

Applications of Green functions on metric measure spaces with Ricci curvature bounded below

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1 Introduction

This article mainly concerns about a special class of metric measure spaces, namely RCD spaces. Roughly speaking, we say a metric measure space is $\text{RCD}(K, N)$ if it is with Ricci curvature bounded below by K and dimension bounded above by N in some synthetic sense.

The idea of curvature-dimension bounds for metric measure spaces via language of optimal transport originated from Lott-Sturm-Villani theory [LV09], [S06a, S06b]. Later, Ambrosio-Gigli-Savaré [AGS14b] introduced the infinitesimal Hilbertian assumption for metric measure spaces with curvature-dimension bounds in the sense of Lott-Villani-Sturm, which is the birth of the notion of RCD spaces. See [A19] for a nice survey on the topic of RCD spaces.

In the present article we mainly consider non-compact $\text{RCD}(0, N)$ spaces that admit entire Green functions. Here by entire Green function of an $\text{RCD}(0, N)$ space $(X, \mathbf{d}, \mathbf{m})$ we mean the function $G : X \times X \setminus \{x = y\} \rightarrow \mathbb{R}_+$ defined by the following property: for any $x \in X$, $G(x, \cdot) = G_x(\cdot)$ is the minimal positive solution to the equation $\Delta G_x = -\delta_x$, where δ_x denotes the Dirac distribution centered at x .

We shall introduce two applications of entire Green functions on $\text{RCD}(0, N)$ spaces. The first one is a sharp gradient estimate of a distance-like function constructed with the Green function and the corresponding rigidity results. The second one is an existence result of solutions to Poisson equations.

2 Gradient estimates for Green functions and rigidity

Let us recall a result by Colding [C12, Theorem 3.1] on Riemannian manifolds:

Theorem 2.1. *Assume (M^N, g) ($N \in \mathbb{N}_{\geq 3}$) is a Riemannian manifold with non-negative Ricci curvature that admits an entire Green function G . Fix any reference point $x \in M^N$, define the function*

$$\mathbf{b}_x := G_x^{\frac{1}{2-N}}, \quad (2.1)$$

the following propositions holds true:

1. (Sharp gradient estimate) For any $y \in M^n$,

$$|\nabla \mathbf{b}_x|(y) \leq (N(N-2))^{\frac{1}{N-2}} \omega_N^{\frac{1}{N-2}} =: \mathcal{C}_N \omega_N^{\frac{1}{N-2}} \quad (2.2)$$

2. (Rigidity) The following statements are equivalent:

- (a) There is $y \in M^n \setminus \{x\}$ such that equality in (2.2) is attained.
- (b) (M^n, g) is isometric to the Euclidean space \mathbb{R}^n .
- (c) $\mathbf{b}_x = \mathcal{C}_N \omega_N^{\frac{1}{N-2}} \mathbf{d}_x$, where \mathbf{d}_x is the distance function from x .

This result is interesting in the author's opinion, because we can determine the geometry of the whole space by the value of a certain function (namely $|\nabla \mathbf{b}_x|$) at a single point.

On $\text{RCD}(0, N)$ spaces, thanks to Bruè-Semola's work [BS19], we also know that Green functions are well-defined on an $\text{RCD}(0, N)$ ($N > 2$) space $(X, \mathbf{d}, \mathbf{m})$ if the following non-parabolic assumption is fulfilled:

$$\int_1^\infty \frac{r}{\mathbf{m}(B_r(x))} \, \mathbf{d}r < \infty, \quad \forall x \in X. \quad (2.3)$$

Therefore we are curious whether we can extend Colding's result to the RCD setting. In a joint work by Prof. Shouhei Honda and the author [HP25], we managed to prove the following result as a counterpart of Colding's result:

Theorem 2.2. *Let $(x, \mathbf{d}, \mathbf{m})$ be an $\text{RCD}(0, N)$ ($N \in \mathbb{R}_{>2}$) space that admits an entire Green function $G(\cdot, *)$. Fix a reference point $x \in X$ with*

$$\nu_x := \lim_{r \downarrow 0} \frac{\mathbf{m}(B_r(x))}{r^N} < \infty, \quad (2.4)$$

Define $\mathbf{b}_x := G_x^{\frac{1}{N-2}}$ similarly as above. Then

1. There is an upper semi-continuous representative of $|\nabla \mathbf{b}_x|$. (Hereafter we always assume $|\nabla \mathbf{b}_x|$ is upper semi-continuous so that the pointwise upper bound of $|\nabla \mathbf{b}_x|$ is well-defined.)
2. (Sharp gradient estimate) For any $y \in X$,

$$|\nabla \mathbf{b}_x|(y) \leq \mathcal{C}_N \nu_x^{\frac{1}{N-2}}. \quad (2.5)$$

3. (Rigidity) If the sharp upper bound of $|\nabla \mathbf{b}_x|$ is attained at some point $y \neq x$, then the space $(X, \mathbf{d}, \mathbf{m}, x)$ is isomorphic to an $\text{RCD}(0, N)$ -metric measure cone (or more precisely, $(X, \mathbf{d}, \mathbf{m}, x)$ is isomorphic to a tangent cone at x).¹

Please note that the conclusion 1 is necessary, as $|\nabla \mathbf{b}_x|$ is only defined in almost-everywhere sense a priori in the RCD setting so that the pointwise upper bound does not make sense. And we emphasize that the key reason for the fact the sharp gradient estimate and rigidity are different from those by Colding in manifold case is that tangent spaces at a reference point on an RCD space are not necessarily Euclidean. The parameter ν_x actually carries the geometric information of tangent spaces at x .

Besides, we also show a quantitative version of the rigidity, or the so-called almost rigidity theorem.

Theorem 2.3 (Almost rigidity). *For all $N > 2$, $0 < \varepsilon < 1$, $v > 0$, $0 < r < R < \infty$ and $\varphi \in L^1([0, \infty), \mathcal{H}^1)$ there exists $\delta := \delta(N, \varepsilon, r, R, \varphi) > 0$ such that if a pointed non-parabolic $\text{RCD}(0, N)$ space $(X, \mathbf{d}, \mathbf{m}, x)$ satisfies $\nu_x < v$,*

$$\frac{s}{\mathbf{m}(B_s(x))} \leq \varphi(s), \quad \text{for } \mathcal{H}^1\text{-a.e. } s \in [1, \infty) \quad (2.6)$$

¹An $\text{RCD}(0, N)$ -metric measure cone means the warped product of $[0, \infty)$ and a compact $\text{RCD}(N-2, N-1)$ space, where the warp factor is r . By De Philippis-Gigli [DG16], if the volume density ν_x is finite, then any tangent cone at x must be a metric measure cone.

and

$$\mathcal{C}_N \nu_x^{\frac{1}{N-2}} - |\nabla \mathbf{b}_x|^*(z) \leq \delta \quad (2.7)$$

hold for some $z \in B_R(x) \setminus B_r(x)$, then $(X, \mathbf{d}, \mathbf{m}, x)$ is ε -pmGH close to the N -metric measure cone over an $\text{RCD}(N-2, N-1)$ space.

This result is new even restricted to the smooth setting.

3 Solvability of Poisson equations

Recall that for given suitable f on \mathbb{R}^n , the (entire) Poisson equation $\Delta u = f$ is solved by

$$u(x) = - \int_{\mathbb{R}^n} f(y) G^{\mathbb{R}^n}(x, y) \, dy \quad (3.1)$$

if the integral exists. On Riemannian manifolds admitting an entire Green function, similar existence results are also known, for instance,

Theorem 3.1 (Theorem 1.1 of [NST01]). *Let (M^n, g, p) be a pointed complete non-compact Riemannian manifold with non-negative Ricci curvature admitting an entire Green function G^{M^n} satisfying that there exists $\sigma > 0$ such that*

$$\sigma^{-1} \frac{r^2}{\text{Vol}_g(B_r(x))} \leq G^{M^n}(x, y) \leq \sigma \frac{r^2}{\text{Vol}_g(B_r(x))}, \quad \forall x, y \in M^n, \quad x \neq y, \quad (3.2)$$

where $r := \mathbf{d}_g(x, y)$, then for any Hölder continuous function $f \geq 0$ satisfying

$$K_f(t) := \text{Vol}_g(B_t(p))^{-1} \int_{B_t(p)} f \, d\text{Vol}_g \in L^1((0, \infty)), \quad (3.3)$$

there exists a continuous solution to the Poisson equation $\Delta u = f$.

Since as we mentioned above, the concept of entire Green function is also well-defined on non-parabolic $\text{RCD}(0, N)$ spaces, we wonder whether similar results can be verified in more general setting. In an unpublished work, the author proved that

Theorem 3.2. *Assume $(X, \mathbf{d}, \mathbf{m}, x_0)$ is a pointed $\text{RCD}(0, N)$ ($N > 2$) and there exists $\sigma > 0$ such that*

$$\int_r^\infty \frac{s}{\mathbf{m}(B_s(x))} \, ds \leq \sigma \frac{r^2}{\mathbf{m}(B_r(x))}, \quad \forall x \in X, \quad \forall r \in (0, \infty). \quad (3.4)$$

For any $f \in C(X, \mathbf{d})$ with

$$K_f(t) := \int_{B_t(x_0)} |f| \, d\mathbf{m} \in L^1((0, \infty)), \quad (3.5)$$

there exists a continuous solution to the entire Poisson equation $\Delta u = f$.

The construction of the solution is done by taking limit of local solutions. Let us introduce it briefly. Consider an exhaustion of X by $\{\Omega_i = B_{r_i, \delta}(x_0)\}_i$, where $r_i \rightarrow \infty$, $\delta > 0$ is fixed, and

$$B_{r_i, \delta}(x_0) := \{x \in B_{r_i}(x_0) \mid \mathbf{d}(x, X \setminus B_{r_i}(x_0)) > \delta\}. \quad (3.6)$$

Note that here we consider balls with shrinkage instead of simply balls because balls might be non-regular in barrier sense, which may cause non-existence of classic solutions to some elliptic

PDEs; while after the shrinkage, the barrier regularity is guaranteed, and thus we can apply Cucinotta-Mondino's result [CM24] to conclude that there is a unique $C(\overline{\Omega}_i)$ -local Dirichlet Green function $G^{\Omega_i}(\cdot, *)$, i.e., for any $x \in \Omega_i$, $G_x^{\Omega_i}$ is the unique solution to the Dirichlet problem

$$\begin{cases} G_x^{\Omega_i} = 0, & \text{on } \partial\Omega_i \\ \Delta G_x^{\Omega_i} = -\delta_x, & \text{in } \Omega_i \end{cases} \quad (3.7)$$

And the unique local solution u_i to the Dirichlet problem

$$\begin{cases} u_i = 0, & \text{on } \partial\Omega_i \cup \{x_0\} \\ \Delta u_i = f, & \text{in } \Omega_i \end{cases} \quad (3.8)$$

can be expressed by

$$u_i(x) = \int_{\Omega_i} f(y) \left(G^{\Omega_i}(x_0, y) - G^{\Omega_i}(x, y) \right) \, d\mathbf{m}(y). \quad (3.9)$$

With the help of assumptions on f and convergence of local Green functions, we can check the local uniform boundedness and equi-continuity of $\{u_i\}_i$. Then applying Arzelà-Ascoli theorem, we obtain a solution to the global Poisson equation.

Recall that in the previous result we assume the growth bound restrictions to f by the integrability of average $K_f(r)$ of f . This assumption can be interpreted as a restriction bound not exceeding the degree of r^{-1} of f in the weak sense. If we replace it with a stronger pointwise growth bound, we can obtain a solution with better regularity:

Theorem 3.3. *Adopt the assumptions as in Theorem 3.2. If we assume further that there exists $\varepsilon, L > 0$ such that*

$$f(x) \leq \frac{L}{\mathbf{d}(x, x_0)^{1+\varepsilon}}, \quad \forall x \in X, \quad (3.10)$$

then the solution constructed as above by limit of local solutions is Lipschitz continuous, with

$$|\nabla u| \leq C(N, L, \varepsilon). \quad (3.11)$$

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