in the timelike geodesically complete case. We refer to [25] for a simplified proof (see also [3, Chapter 14]), and to [19, 24] for surveys including some similarities and differences between Theorem 6.1 and the Cheeger–Gromoll splitting theorem. Some generalizations to weighted spacetimes can be found in [14, 58, 59].

The Cheeger–Gromoll splitting theorem was generalized to measured Finsler manifolds in [47]. Therefore, it is natural to expect a (weighted) Finsler version of Theorem 6.1. The approach to timelike splitting theorem is common to the positive-definite case to some extent; we employ the *Busemann function*

$$\mathbf{b}_\eta(x) := \lim_{t \to \infty} \bigl\{ t - \tau\bigl(x, \eta(t)\bigr) \bigr\}$$

associated with a timelike straight line $\eta\colon \mathbb{R} \longrightarrow M$. There are, however, a number of difficulties due to the Lorentzian signature of the metric L. One of the most notable differences from the positive-definite case is the hyperbolicity of the d'Alembertian $\Box_{\mathfrak{m}}$, which prevents us from applying the maximum principle to the Busemann function. For this reason, in our first attempt [35] following the argument in [25], we needed to put strong assumptions.

Recently, a breakthrough in [4, 9] (see also [8, 36, 43]) provided a way to overcome this difficulty, by introducing the *p-d'Alembertian*

$$\Box_{\mathfrak{m},p}(-f) := \mathrm{div}_{\mathfrak{m}}\big[F^*(-\mathrm{d}f)^{p-2}\cdot\nabla(-f)\big]$$

for $p<0$ or $p\in(0,1)$, where $F^*(\omega) := \sqrt{-2L^*(\omega)}$ for $\omega\in\Omega^* := \bigcup_{x\in M}\Omega^*_x$. Note that this deformation is similar to the p-Laplacian in the positive-definite case, but there we take $p>1$. The p-d'Alembertian is associated with the *q-Lagrangian* $L_q\colon TM\longrightarrow(-\infty,0]\cup\{\infty\}$ defined as

$$L_q(v) := \begin{cases} -\frac{1}{q}\big(-2L(v)\big)^{q/2} & \text{if } v\in\overline{\Omega}, \\ \infty & \text{otherwise,}\end{cases}$$

where $p^{-1}+q^{-1}=1$ (thus $q\in(0,1)$ or $q<0$). Observe that $L_q(v)=-F(v)^q/q$ for $v\in\overline{\Omega}$. Though the original Lagrangian $L=L_2$ is not convex in radial directions, taking the $(q/2)$-th power as above makes L_q convex on T_xM and strictly convex on Ω_x for each $x\in M$. (In the 1-dimensional analysis, this modification turns a concave function $g(t)=-t^2$ for $t>0$ into a strictly convex function $-\frac{1}{q}(-g(t))^{q/2}=-\frac{1}{q}t^q$).

Then, $\Box_{\mathfrak{m},p}$ is the generator of gradient flow for the *p-energy functional*

$$\mathcal{E}_p(-f) := \int_M H_p(-\mathrm{d}f)\,\mathrm{d}\mathfrak{m},$$

where f is a temporal function and $H_p\colon T^*M\longrightarrow[0,\infty]$ is the *p-Hamiltonian* defined as the convex dual to L_q:

$$H_p(\omega) := \begin{cases} -\frac{1}{p}\big(-2L^*(\omega)\big)^{p/2} & \text{if } \omega\in\Omega^*, \\ \infty & \text{otherwise.}\end{cases}$$

Thanks to the convexity of $\mathcal{E}_p$ in the cone consisting of temporal functions, we can apply the maximum principle and prove the following splitting theorem [13, Theorem 1.1] along the lines of [9] in the smooth Lorentzian setting (see also [10] for a generalization to less regular Lorentzian spacetimes).

Theorem 6.2 (Splitting theorem) *Let $(M,L,\mathfrak{m})$ be a weighted Finsler spacetime and suppose the following:*

(1) *(M,L) is either timelike geodesically complete, or Berwald and globally hyperbolic;*

(2) *there is a timelike straight line $\eta\colon\mathbb{R}\longrightarrow M$;*

(3) *$(M,L,\mathfrak{m})$ satisfies $\mathrm{Ric}_N\ge 0$ in timelike directions for some $N\in(-\infty,0)\cup[n,\infty]$;*

(4) *in the case of $N\in(-\infty,0)\cup\{\infty\}$, (M,L) satisfies the timelike ϵ-completeness in the sense of [13] for some ϵ in the range (3.3) associated with N.*

Then, the Lorentzian manifold $(M, g_{\nabla(-\mathbf{b}_\eta)})$ *isometrically splits in the sense that there exists an* $(n-1)$*-dimensional Riemannian manifold* (Σ, h) *such that* $(M, g_{\nabla(-\mathbf{b}_\eta)})$ *is isometric to* $(\mathbb{R} \times \Sigma, -\mathrm{d}t^2 + h)$*. Moreover,* (Σ, h) *is equipped with a measure* $\mathfrak{n}$ *for which* $\mathfrak{m}$ *coincides with the product of the Lebesgue measure* $\mathrm{d}t$ *on* $\mathbb{R}$ *and* $\mathfrak{n}$*.*

We remark that, for technical reasons, the timelike ϵ-completeness given in [13, Definition 2.13] is slightly different from Definition 3.4.

Outline of the proof of Theorem 6.2 goes as follows.

- The d'Alembertian comparison theorem (Theorem 4.6) under $\mathrm{Ric}_N \geq 0$ implies the *p-superharmonicity* $\overleftarrow{\Box}_{\mathfrak{m},p}\mathbf{b}_\eta = -\Box_{\mathfrak{m},p}(-\mathbf{b}_\eta) \leq 0$ in the weak sense near $\eta(\mathbb{R})$. Similarly, the Busemann function

$$\overline{\mathbf{b}}_\eta(x) := \lim_{t\to\infty} \{t - \tau(\eta(-t), x)\}$$

  in the past direction satisfies $\Box_{\mathfrak{m},p}\overline{\mathbf{b}}_\eta \leq 0$.

- By comparing the above p-superharmonicity and $\mathbf{b}_\eta + \overline{\mathbf{b}}_\eta \geq 0$ from the reverse triangle inequality, maximum principle yields $\mathbf{b}_\eta + \overline{\mathbf{b}}_\eta = 0$ and the *p-harmonicity* $\Box_{\mathfrak{m},p}(-\mathbf{b}_\eta) = \Box_{\mathfrak{m},p}\overline{\mathbf{b}}_\eta = 0$ in the weak sense, as well as $\mathbf{b}_\eta \in C^{1,1}_{\mathrm{loc}}$ near $\eta(\mathbb{R})$.

- *Bochner-type identity* shows $\nabla^2(-\mathbf{b}_\eta) = 0$ and $\mathbf{b}_\eta \in C^\infty$ near $\eta(\mathbb{R})$.

- Finally, with $\nabla^2(-\mathbf{b}_\eta) = 0$ in hand, one can build the desired splitting structure in a standard way.

We remark that Σ is given as the level surface $\mathbf{b}_\eta^{-1}(0)$ (in other words, $\mathbf{b}_\eta$ coincides with the projection from $M = \mathbb{R} \times \Sigma$ to $\mathbb{R}$), and $(\Sigma, L|_{T\Sigma})$ is spacelike if L is reversible and $\dim M \geq 3$, though it is unclear whether $L|_{T\Sigma}$ is positive-definite (strongly convex); see [13, Corollary 6.1] for details. The Berwald condition in the globally hyperbolic case is imposed merely for technical reasons and could be redundant.

A distinct feature of the Finsler version is that the splitting is done for the Lorentzian metric $g_{\nabla(-\mathbf{b}_\eta)}$ and the original Lorentz–Finsler structure L itself does not possess a product structure. This is a similar phenomenon to the positive-definite Finsler setting; note that normed spaces do not isometrically split in general. Moreover, in the Lorentz–Finsler setting, we can modify L in spacelike directions without changing it in timelike directions, thereby Theorem 6.2 is actually the most we can expect.

Nonetheless, we can show the following [13, Theorem 1.3] in the same spirit as [47, §5], thanks to the more rigid structure of Berwald spacetimes.

Theorem 6.3 (Isometric translations in Berwald spacetimes) *Let* $(M, L, \mathfrak{m})$ *be a weighted Berwald spacetime satisfying the hypotheses in Theorem 6.2. Then, we have the following.*

(i) *In the product structure* $M = \mathbb{R} \times \Sigma$*, the translations*

$$\Phi_t(s, x) := (s + t, x), \qquad (s, x) \in \mathbb{R} \times \Sigma, \;\; t \in \mathbb{R},$$

are isometric transformations of (M, L) *and preserve the measure* $\mathfrak{m}$*.*

(ii) *The geodesic equation of M splits into those of $\mathbb{R}$ and Σ. Precisely, a curve in M is geodesic if and only if its projections to $\mathbb{R}$ and Σ are geodesic.*

7 Timelike curvature-dimension condition

In this final section, we briefly review a characterization of lower bounds of the weighted Ricci curvature Ric_N in timelike directions in terms of optimal transport theory. This provides a *synthetic geometric* way (i.e., without relying on differentiable structures) of studying spacetimes of Ricci curvature bounded below. Synthetic theory has achieved great success in the positive-definite setting, where one of the main purposes is to analyze singular (metric measure) spaces appearing as limit spaces of Riemannian manifolds. Recently, there has been growing interest in a Lorentzian counterpart, in connection with the appearance of less regular spacetimes in general relativity (e.g., as solutions to the Einstein equation).

Inspired by the theory of curvature-dimension condition in the positive-definite case (see [17, 32, 52, 55, 56] among others), a synthetic condition characterizing lower Ricci curvature bounds will be written as the convexity of a certain entropy functional in the space of probability measures. Denote by $\mathcal{P}(M)$ the set of all Borel probability measures on M, and by $\mathcal{P}_c(M)$ (resp. $\mathcal{P}^{\mathrm{ac}}(M)$) the set of all $\mu \in \mathcal{P}(M)$ with compact support (resp. absolutely continuous with respect to $\mathfrak{m}$, written as $\mu \ll \mathfrak{m}$). We also set $\mathcal{P}_c^{\mathrm{ac}}(M) := \mathcal{P}_c(M) \cap \mathcal{P}^{\mathrm{ac}}(M)$. We will consider the following entropies, similarly to the positive-definite case (see [48, 49] for $N \le 0$).

Definition 7.1 (Entropies) Given $\mu \in \mathcal{P}(M)$, denote by $\mu = \rho\mathfrak{m} + \mu_s$ the Lebesgue decomposition into its absolutely continuous and singular parts with respect to $\mathfrak{m}$.

(1) The *relative entropy* of μ with respect to $\mathfrak{m}$ is defined by

$$\mathrm{Ent}_{\mathfrak{m}}(\mu) := \int_M \rho \log \rho \, d\mathfrak{m}$$

if $\mu \ll \mathfrak{m}$ (i.e., $\mu_s(M) = 0$) and $\int_{\{\rho<1\}} \rho \log \rho \, d\mathfrak{m} > -\infty$, otherwise $\mathrm{Ent}_{\mathfrak{m}}(\mu) := \infty$.

(2) For $N \in [n, \infty)$, define the *N-Rényi–Tsallis entropy* of μ by

$$S_{\mathfrak{m}}^N(\mu) := -\int_M \rho^{(N-1)/N} \, d\mathfrak{m}.$$

(3) For $N < 0$, we define

$$S_{\mathfrak{m}}^N(\mu) := \int_M \rho^{(N-1)/N} \, d\mathfrak{m}$$

if $\mu \ll \mathfrak{m}$, otherwise $S_{\mathfrak{m}}^N(\mu) := \infty$.

(4) Finally, $S_{\mathfrak{m}}^0(\mu) := \|\rho\|_{L^\infty}$ if $\mu \ll \mathfrak{m}$, otherwise $S_{\mathfrak{m}}^0(\mu) := \infty$.

Now, we need to recall some notions in optimal transport theory, taking causality issues into account (we refer to [11, 15, 36] for more details). Let (M, L) be globally hyperbolic in the sequel. Given $\mu, \nu \in \mathcal{P}(M)$, let $\Pi(\mu, \nu) \subset \mathcal{P}(M \times M)$ be the set of *couplings* of (μ, ν), i.e., for $\pi \in \Pi(\mu, \nu)$ we have

$$\pi(A \times M) = \mu(A), \qquad \pi(M \times A) = \nu(A)$$

for all Borel sets $A \subset M$. Then, for each $q \in (0, 1)$, define the *q-Lorentz–Kantorovich–Wasserstein distance* by

$$W_q(\mu, \nu) := \sup_{\pi \in \Pi(\mu,\nu)} \left(\int_{M \times M} \tau(x, y)^q \, \pi(\mathrm{d}x\mathrm{d}y) \right)^{1/q},$$

and $\pi \in \Pi(\mu, \nu)$ attaining the supremum is said to be *q-optimal*. A curve $(\mu_t)_{t\in[0,1]}$ in $\mathcal{P}_c(M)$ is called a *q-geodesic* if $0 < W_q(\mu_0, \mu_1) < \infty$ and $W_q(\mu_s, \mu_t) = (t - s)W_q(\mu_0, \mu_1)$ for all $s, t \in [0, 1]$ with $s < t$.

We say that $(\mu, \nu) \in \mathcal{P}_c(M) \times \mathcal{P}_c(M)$ is *q-separated* if there are $\pi \in \Pi(\mu, \nu)$ and lower semi-continuous functions $\alpha\colon \operatorname{supp}(\mu) \longrightarrow (-\infty, \infty]$, $\beta\colon \operatorname{supp}(\nu) \longrightarrow (-\infty, \infty]$ such that $\tau(x, y)^q/q \le \alpha(x) + \beta(y)$ on $\operatorname{supp}(\mu) \times \operatorname{supp}(\nu)$ and that the set

$$E := \bigl\{(x, y) \in \operatorname{supp}(\mu) \times \operatorname{supp}(\nu) \mid \alpha(x) + \beta(y) = \tau(x, y)^q/q\bigr\}$$

satisfies $\operatorname{supp}(\pi) \subset E \subset \{(x, y) \in M \times M \mid x \ll y\}$. An archetypal case is the following: If $\mu, \nu \in \mathcal{P}_c(M)$ satisfy $x \ll y$ for all $(x, y) \in \operatorname{supp}(\mu)\times\operatorname{supp}(\nu)$, then (μ, ν) is q-separated ([11, Lemma 4.10]).

Using $\mathbf{s}_\kappa$ in (4.8), we define the function

$$\boldsymbol{\tau}^{(t)}_{K,N}(r) := t^{1/N} \left(\frac{\mathbf{s}_{K/(N-1)}(tr)}{\mathbf{s}_{K/(N-1)}(t)} \right)^{(N-1)/N}$$

for $K \in \mathbb{R}$, $N \in (-\infty, 0) \cup [n, \infty)$ and $t \in (0, 1)$, where $r > 0$ when $K/(N - 1) \le 0$ and $r \in (0, \pi\sqrt{(N - 1)/K})$ when $K/(N - 1) > 0$. We also set $\boldsymbol{\tau}^{(t)}_{K,N}(0) := t$. Then, we generalize the curvature-dimension condition to the Lorentzian setting as follows (see [15, 36, 43] and [11, Definition 5.7]).

Definition 7.2 (Timelike curvature-dimension condition) Let $q \in (0, 1)$, $K \in \mathbb{R}$, and $N \in (-\infty, 0] \cup [n, \infty]$. We say that a weighted Finsler spacetime $(M, L, \mathfrak{m})$ satisfies the *timelike curvature-dimension condition* $\mathrm{TCD}_q(K, N)$ if, for every q-separated pair $(\mu_0, \mu_1) \in \mathcal{P}^{\mathrm{ac}}_c(M) \times \mathcal{P}^{\mathrm{ac}}_c(M)$, there exist a q-geodesic $(\mu_t)_{t\in[0,1]}$ from μ_0 to μ_1 and a q-optimal coupling $\pi \in \Pi(\mu_0, \mu_1)$ such that the following hold.

(1) When $N = \infty$, for every $t \in (0, 1)$, we have

$$\mathrm{Ent}_{\mathfrak{m}}(\mu_t) \le (1-t)\,\mathrm{Ent}_{\mathfrak{m}}(\mu_0) + t\,\mathrm{Ent}_{\mathfrak{m}}(\mu_1) - \frac{K}{2}t(1-t)\int_{M\times M} \tau(x, y)^2 \, \pi(\mathrm{d}x\mathrm{d}y). \quad (7.1)$$

(2) When $N \in [n, \infty)$, for all $N' \in [N, \infty)$ and $t \in (0,1)$, we have

$$\begin{aligned} S_{\mathfrak{m}}^{N'}(\mu_t) \le &- \int_{M\times M} \boldsymbol{\tau}_{K,N'}^{(1-t)}\big(\tau(x,y)\big)\rho_0(x)^{-1/N'}\,\pi(\mathrm{d}x\mathrm{d}y) \\ &- \int_{M\times M} \boldsymbol{\tau}_{K,N'}^{(t)}\big(\tau(x,y)\big)\rho_1(y)^{-1/N'}\,\pi(\mathrm{d}x\mathrm{d}y), \end{aligned} \tag{7.2}$$

where $\mu_0 = \rho_0\mathfrak{m}$ and $\mu_1 = \rho_1\mathfrak{m}$.

(3) When $N < 0$, for all $N' \in [N, 0)$ and $t \in (0,1)$, we have

$$\begin{aligned} S_{\mathfrak{m}}^{N'}(\mu_t) \le &\int_{M\times M} \boldsymbol{\tau}_{K,N'}^{(1-t)}\big(\tau(x,y)\big)\rho_0(x)^{-1/N'}\,\pi(\mathrm{d}x\mathrm{d}y) \\ &+ \int_{M\times M} \boldsymbol{\tau}_{K,N'}^{(t)}\big(\tau(x,y)\big)\rho_1(y)^{-1/N'}\,\pi(\mathrm{d}x\mathrm{d}y), \end{aligned} \tag{7.3}$$

where we set $\boldsymbol{\tau}_{K,N'}^{(t)}(r) := \infty$ if $K < 0$ and $r \ge \pi\sqrt{(N'-1)/K}$.

(4) When $N = 0$, for every $t \in (0,1)$, we have

$$S_{\mathfrak{m}}^{0}(\mu_t) \le \pi\text{-}\operatorname*{ess\,sup}_{(x,y)} \max\left\{ \frac{\mathbf{s}_{-K}((1-t)\tau(x,y))}{(1-t)\mathbf{s}_{-K}(\tau(x,y))}\rho_0(x), \frac{\mathbf{s}_{-K}(t\tau(x,y))}{t\mathbf{s}_{-K}(\tau(x,y))}\rho_1(y) \right\}, \tag{7.4}$$

where we set $\mathbf{s}_{-K}(t\tau(x,y))/(t\mathbf{s}_{-K}(\tau(x,y))) := 1$ if $\tau(x,y) = 0$.

When $K = 0$, we have $\boldsymbol{\tau}_{0,N}^{(t)}(r) = t$ and (7.2) and (7.3) are simply the convexity:

$$S_{\mathfrak{m}}^{N'}(\mu_t) \le (1-t)S_{\mathfrak{m}}^{N'}(\mu_0) + tS_{\mathfrak{m}}^{N'}(\mu_1),$$

while (7.4) implies the quasi-convexity:

$$S_{\mathfrak{m}}^{0}(\mu_t) \le \max\{S_{\mathfrak{m}}^{0}(\mu_0), S_{\mathfrak{m}}^{0}(\mu_1)\}.$$

We remark that, since π is a q-optimal coupling, $\int_{M\times M} \tau(x,y)^2\,\pi(\mathrm{d}x\mathrm{d}y)$ in the last term of (7.1) is not directly a transport cost.

We are ready to state a characterization of $\mathrm{Ric}_N \ge K$ in timelike directions established in [11, Theorems 5.9, 6.1] along the lines of [36].

Theorem 7.3 ($\mathrm{Ric}_N \ge K \Longleftrightarrow \mathrm{TCD}_q(K,N)$) *Let $(M, L, \mathfrak{m})$ be a globally hyperbolic, weighted Finsler spacetime, $K \in \mathbb{R}$ and $N \in (-\infty, 0] \cup [n, \infty]$. Then, the following are equivalent.*

(A) $\mathrm{Ric}_N \ge K$ *in timelike directions.*

(B) $\mathrm{TCD}_q(K,N)$ *holds for some $q \in (0,1)$.*

(C) $\mathrm{TCD}_q(K,N)$ *holds for all $q \in (0,1)$.*

McCann [36] first established the equivalence between lower Ricci curvature bounds and the timelike curvature-dimension condition, followed by an independent work by Mondino–Suhr [43] which includes a characterization of the Einstein equation as well. Precisely, they considered a variant $\mathrm{TCD}^e_q(K,N)$, called the *entropic timelike curvature-dimension condition*, which is (by definition) slightly weaker than $\mathrm{TCD}_q(K,N)$ when $K \neq 0$. The above $\mathrm{TCD}_q(K,N)$ was first studied in [7]. Thus, the equivalence in Theorem 7.3 with $K \neq 0$ was new in [11] even for Lorentzian spacetimes (and required more involved calculations than the positive-definite case). In addition, the case of $N \leq 0$ in the Lorentzian setting was first investigated in [11].

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References

[1] D. Bao, S.-S. Chern and Z. Shen, An introduction to Riemann–Finsler geometry. Springer-Verlag, New York, 2000.

[2] J. K. Beem, Indefinite Finsler spaces and timelike spaces. Can. J. Math. **22** (1970), 1035–1039.

[3] J. K. Beem, P. E. Ehrlich and K. L. Easley, Global Lorentzian Geometry. Marcel Dekker Inc., New York, 1996.

[4] T. Beran, M. Braun, M. Calisti, N. Gigli, R. J. McCann, A. Ohanyan, F. Rott and C. Sämann, A nonlinear d'Alembert comparison theorem and causal differential calculus on metric measure spacetimes. Preprint (2024). Available at arXiv:2408.15968

[5] T. Beran, D. Erös, S. Ohta and F. Rott, Concavity of spacetimes. Preprint (2025). Available at arXiv:2509.26196

[6] T. Beran, M. Kunzinger and F. Rott, On curvature bounds in Lorentzian length spaces. J. Lond. Math. Soc. (2) **110** (2024), Paper No. e12971.

[7] M. Braun, Rényi's entropy on Lorentzian spaces. Timelike curvature-dimension conditions. J. Math. Pures Appl. (9) **177** (2023), 46–128.

[8] M. Braun, Exact d'Alembertian for Lorentz distance functions. Preprint (2024). Available at arXiv:2408.16525

[9] M. Braun, N. Gigli, R. J. McCann, A. Ohanyan and C. Sämann, An elliptic proof of the splitting theorems from Lorentzian geometry. Preprint (2024). Available at arXiv:2410.12632

[10] M. Braun, N. Gigli, R. J. McCann, A. Ohanyan and C. Sämann, A Lorentzian splitting theorem for continuously differentiable metrics and weights. Preprint (2025). Available at arXiv:2507.06836

[11] M. Braun and S. Ohta, Optimal transport and timelike lower Ricci curvature bounds on Finsler spacetimes. Trans. Amer. Math. Soc. **377** (2024), 3529–3576.

[12] H. Busemann, Spaces with non-positive curvature. Acta Math. **80** (1948), 259–310.

[13] E. Caponio, A. Ohanyan and S. Ohta, Splitting theorems for weighted Finsler spacetimes via the p-d'Alembertian: beyond the Berwald case. Preprint (2024). Available at arXiv:2412.20783

[14] J. S. Case, Singularity theorems and the Lorentzian splitting theorem for the Bakry–Emery–Ricci tensor. J. Geom. Phys. **60** (2010), 477–490.

[15] F. Cavalletti and A. Mondino, Optimal transport in Lorentzian synthetic spaces, synthetic timelike Ricci curvature lower bounds and applications. Camb. J. Math. **12** (2024), 417–534.

[16] J. Cheeger and D. Gromoll, The splitting theorem for manifolds of nonnegative Ricci curvature. J. Differential Geometry **6** (1971/72), 119–128.

[17] D. Cordero-Erausquin, R. J. McCann and M. Schmuckenschläger, A Riemannian interpolation inequality à la Borell, Brascamp and Lieb. Invent. Math. **146** (2001), 219–257.

[18] D. Erös and S. Gieger, A synthetic Lorentzian Cartan–Hadamard theorem. Preprint (2025). Available at `arXiv:2506.22197`

[19] J. L. Flores, The Riemannian and Lorentzian splitting theorems. Topics in modern differential geometry, 1–20, Atlantis Trans. Geom., **1**, Atlantis Press, [Paris], 2017.

[20] A. Fuster, S. Heefer, C. Pfeifer and N. Voicu, On the non metrizability of Berwald Finsler spacetimes. Universe **6** (2020), no. 5: 64. https://doi.org/10.3390/universe6050064

[21] A. Fuster and C. Pabst, Finsler *pp*-waves. Phys. Rev. D **94** (2016), no. 10, 104072, 5 pp.

[22] A. Fuster, C. Pabst and C. Pfeifer, Berwald spacetimes and very special relativity. Phys. Rev. D **98** (2018), no. 8, 084062, 14 pp.

[23] G. J. Galloway, The Lorentzian splitting theorem without the completeness assumption. J. Differential Geometry **29** (1989), 373–387.

[24] G. J. Galloway, The Lorentzian version of the Cheeger–Gromoll splitting theorem and its application to general relativity. Differential geometry: geometry in mathematical physics and related topics, 249–257, Proc. Sympos. Pure Math., **54**, Part 2, Amer. Math. Soc., Providence, RI, 1993.

[25] G. J. Galloway and A. Horta, Regularity of Lorentzian Busemann functions. Trans. Amer. Math. Soc. **348** (1996), 2063–2084.

[26] S. W. Hawking and G. F. R. Ellis, The large scale structure of space-time. Cambridge Monographs on Mathematical Physics, No. 1. Cambridge University Press, London-New York, 1973.

[27] M. Kunzinger and C. Sämann, Lorentzian length spaces. Ann. Global Anal. Geom. **54** (2018), 399–447.

[28] K. Kuwae and X.-D. Li, New Laplacian comparison theorem and its applications to diffusion processes on Riemannian manifolds. Bull. Lond. Math. Soc. **54** (2022), 404–427.

[29] K. Kuwae and Y. Sakurai, Rigidity phenomena on lower N-weighted Ricci curvature bounds with ε-range for nonsymmetric Laplacian. Illinois J. Math. **65** (2021), 847–868.

[30] K. Kuwae and Y. Sakurai, Comparison geometry of manifolds with boundary under lower N-weighted Ricci curvature bounds with ε-range. J. Math. Soc. Japan **75** (2023), 151–172.

[31] K. Kuwae and Y. Sakurai, Lower N-weighted Ricci curvature bound with ε-range and displacement convexity of entropies. J. Topol. Anal. **17** (2025), 105–130.

[32] J. Lott and C. Villani, Ricci curvature for metric-measure spaces via optimal transport. Ann. of Math. (2) **169** (2009), 903–991.

[33] Y. Lu, E. Minguzzi and S. Ohta, Geometry of weighted Lorentz–Finsler manifolds I: Singularity theorems. J. Lond. Math. Soc. **104** (2021), 362–393.

[34] Y. Lu, E. Minguzzi and S. Ohta, Comparison theorems on weighted Finsler manifolds and spacetimes with ϵ-range. Anal. Geom. Metr. Spaces **10** (2022), 1–30.

[35] Y. Lu, E. Minguzzi and S. Ohta, Geometry of weighted Lorentz–Finsler manifolds II: A splitting theorem. Internat. J. Math. **34** (2023), no. 1, Paper No. 2350002.

[36] R. J. McCann, Displacement convexity of Boltzmann's entropy characterizes the strong energy condition from general relativity. Camb. J. Math. **8** (2020), 609–681.

[37] E. Minguzzi, Convex neighborhoods for Lipschitz connections and sprays. Monatsh. Math. **177** (2015), 569–625.

[38] E. Minguzzi, Light cones in Finsler spacetimes. Comm. Math. Phys. **334** (2015), 1529–1551.

[39] E. Minguzzi, Raychaudhuri equation and singularity theorems in Finsler spacetimes. Class. Quantum Grav. **32** (2015), no. 18, 185008, 26pp.

[40] E. Minguzzi, Special coordinate systems in pseudo-Finsler geometry and the equivalence principle. J. Geom. Phys. **114** (2017), 336–347.

[41] E. Minguzzi, Causality theory for closed cone structures with applications. Rev. Math. Phys. **31** (2019), no. 5, 1930001, 139pp.

[42] E. Minguzzi, Lorentzian causality theory. Living Reviews in Relativity **22**, 3 (2019). https://doi.org/10.1007/s41114-019-0019-x.

[43] A, Mondino and S. Suhr, An optimal transport formulation of the Einstein equations of general relativity. J. Eur. Math. Soc. **25** (2023), 933–994.

[44] R. P. A. C. Newman, A proof of the splitting conjecture of S.-T. Yau. J. Differential Geometry **31** (1990), 163–184.

[45] S. Ohta, Finsler interpolation inequalities. Calc. Var. Partial Differential Equations **36** (2009), 211–249.

[46] S. Ohta, Vanishing S-curvature of Randers spaces. Differential Geom. Appl. **29** (2011), 174–178.

[47] S. Ohta, Splitting theorems for Finsler manifolds of nonnegative Ricci curvature. J. Reine Angew. Math. **700** (2015), 155–174.

[48] S. Ohta, (K, N)-convexity and the curvature-dimension condition for negative N. J. Geom. Anal. **26** (2016), 2067–2096.

[49] S. Ohta, Needle decompositions and isoperimetric inequalities in Finsler geometry. J. Math. Soc. Japan **70** (2018), 651–693.

[50] S. Ohta, Comparison Finsler geometry. Springer Monographs in Mathematics, Springer, Cham, 2021.

[51] B. O'Neill, Semi-Riemannian geometry: With applications to relativity. Academic Press, Inc., New York, 1983.

[52] M.-K. von Renesse and K.-T. Sturm, Transport inequalities, gradient estimates, entropy and Ricci curvature. Comm. Pure Appl. Math. **58** (2005), 923–940.

[53] M. Sánchez, On the foundations and applications of Lorentz–Finsler geometry, Preprint (2025). Available at arXiv:2511.04645

[54] Z. Shen, Lectures on Finsler geometry. World Scientific Publishing Co., Singapore, 2001.

[55] K.-T. Sturm, On the geometry of metric measure spaces. I. Acta Math. **196** (2006), 65–131.

[56] K.-T. Sturm, On the geometry of metric measure spaces. II. Acta Math. **196** (2006), 133–177.

[57] Z. I. Szabó, Positive definite Berwald spaces. Structure theorems on Berwald spaces. Tensor (N.S.) **35** (1981), 25–39.

[58] E. Woolgar and W. Wylie, Cosmological singularity theorems and splitting theorems for N-Bakry–Émery spacetimes. J. Math. Phys. **57** (2016), 022504, 1–12.

[59] E. Woolgar and W. Wylie, Curvature-dimension bounds for Lorentzian splitting theorems. J. Geom. Phys. **132** (2018), 131–145.

[60] W. Wylie, A warped product version of the Cheeger–Gromoll splitting theorem. Trans. Amer. Math. Soc. **369** (2017), 6661–6681.

[61] W. Wylie and D. Yeroshkin, On the geometry of Riemannian manifolds with density. Preprint (2016). Available at arXiv:1602.08000

[62] S. T. Yau, Problem section. Seminar on Differential Geometry, pp. 669–706, Ann. of Math. Stud., **102**, Princeton Univ. Press, Princeton, N.J., 1982.