RECENT PROGRESS ON LYZ EQUATION

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Abstract. We survey some recent progress on the LYZ equation and the heat flows associated to the LYZ equation.

1. Introduction

Let (X, ω) be a compact Kähler manifold of complex dimension n and L a holomorphic line bundle over X. Given a metric h on L, we define a complex function ζ on X by

$$\zeta := \frac{(\omega - F)^n}{\omega^n}$$

where F is the curvature of the Chern connection with respect to the metric h. It is easy to see that the average of this function is a fixed complex number

$$Z_{L,[\omega]} := \int_X \zeta \frac{\omega^n}{n!}$$

depending only on the cohomology classes $c_1(L)$ and $[\omega] \in H^{1,1}(X, \mathbb{R})$. Let $\hat{\theta}$ be the argument of $Z_{L,[\omega]}$.

Jacob and Yau [20] introduced the deformed Hermitian Yang-Mills metric h on the holomorphic line bundle L over X.

Definition 1.1. *A Hermitian metric h on L is said to be deformed Hermitian-Yang-Mills (dHYM) metric if it satisfies*

$$\mathrm{Im}(\omega - F)^n = \tan(\hat{\theta}) \,\mathrm{Re}(\omega - F)^n. \tag{1.1}$$

Let λ_i be the eigenvalues of $\sqrt{-1}\omega^{-1}F \in \mathrm{End}(T^{1,0}X)$. Then

$$\zeta = \prod_{i=1}^{n} (1 + \sqrt{-1}\lambda_i).$$

It is easy to see that the argument of ζ is

$$\theta = \sum_i \arctan(\lambda_i)$$

We define the Lagrangian phase operator $\theta : \wedge^{1,1}X \to \mathbb{R}$ by

$$\theta(F) = \sum_i \arctan(\lambda_i) \tag{1.2}$$

Then the equation (1.1) is equivalent to

$$\theta(F) = \hat{\theta} \ (\mathrm{mod}.2\pi). \tag{1.3}$$

AMS Mathematics Subject Classification. 53C55, 32W20.

This equation is the complex version of the special Lagrangian equation for graphs, with θ the analog of the Lagrangian angle and λ_j analogs of the eigenvalues of the Hessian of the generating function of the graph (see for instance [6], [32], [33]).

As discussed in [5], looking for the dHYM metric is equivalent to looking for one real, closed $(1,1)$ form $\chi \in 2\pi c_1(L)$ such that

$$\operatorname{Im}(\omega + \sqrt{-1}\chi)^n = \tan(\hat{\theta}) \operatorname{Re}(\omega + \sqrt{-1}\chi)^n. \tag{1.4}$$

In this case, Lagrangian phase operator also can be written as $\theta(\chi)$.

In the special Lagrangian case certain bounds on the phase affect the behavior of the equation. In this setting the bounds of the phase also play an important role.

Definition 1.2. *We say θ satisfies the supercritical phase condition if*

$$|\theta| > \frac{(n-2)\pi}{2}.$$

θ satisfies the hypercritical phase condition if

$$|\theta| > \frac{(n-1)\pi}{2}.$$

The Lagrangian phase operator is always elliptic and is concave in the hypercritical case. The dHYM equation was first discovered by Marino et all [26] as the requirement for a D-brane on the B-model of mirror symmetry to be supersymmetric. This is explained by Leung-Yau-Zaslow [24] from mathematical side. Thus the equation (1.3) is also called **LYZ equation**. Recently, it has been studied actively(e.g. [5], [7], [8] [9], [11] [12], [13], [19],[20], [27], [29], [30] etc). The purpose of this article is to survey recent progress about the existence of LYZ equation and the associated heat flow.

2. Existence of LYZ euqation

In this section, we are towards to understand the existence of the solution of LYZ equation. First Jacob-Yau proved the following uniqueness result:

Theorem 2.1. *(Jacob-Yau [20]) Let L be a holomorphic line bundle over a compact Kähler manifold X. Suppose there exists a metric h on L that solves (1.3). Then any other solution is a real constant multiple of h.*

In [20] Jacob-Yau gave a necessary and sufficient condition for the existence of LYZ equation on the Kähler surface by observing that in this case, the LYZ equation is equivalent to the complex Monge-Ampère equation. They proved that :

Theorem 2.2. *(Jacob-Yau [20]) Let L be a holomorphic line bundle over a Kähler surface X. Then L admits a solution to (1.3) if and only if there exists a metric h on L such that $\Omega > 0$, where $\Omega = \cot(\hat{\theta})\omega + iF$.*

When $n \geq 3$, Collins-Jacob-Yau proved the solvability of the LYZ equation under the assumption of the existence of a supercritical $\mathcal{C}$-subsolution Precisely,

Theorem 2.3. *(Collins-Jacob-Yau [5]) Fix $\omega_0 \in 2\pi c_1(L)$ and suppose $\hat{\theta}$ satisfies the critical phase condition*

$$\hat{\theta} > (n-2)\frac{\pi}{2}. \tag{2.1}$$

Suppose there exists $\chi := \omega_0 + \sqrt{-1}\partial\bar{\partial}u \in 2\pi c_1(L)$ *defining a subsolution in the sense of Definition* 3.2 *in* [5]. *Assume*

$$\theta(\chi) > (n-2)\frac{\pi}{2}. \tag{2.2}$$

Then there exists a unique smooth $(1,1)$ *form* α *solving (1.4).*

Collins–Jacob–Yau conjectured that their existence theorem is true without condition (2.2) . When $n = 3$, Pingali [27] can solve the LYZ equation without condition (2.2) by translating the LYZ equation into a mixed Monge Ampére type equation.

When $n = 4$, Lin [21] confirmed their conjecture without condition (2.2). Very recently, Lin [23] claimed that he solved the LYZ equation without condition (2.2) for all n.

Theorem 2.4. *(Lin* [23]*) Let* (M,ω) *be a compact Kähler manifold of dimension n and* χ *a closed real* $(1,1)$*-form on* M *with* $\hat{\theta} > (n-2)\frac{\pi}{2}$*. If there exists a subsolution* χ *of LYZ equation in the sense of Definition* 3.2 *in* [5]*, then there exists a unique smooth solution of (1.4).*

The LYZ equation has been generalized to non Kähler case. Lin [22] generalized Collins-Jacob-Yau's result to the Hermitian case (M,ω) with $\partial\bar{\partial}\omega = \partial\bar{\partial}\omega^2 = 0$. Huang-Zhang-Zhang [18] consider the LYZ equation on a compact almost Hermitian manifold for the hypercritical case and the supercritical case was solved by Huang-Zhang [17].

3. Parabolic flow

In order to study the existence of dHYM metrics on high dimensional Kähler manifolds, Jacob-Yau [20] introduced a parabolic evolution flow for the metric h on the line bundle L.

Definition 3.1. *(Line bundle MCF) Given one Hermitian metric* h_0 *on* L*, we define a flow of metrics* $h_t = e^{-u_t}h_0$ *by the following equation:*

$$\frac{d}{dt}u_t = \theta(F_{h_t}) - \hat{\theta} \tag{3.1}$$

where $F_{h_t} = F_{h_0} + \partial\bar{\partial}u_t$ *is the Chern curvature of* L *with respect to* h_t*.*

The flow (3.1) can be regarded as the complex version of the mean curvature flow for the Lagrangian graph. Thus it is also called line bundle mean curvature flow (line bundle MCF). As shown in [20], this equation is parabolic and exists in a short time $[0,\varepsilon)$. They also proved the following theorem on extension of line bundle MCF.

Theorem 3.1 (Jacob-Yau [20])**.** *Let* L *be a holomorphic line bundle over the compact Kähler manifold* X *and* h_t *be a path of metrics on* L *solving (3.1). Assume that* $Z_{L,[\omega]} \neq 0$ *and* $|\nabla F_t|_g^2 \leq C$ *uniformly in time* $[0,T)$*, then all derivatives* $|\nabla^k F_t|_g$ *are bounded by some constant* C_k*. Furthermore, if* T *is finite, then the flow can be extended to* $T+\varepsilon$*. If* $T=\infty$*, then the flow subsequently converges to a smooth solution of (1.3).*

In [20], Jacob-Yau also studied the long time existence and convergence of the line bundle MCF under some assumptions.

Theorem 3.2 (Jacob-Yau [20]). *Let (X, ω) be a Kähler manifold with non-negative orthogonal bisectional curvature and L be an ample line bundle. Then there exists a natural number k such that $L^{\otimes k}$ admits a dHYM metric and it is constructed via a smoothly converging family of metrics along the line bundle MCF.*

Indeed, the condition on k in Theorem 3.2 guarantees that the initial data u_0 satisfies the so-called hypercritical condition, i.e. $\theta(F_{u_0}) > \frac{(n-1)\pi}{2}$. As a consequence of this hypercritical condition, $F_{u_t} > 0$ for all $t \geq 0$. Then the operator $\theta(F_{u_t})$ is concave and the Evans-Krylov theory works for higher order estimates.

In [5], Collins-Jacob-Yau also get the following result for the line bundle MCF.

Theorem 3.3 (Collins-Jacob-Yau [5]). *If $\hat{\theta} > \frac{(n-1)\pi}{2}$ and u_0 is a subsolution with $\theta(F_{u_0}) > \frac{(n-1)\pi}{2}$, then the line bundle MCF starting from u_0 converges smoothly to a dHYM metric.*

In [31], Takahashi introduced the tangent Lagrangian phase flow (TLPF) on the space of almost calibrated $(1,1)$-forms:

$$\frac{d}{dt} u_t = \tan(\theta(\chi_u) - \hat{\theta}), \tag{3.2}$$

where $\chi(u) = \chi + \sqrt{-1}\partial\bar{\partial}u$.

He showed that the TLPF starting from any initial data exists for all positive time and converges smoothly to a dHYM metric assuming the existence of a $\mathcal{C}$-subsolution.

Theorem 3.4. *(Takahashi [31]) Let X be a compact complex manifold with a Kähler form ω, and χ a closed real $(1,1)$-form. Assume $\hat{\theta} > (n-1)\frac{\pi}{2}$. Then the flow u_t exists for all positive time. Moreover, if there is a $\mathcal{C}$-subsolution, then the flow converges to the deformed Hermitian Yang–Mills metric in the C^∞-topology.*

There are some important progress made by G.Chen [2] on the J-equation and the dHYM equation. Motivated by the concavity of $\cot(\theta(\chi_u))$ by Chen, Fu-Yau-Zhang consider the cotangent Lagrangian phase flow

$$\frac{d}{dt} u_t = \cot(\theta(\chi_u)) - \cot(\hat{\theta}). \tag{3.3}$$

The advantage of this new flow is that the imaginary part of the Calabi-Yau functional is constant along the flow. They showed the long time existence of the flow in hypercritical case and the longtime solution converges to the solution of the deformed Hermitian Yang-Mills equation under the Collins-Jacob-Yau's condition on the subsolution. Hence they reproved the Collins-Jacob-Yau's existence theorem 3.3.

Theorem 3.5. *(Fu-Yau-Zhang [10]) Let X be a compact complex manifold with a Kähler form ω, and χ a closed real $(1,1)$-form. Assume $\hat{\theta} > (n-2)\frac{\pi}{2}$ and $\theta(\chi_{u_0}) > (n-2)\frac{\pi}{2}$, then the flow (3.3) exists for all time. Moreover, if there is a $\mathcal{C}$-subsolution, then the flow converges to the deformed Hermitian Yang–Mills metric in the C^∞-topology.*

A desirable property of a geometric flow is the stability of stationary points. That is, if the initial date is sufficiently close to one stationary point, then the flow exists for long time and converges to the stationary point. The stability result gives more evidences that the method of flows will work to find the stationary point.

Han-Jin considered the stability of the line bundle MCF in [14]. That is, if assume that there exists one dHYM metric on holomorphic line bundle over a compact Kähler manifold and the initial metric is C^2 close to this dHYM metric, then the flow will admit long-time solution and exponentially converge to this given dHYM metric.

Theorem 3.6. *(Han-Jin* [14]*) Let (X, ω) be a compact Kähler manifold of complex dimension n and L be a holomorphic line bundle over X. Assume $\hat{h}$ is a dHYM metric on L and $h(t) = e^{-u_t}\hat{h}$ satisfy the line bundle MCF. There exists a constant $\delta_0 > 0$ such that if the smooth initial data u_0 satisfies $||D^2 u_0||_{L^\infty} \leq \delta_0$, then the line bundle MCF exists for long time and u_t converges to a constant exponentially.*

Here we do not need any assumption on the phase $\hat{\theta}$ and the positivity of F_{u_t} which are crucial to guarantee the concavity of the operator $\theta(F)$.

4. Chern number inequality

The Chern number inequalities play very important roles in the study of canonical metrics. For example, in the study of Hermian-Einstein metrics, Bogomolov [1] obtained the Bogomolov inequality for semi-stable holomorphic vector bundle and Simpson [28] proved the Bogomolov inequality for stable Higgs bundles on compact Kähler manifolds by constructing Higgs Hermitian-Einstein metrics. Furthermore, in the study of Kähler-Einstein metrics, Miyaoka [25] and Yau [34] proved the famous Miyaoka-Yau inequality. In this section we survey the chern number inequalities about the dHYM metric. First we define the algebraic lifted angle as follow.

Definition 4.1. *Suppose that (X, ω) is a compact n-dimensional Kähler manifold. For $[\alpha] \in H^{1,1}(X, \mathbb{R})$ and p-dimensional irreducible subvariety $V \subset X$, define*

$$\begin{cases} Z_{V,[\alpha]}(t) = -\int_V e^{-\sqrt{-1}(t\omega + \sqrt{-1}\alpha)} = -\frac{(-\sqrt{-1})^p}{p!}\int_V (t\omega + \sqrt{-1}\alpha)^p, \\ Z_V([\alpha]) = Z_{V,[\alpha]}(1) \end{cases}$$

for $t \in [1, +\infty)$. Suppose that $Z_{V,[\alpha]}(t) \in \mathbb{C}^*$ for all $t \in [1, +\infty)$.

(i) The algebraic lifted angle $\hat{\theta}_V([\alpha])$ is defined as the winding angle of the curve $Z_{V,[\alpha]}(t)$ as t runs from $+\infty$ to 1 .

(ii) The slicing angle $\varphi_V([\alpha])$ is defined as

$$\varphi_V([\alpha]) = \hat{\theta}_V([\alpha]) - (p-2) \cdot \frac{\pi}{2}.$$

Collins-Yau [9] proved the following Chern number inequality for $\dim_{\mathbb{C}} X = 3$.

Theorem 4.1 (Collins-Yau [9])**.** *Suppose (X, ω) is 3-dimensional Kähler manifold and L admits a dHYM metric with analytic lifted angle $\hat{\theta} \in (\frac{\pi}{2}, \frac{3\pi}{2})$. Then the following hold*

(i) The Chern number satisfies

$$\left(\int_X \alpha^3\right)\left(\int_X \omega^3\right) < 9\left(\int_X \alpha \wedge \omega^2\right)\left(\int_X \alpha^2 \wedge \omega\right)$$

in particular, the algebraic lifted angle $\hat{\theta}_X([\alpha])$ is well-defined.

(ii) $\mathrm{Im}\,(Z_X([\alpha])) > 0$ *and* $\varphi_X([\alpha]) \in \left(\frac{\pi}{2}, \pi\right)$.

(iii) For any irreducible subvariety $V \subsetneq X$,

$$\operatorname{Im}(Z_V([\alpha])) > 0, \quad \varphi_V([\alpha]) > \varphi_X([\alpha]),$$

where $Z_X([\alpha]), Z_V([\alpha])$ and $\varphi_X([\alpha]), \varphi_V([\alpha])$ are defined in Definition 4.1.

Later, based on a Nakai–Moishezon type criterion proved by Chu-Lee-Takahashi [4], Chu-Lee [3] proved the converse of the above theorem is true in the hypercritical case.

Theorem 4.2. *(Chu-Lee [3]) The converse of Theorem 4.1 is true.*

In [9] Collins-Yau proposed the following conjecture of Chern number inequalities on the 4-dimension case.

Conjecture 4.3 (Collins-Yau [9])**.** *Suppose (X, ω) is a four dimensional compact Kähler manifold and L admits a dHYM metric F with constant angle $\hat{\theta} \in (\frac{3\pi}{2}, 2\pi)$. Then the following Chern number inequalities hold*

$$\frac{c_1(L)^3 \cdot \omega}{c_1(L) \cdot \omega^3} > 1$$

and

$$\frac{(c_1(L)^3 \cdot \omega)(\omega^4)}{c_1(L) \cdot \omega^3} - 6(c_1(L)^2 \cdot \omega^2) + \frac{(c_1(L) \cdot \omega^3)(c_1(L)^4)}{c_1(L)^3 \cdot \omega} < 0.$$

Han-Jin [15] proved this conjecture under the assumption $\hat{\theta} \in (\pi, 2\pi)$.

Theorem 4.4. *(Han-Jin [15]) Suppose $\hat{\theta} \in (\pi, 2\pi)$, then the Chern number inequalities in Conjecture 4.3 hold.*

They [16] also consider the converse of the above theorem.

Theorem 4.5. *(Han-Jin [16]) Suppose that (X^4, ω) is a compact four-dimensional Kähler manifold and $[\alpha] \in H^{1,1}(X, \mathbb{R})$. If the following hold,*
(i) The Chern number inequalities in Conjecture 4.3 hold,
(ii) For any 3-dimensional irreducible subvariety $V \subset X$,

$$\left(\int_V \alpha^3\right)\left(\int_V \omega^3\right) < 9\left(\int_V \alpha \wedge \omega^2\right)\left(\int_V \alpha^2 \wedge \omega\right)$$

(iii) $\operatorname{Im}(Z_X([\alpha])) > 0$ and $\varphi_X([\alpha]) \in \left(\frac{\pi}{2}, \pi\right)$.
(iv) For any irreducible subvariety $V \subsetneq X$,

$$\operatorname{Im}(Z_V([\alpha])) > 0, \quad \varphi_V([\alpha]) > \varphi_X([\alpha]),$$

where $Z_X([\alpha]), Z_V([\alpha])$ and $\varphi_X([\alpha]), \varphi_V([\alpha])$ are defined in Definition 4.1, then the dHYM equation admits a solution with $\hat{\theta} \in \left(\frac{3\pi}{2}, 2\pi\right)$.

Acknowledgement

The first author is supported by National Key R&D Program of China 2022YFA1005400.

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