A direction energy approach to geometric variational problems

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Abstract

This short review paper summarizes our recent works on a direction energy approach for treating infinite-length curves in geometric variational problems involving the length functional, both in stationary and dynamical contexts. Applications include elastica, curve shortening flow, and elastic flow. We also present some open problems.

1 Introduction

A central difficulty in geometric variational problems on non-compact complete curves lies in the fact that the length functional

$$L[\gamma] = \int_\gamma ds,$$

is typically infinite and therefore, in contrast to the compact case, cannot be used as a quantitative control in either stationary or dynamical settings.

The purpose of this note is to review our recent works [11–13] based on the simple idea of *replacing length by a direction energy*, and to explain how this idea enables variational methods for infinite-length curves in both Euler's elastica problem and (possibly higher-order) geometric flows such as the curve shortening flow and the elastic flow. Some open problems, mainly concerning gradient flows, are also discussed.

For the sake of brevity, we do not attempt to provide a comprehensive list of related references in this note. Further background and additional literature can be found in the references cited herein and in the works cited therein.

2 Direction energy

Fix a unit vector $e_1 \in \mathbf{R}^n$. For an immersed curve γ with arclength parameter s and unit tangent $\partial_s\gamma$, we define the *direction energy*

$$D[\gamma] := \frac{1}{2} \int_\gamma |\partial_s\gamma - e_1|^2 ds.$$

Expanding the squared norm gives

$$D[\gamma] = \int_\gamma (1 - \langle \partial_s \gamma, e_1 \rangle) ds,$$

and hence, for finite-length curves,

$$D[\gamma] = L[\gamma] - \langle \gamma, e_1 \rangle \big|_{\text{endpoints}}.$$

Thus $D = L$ for closed curves, while for complete non-closed curves the direction energy D may remain finite even when $L = +\infty$. Moreover, since the term $\partial_s \langle \gamma, e_1 \rangle$ is a null Lagrangian, the functionals D and L formally have the same first variation; in particular, they have the same critical points, and their gradient flows formally coincide.

This observation allows one to reinterpret "curve shortening" as "tangent aligning," providing a suitable finite quantity for non-compact curves. It is also conceptually interesting that the direction energy provides an *anisotropic* substitute for the *isotropic* length functional.

Notice that the direction energy takes a form similar to the *tilt-excess*, a local quantity in geometric measure theory that measures the deviation of tangent planes from a fixed reference plane; see, e.g., [1, 15]. A dynamical analogue of the tilt-excess appears in the theory of weak solutions to geometric flows and sharp interface models in the form of the *relative entropy* (or *relative energy*); see, e.g., [3–5]. The tilt-excess component of the relative entropy measures the deviation between two appropriate, evolving (non-constant) normal fields, and is mainly used to control *local* geometric errors. By contrast, the direction energy is used as a quantity that *globally* coincides with the length functional up to a null Lagrangian.

We also remark that the idea of subtracting a null Lagrangian in order to improve the analytic treatment of a variational problem on a non-compact domain already appears in a different context, namely in Melcher's study of two-dimensional chiral magnetic skyrmions [6, Section 2, (b)]. We stress that Melcher's modification enlarges the admissible class in a *functional-analytic* sense, by allowing slower decay of the tails. In contrast, the direction energy enlarges the admissible class in a truly *geometric* sense, extending admissible configurations from finite to infinite total length.

3 Stationary applications: Euler's elastica

The direction energy is first applied in the author's work [7] to the study of Euler's elastica when analyzing global minimizers of the elastic energy

$$\hat{E}_\varepsilon[\gamma] := \varepsilon^2 B[\gamma] + L[\gamma],$$

among planar curves subject to the clamped boundary condition. Here $B[\gamma] = \frac{1}{2} \int_\gamma |\kappa|^2 ds$ is the bending energy (note that [7] adopts the normalization without

the factor $\frac{1}{2}$), and $\kappa = \partial_s^2 \gamma$ denotes the curvature vector. For this classical problem, the existence and regularity of minimizers are standard (see, e.g., [9]), whereas uniqueness and finer properties are considerably more delicate. In [7], a detailed asymptotic analysis as $\varepsilon \to 0$ is developed by revealing the fact that the elastic energy $\hat{E}_\varepsilon$ agrees, up to an additive constant, with

$$\varepsilon^2 B[\gamma] + D[\gamma] = \frac{\varepsilon^2}{2} \int |\partial_s \theta|^2 ds + \int (1 - \cos \theta) ds,$$

where θ is the tangential angle. This representation allows for an argument analogous to Modica–Mortola's celebrated analysis [14], which was later applied to the van der Waals–Cahn–Hilliard theory of phase transitions. Already (implicitly) used in this approach is the important fact that L and D differ only by a null Lagrangian.

The direction energy method is extended in the author and Wheeler's work [13] in order to variationally treat infinite-length elasticae. More precisely, in [13] we discovered a simple geometric criterion for unique minimality: every planar elastica with non-constant monotone curvature is a unique minimizer of the bending energy under the fixed-length constraint and clamped boundary condition. The proof introduces a new idea of constructing a family of free-boundary problems in which each monotone-curvature elastica arises as a unique minimizer (or its subarc). Each free-boundary condition is prescribed by right-angle conditions on two parallel lines, and this covers all cases except the so-called *borderline elastica*. To characterize the minimality of the borderline elastica, an appropriate variational framework that admits infinite-length curves is required, and this is achieved through the use of the direction energy. Indeed, in [7] the direction energy already played a crucial role in providing a suitable variational treatment of the borderline elastica.

4 Dynamical applications: Gradient flows

The direction energy method also extends to the study of geometric flows that decrease energy functionals involving length, even for infinite-length curves. Applications include the curve shortening flow, surface diffusion flow, Chen's flow, and elastic flow for curves in $\mathbf{R}^n$, where $n \geq 2$.

The author and Rupp's recent work [11] is the first to apply the direction energy method to flows of infinite-length curves. Roughly speaking, we developed the following framework:

- (Energy identity) A global identity that propagates the finite energy structure is established: If $\gamma : [0, T) \times \mathbf{R} \to \mathbf{R}^n$ is obtained as a gradient flow of an energy functional $\hat{\mathcal{E}}$ involving the length L, then for the (finite) energy $\mathcal{E}$ with L replaced by D,

$$\mathcal{E}[\gamma(\tau, \cdot)] + \int_0^\tau \int_{\gamma(t, \cdot)} |\partial_t \gamma|^2 \, ds \, dt = \mathcal{E}[\gamma(0, \cdot)] \qquad (0 \leq \tau < T).$$

- (Curvature interpolation) A localized, weighted version of the Dziuk–Kuwert–Schätzle interpolation estimate [2] is also established: If $P_b^{a,c}$ denotes a curvature polynomial consisting of b factors $\partial_s^{i_1}\kappa * \cdots * \partial_s^{i_b}\kappa$ with $i_1 + \cdots + i_b = a$ and $i_j \leq c$, and $\zeta \geq 0$ is a compactly supported cutoff, then we obtain a general interpolation estimate of the form

$$\int |P_b^{a,c}|\zeta^{a+\frac{b}{2}-1}ds$$

$$\leq \varepsilon \int |\nabla_s^k \kappa|^2 \zeta^{2k} ds + C_\varepsilon \left(\int_{[\zeta>0]} |\kappa|^2 ds \right)^{\frac{b-\delta}{2-\delta}} + C_\varepsilon \left(\int_{[\zeta>0]} |\kappa|^2 ds \right)^{\frac{b}{2}}$$

for k such that $c \leq k$ and $\delta := \frac{1}{k}(a + \frac{b}{2} - 1) < 2$. The key advantage is that we can precisely keep track of the powers of the cutoff, to proceed with interpolation estimates under spatial localization. A time cutoff then globalizes the estimate along the flow, giving uniform control of all curvature derivatives in terms of the bending energy. (This part is independent of the direction energy.)

Here we highlight two representative applications within the above framework. The first concerns the curve shortening flow, which is the second-order $L^2(ds)$-gradient flow of the length functional L. The second concerns the elastic flow, which is the fourth-order $L^2(ds)$-gradient flow of the elastic energy $\hat{E} := B + L$.

4.1 Curve shortening flow

We begin with the curve shortening flow [11, Theorem 2.2]. Let $\gamma_0 : \mathbf{R} \to \mathbf{R}^n$ be a smooth initial curve satisfying $\sup |\partial_x^m \gamma_0| < \infty$ for all $m \geq 1$ and $\inf |\partial_x \gamma_0| > 0$. Then there exists a unique maximal solution $\gamma : [0, T) \times \mathbf{R} \to \mathbf{R}^n$ to the curve shortening flow starting from γ_0. The flow satisfies the energy identity

$$D[\gamma(\tau, \cdot)] + \int_0^\tau \int_{\gamma(t,\cdot)} |\partial_t \gamma|^2 ds dt = D[\gamma_0] \qquad (0 \leq \tau < T).$$

Moreover, the following dichotomy holds:

- If $T < \infty$, then the bending energy $B[\gamma(t, \cdot)]$ blows up as $t \to T$.

- If $T = \infty$, then the solution converges locally smoothly to a horizontal line (after translation and arclength reparametrization). In particular, no singularities occur in infinite time.

In the case $T = \infty$, we also have

$$B[\gamma(t, \cdot)] \to 0, \quad \|\partial_s \gamma(t, \cdot) - e_1\|_{L^\infty} \to 0, \quad \|\partial_s^m \kappa(t, \cdot)\|_{L^\infty} \to 0 \text{ for all } m \geq 0,$$

as $t \to \infty$. It remains open whether the same is true for the direction energy.

Problem 4.1 (Direction energy convergence). In [11, Theorem 2.2 (ii)], does the direction energy $D[\gamma(t, \cdot)]$ also converge to zero as $t \to \infty$?

4.2 Elastic flow I: Global existence and asymptotic analysis

We now turn to the elastic flow [11, Theorem 2.8]. In this case, the direction energy method yields nontrivial uniform control of the bending energy along the flow, allowing us to obtain a global existence theory. Let $\gamma_0 : \mathbf{R} \to \mathbf{R}^n$ be a smooth initial curve as above. Then there exists a unique global solution $\gamma : [0, \infty) \times \mathbf{R} \to \mathbf{R}^n$ to the elastic flow starting from γ_0. The flow satisfies the energy identity for the adapted elastic energy

$$E := B + D,$$

namely

$$E[\gamma(\tau, \cdot)] + \int_0^\tau \int_{\gamma(t, \cdot)} |\partial_t \gamma|^2 ds dt = E[\gamma_0] \qquad (0 \le \tau < \infty).$$

In addition, the solution sub-converges to either a horizontal line or a borderline elastica with horizontal ends (after translation and arclength reparametrization). Here "sub-convergence" means convergence along a time subsequence $t_j \to \infty$ in the locally smooth sense. It is unknown whether convergence holds without passing to a subsequence.

Problem 4.2 (Full convergence). Under the assumptions of [11, Theorem 2.8], does $\gamma(t, \cdot)$ converge as $t \to \infty$ without passing to a subsequence? More precisely, letting $\Phi_t : \mathbf{R} \to \mathbf{R}$ denote the inverse map of $x \mapsto \int_0^x |\partial_x \gamma(t, x')| dx'$, does the arclength-reparametrized and translated curve $\gamma(t, \Phi_t(\cdot)) - \gamma(t, 0)$ converge locally smoothly as $t \to \infty$?

Furthermore, we expect that the solution decomposes, in the limit, into a finite number of borderline elasticae. Since each borderline elastica has energy $E = 8$, we expect the following energy quantization phenomenon.

Problem 4.3 (Energy quantization conjecture, cf. [11]). Under the assumptions of [11, Theorem 2.8], does the solution satisfy

$$\lim_{t \to \infty} E[\gamma(t, \cdot)] \in 8\mathbf{Z} ?$$

Remark 4.4. The above conjecture was already stated in the planar case $n = 2$ in [11, Conjecture 7.18]. In that setting, we further expect that the limiting energy is determined by the rotation number $N[\gamma_0]$ of the initial curve, namely

$$\lim_{t \to \infty} E[\gamma(t, \cdot)] = 8|N[\gamma_0]|.$$

4.3 Elastic flow II: Positivity-preserving properties

Finally, we briefly mention some further results on positivity preservation for complete elastic flows obtained in [12], building on the previous works [11, 13].

Since higher-order flows do not possess suitable maximum principles, they may lose various positivity properties during the evolution. Nevertheless, certain positivity-preserving phenomena can still be recovered under suitable smallness assumptions on the energy. For instance, in the closed-curve case we derived optimal energy thresholds for the preservation of embeddedness along the elastic flow [10]. In contrast, for complete curves there is no canonical decreasing finite energy, and therefore no direct analogue of such a result is available. This difficulty, however, can be overcome by exploiting the direction energy method developed above.

A first application concerns the preservation of graphicality. In [12, Theorem 1.4] the authors identify an explicit planar curve γ_S, the *elastic serpent*, obtained by gluing two pieces of a borderline elastica, and show that γ_S is the unique (up to the natural invariances) minimizer of E among non-graphical complete curves. In particular, its energy is given by

$$E[\gamma_S] = 8 - 4\sqrt{2},$$

and this value provides an optimal graphicality threshold: if $E[\gamma_0] \leq 8 - 4\sqrt{2}$ and γ_0 is graphical, then the elastic flow starting from γ_0 remains graphical for all time. This threshold is optimal.

A second application concerns planar embeddedness. In [12, Theorem 1.1] we determined another explicit shape γ_P, the *elastic pendant*, obtained by gluing an elastic teardrop to two pieces of a borderline elastica. This curve is the unique minimizer of E among complete planar curves with rotation number zero that have a tangential self-intersection. Its energy $E[\gamma_P]$, numerically

$$E[\gamma_P] \approx 10.90658,$$

yields the optimal embeddedness threshold: if $E[\gamma_0] \leq E[\gamma_P]$ and γ_0 is embedded in $\mathbf{R}^2$, then the elastic flow keeps $\gamma(t, \cdot)$ embedded for all $t \geq 0$. The threshold is again optimal.

The embeddedness preservation threshold in higher codimension remains open, in contrast to the closed-curve case [10]. One reason is that the following Li–Yau type inequality is available only in the planar case. More precisely, if $\gamma : \mathbf{R} \to \mathbf{R}^2$ is a complete planar curve with a self-intersection, then [12, Theorem 5.6] shows that

$$E[\gamma] \geq 8,$$

with equality if and only if γ is a suitable borderline elastica (with horizontal ends). In particular, the condition $E < 8$ forces global embeddedness. It is natural to expect that such a Li–Yau type inequality persists in all codimensions, as already stated in [12, Conjecture 5.9].

Problem 4.5 (Li–Yau type inequality in higher codimension [12])**.** Let $n \geq 3$ and let $\gamma : \mathbf{R} \to \mathbf{R}^n$ be a complete immersed curve with a self-intersection. Does the adapted elastic energy satisfy

$$E[\gamma] \geq 8,$$

with equality if and only if γ is a (suitable) borderline elastica?

In the closed-curve case, an analogous Li–Yau type inequality is established in arbitrary codimension [8]. The same method, however, does not extend to the complete-curve setting. If the above problem is affirmatively resolved, the borderline elastica would then provide the optimal energy threshold for the preservation of embeddedness along complete elastic flows in $\mathbf{R}^n$ for every $n \geq 3$.

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