HEINTZE-KARCHER-TYPE INEQUALITIES AND APPLICATIONS ON ALEXANDROV-TYPE THEOREMS

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Abstract. In this paper, we shall give a brief survey on several Heintze-Karcher-type inequalities and their applications on Alexandrov-type theorems.

1. Introduction

In Riemannian geometry, the celebrated Bishop-Gromov's comparison theorem states that, for a given Riemannian manifold (M^n, g) with $Ric \geq (n-1)Kg$ for some constant $K \in \mathbb{R}$, the volume of a geodesic ball of radius r in M can be bounded above by the volume of a geodesic ball of the same radius in a space form of curvature K. Heintze-Karcher [6] proves another comparison theorem concerning tubes around hypersurfaces in M. For the purpose of this survey, we state below a simple version of Heintze-Karcher's comparison theorem for a bounded domain with smooth boundary in manifolds with $Ric \geq 0$.

Theorem 1.1 (Heintze-Karcher [6]). *Let Σ be an embedded closed hypersurface, which encloses a bounded domain Ω, in an $(n+1)-$dimensional Riemannian manifold with $\mathrm{Ric} \geq 0$, then*

$$\mathrm{Vol}(\Omega) \leq \int_\Sigma \int_0^{c(p)} \left(1 - \frac{H(p)}{n} t\right)^n dt dA(p).$$

where $c(p)$ is the length to reach the first focal point of Σ from p by the normal exponential map and $H(p)$ is the mean curvature of Σ at p.

The key ingredient for Heintze-Karcher is to give the following Jacobian estimate for exponential map from Σ,

$$\mathrm{Jacobian}(d\exp_p) \leq \left(1 - \frac{H(p)}{n} t\right)^n.$$

Note also that $c(p)$ can be estimated above by $c(p) \leq \frac{n}{H(p)}$ if $H(p) > 0$. Consequently, one has the following version of sharp Heintze-Karcher's inequality.

Theorem 1.2 (Heintze-Karcher [6], Ros [15]). *Let Σ be an embedded closed hypersurface, which encloses a bounded domain Ω, in an $(n+1)-$dimensional Riemannian manifold with $\mathrm{Ric} \geq 0$. Assume $H > 0$. Then*

$$\int_\Sigma \frac{1}{H} dA \geq \frac{n+1}{n} Vol(\Omega),$$

Equality holds if and only if Ω is a Euclidean ball.

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Another proof for Theorem 1.2 has been given by Ros [15] via Reilly's formula [13]. Using Minkowski-Hsiung's integral formula and the sharp Heintze-Karcher inequality (1.2) together with its equality characterization, one can give an alternative proof of famous Alexandrov's soap-bubble theorem, which says that any closed embedded constant mean curvature hypersurface in the Euclidean space is a round sphere. Recall that Alexandrov's soap-bubble theorem has been first proved by Alexandrov [2] through moving plane method developed by himself.

In this paper, we shall give a brief survey on several Heintze-Karcher-type inequalities and their applications on Alexandrov-type theorems.

2. Closed hypersurfaces in manifolds

In this section, we survey Heintze-Karcher-type inequalities in manifolds with various curvature conditions.

By Heintze-Karcher's idea based on Jacobian estimate for exponential map, Montiel-Ros derived the following inequality for closed hypersurfaces in the hyperbolic space $\mathbb{H}^{n+1}$, the simply connected space form with curvature -1, see Ritore [14, Theorem 4.38].

Theorem 2.1 (Montiel-Ros [11])**.** *Let Σ be an embedded closed hypersurface, which encloses a bounded domain Ω, in $\mathbb{H}^{n+1}$. Assume H is a constant. Then*

$$\frac{1}{H}\int_\Sigma \cosh r dA \geq \frac{n+1}{n}\int_\Omega \cosh r dV,$$

where r is the distance function to an arbitrary point in $\mathbb{H}^{n+1}$. Equality holds if and only if Ω is a geodesic ball in $\mathbb{H}^{n+1}$.

Theorem 2.1 can be used to prove Alexandrov's soap-bubble theorem in the hyperbolic space. Later, Brendle [4] used a modified Heintze-Karcher's idea by considering normal flow with respect to the conformal metric $\tilde{g} = \cosh^{-2} r g$ to improve the above result.

Theorem 2.2 (Brendle [4])**.** *Let Σ be an embedded closed hypersurface, which encloses a bounded domain Ω, in $\mathbb{H}^{n+1}$. Assume $H > 0$. Then*

$$\int_\Sigma \frac{\cosh r}{H} dA \geq \frac{n+1}{n}\int_\Omega \cosh r dV,$$

where r is the distance function to an arbitrary point in $\mathbb{H}^{n+1}$. Equality holds if and only if Ω is a geodesic ball in $\mathbb{H}^{n+1}$.

Recall that Ros gave another proof of Heintze-Karcher's inequality by using Reilly's formula. Qiu and I [12] have established a new Reilly-type formula and proved the Heintze-Karcher-type inequality in manifolds with sectional curvature bounded below.

Theorem 2.3 (Qiu-Xia [12])**.** *Let Σ be an embedded closed hypersurface, which encloses a bounded domain Ω, in an $(n+1)-$dimensional Riemannian manifold (M, g) with sectional curvature bounded below Sect ≥ -1. Assume $H > 0$. Then*

$$\int_\Sigma \frac{\cosh r}{H} dA \geq \frac{n+1}{n}\int_\Omega \Delta \cosh r dV = \frac{n+1}{n}\int_\Sigma \langle \sinh r \partial_r, \nu\rangle dA.$$

where r is the distance function to an arbitrary point in M. Equality holds if and only if Ω is a geodesic ball in $\mathbb{H}^{n+1}$.

Remark 2.1. *It is natural to ask whether above result holds under the assumption of* $\mathrm{Ric} \geq -ng$*.*

Brendle [4] has established Heintze-Karcher-type inequality in general warped product manifolds with certain curvature condition.

Theorem 2.4 (Brendle [4]). *Let M be a warped product manifold $M = [0, \bar{r}) \times N$ equipped with $g = dr^2 + \lambda^2(r) g_N$, where (N, g_N) be a closed n-manifold and λ is a smooth function satisfying the following*
(H1) $\lambda'(0) = 0, \lambda''(0) > 0$,
(H2) $\lambda'(r) > 0$ for $r \in (0, \bar{r})$,
(H3) $\Delta\lambda' g - \nabla^2\lambda' + \lambda' Ric \geq 0$.
Let Σ be an embedded closed hypersurface with $H > 0$ in M. If Σ is null-homologous, then

$$\int_\Sigma \frac{\lambda'(r)}{H} dA \geq \frac{n+1}{n} \int_\Omega \lambda'(r) dV.$$

If Σ is homologous to $\partial M = N \times \{0\}$, then

$$\int_\Sigma \frac{\lambda'(r)}{H} dA \geq \frac{n+1}{n} \left(\int_\Omega \lambda'(r) dV + \frac{1}{n} \lambda(0)^n Vol(N, g_N) \right).$$

Equality holds if and only if Σ is a radial slice $N \times \{r = r_0\}$.

The novelty in Brendle's proof is to use normal flow with respect to the conformal metric $\tilde{g} = \lambda'(r)^{-2} g$. Brendle's Heintze-Karcher inequality leads to Alexandrov-type theorem in certain warped product manifolds, including (Anti-de Sitter) Schwarzschild manifold in General Relativity.

Theorem 2.5 (Brendle [4]). *Let M be as in Theorem 2.4. Let Σ be an embedded closed hypersurface in M. Then Σ is a radial slice $N \times \{r = r_0\}$.*

Li and I [10] established another Reilly-type formula, by which we generalized Brendle's Heintze-Karcher inequality to general sub-static manifolds with no warped product structure. Borghini-Fogagnolo-Pinamonti [3] proved the rigidity part.

Theorem 2.6 (Li-Xia [10], Borghini-Fogagnolo-Pinamonti [3]). *Let (M, g) be a sub-static Riemannian manifold, that is, there exists $f \in C^\infty(M)$ such that $f > 0$ in $\mathrm{int}(M)$, $f = 0$ on ∂M and*

$$\Delta f g - \nabla^2 f + f \mathrm{Ric} \geq 0.$$

Let Σ be an embedded closed hypersurface homologous to ∂M and $H > 0$. Then

$$\int_\Sigma \frac{f}{H} dA \geq \frac{n+1}{n} \left(\int_\Omega f dV + c_{\partial M} \int_{\partial M} |\nabla f| dA \right),$$

where

$$c_{\partial M} = \frac{n}{n+1} \frac{\int_{\partial M} |\nabla f| dA}{\int_{\partial M} |\nabla f| \left[\frac{\Delta f}{f} - \frac{\nabla^2 f}{f} \left(\frac{\nabla f}{f}, \frac{\nabla f}{f} \right) \right] dA}.$$

Equality holds if and only if $\partial\Omega = \Sigma \bigsqcup \partial M$ is isometric to

$$\left([s_0, s_1] \times \partial M, \frac{ds^2}{f(s)^2} + s^2 g_{\partial M} \right).$$

3. Hypersurfaces with boundary

In this section, we survey Heintze-Karcher-type inequalities for hypersurfaces with boundary. The motivation is an Alexandrov-type theorem, proved by Wente [17], for capillary hypersurfaces supported in a half-space. Here a capillary hypersurface means that a hypersurface which intersects the support hyperplane at a constant contact angle.

Theorem 3.1 (Wente [17])**.** *Any compact embedded constant mean curvature capillary hypersurface in a half-space must be a spherical cap.*

Wente used Alexandrov's moving plane method to prove this theorem. It is natural to ask whether there is an integral proof. This motivates us to prove the following Heintze-Karcher-type inequalities for hypersurfaces with boundary supported in a half-space. Denote $\mathbb{R}^{n+1}_+ = \{x_{n+1} > 0\}$ and E_{n+1} the $(n+1)$-th coordinate unit vector.

Theorem 3.2 (Jia-Wang-Xia-Zhang [7])**.** *Let $\Sigma \subset \overline{\mathbb{R}^{n+1}_+}$ be a compact, embedded hypersurface with boundary lying on $\partial\mathbb{R}^{n+1}_+$ such that $H > 0$ and the contact angle $\theta(x) \leq \theta_0$, $\theta_0 \in (0, \pi)$. Let Ω be the domain enclosed by Σ and $\partial\mathbb{R}^{n+1}_+$ and ν be the unit normal to Σ pointing outward Ω. Then*

$$\int_\Sigma \frac{1 - \cos\theta_0 \langle \nu, E_{n+1}\rangle}{H} dA \geq \frac{n+1}{n} \mathrm{Vol}(\Omega),$$

Equality holds if and only if Σ is a θ-capillary spherical cap.

Combining with the Minkowski-Hsiung formula for θ_0-capillary hypersurfaces supported in a half-space (see e.g. [1])

$$\int_\Sigma n(1 - \cos\theta_0 \langle \nu, E_{n+1}\rangle) - H\langle x, \nu\rangle dA = 0,$$

we give an alternative proof of Wente's theorem. The proof of Theorem 3.2 relies on Heintze-Karcher's idea, with a modified normal map

$$\zeta(p, t) = p - t(\nu(p) - \cos\theta_0 E_{n+1}), \qquad \text{for } p \in \Sigma, t \in \mathbb{R}.$$

The approach is quite flexible. In the following we introduce a similar problem in anisotropic setting. For $\Omega \subset \overline{\mathbb{R}^{n+1}_+}$ as in Theorem 3.2, consider the free energy functional

$$\mathcal{E}(\Sigma) = \int_\Sigma F(\nu) dA + \omega_0 |\partial\Omega \cap \partial\mathbb{R}^{n+1}_+|,$$

where $F : \mathbb{S}^n \to \mathbb{R}_+$ is a C^2 positive function on $\mathbb{S}^n$, such that $(\nabla^2 F + F\sigma) > 0$, where σ is the canonical metric on $\mathbb{S}^n$ and ∇^2 is the Hessian on $\mathbb{S}^n$, and $\omega_0 \in \mathbb{R}$ is a given constant. The Cahn-Hoffman map associated with F is given by

$$\Phi : \mathbb{S}^n \to \mathbb{R}^{n+1}, \quad \Phi(x) = \nabla F(x) + F(x)x.$$

The image $W_F = \Phi(\mathbb{S}^n)$ is called Wulff shape. For a hypersurface $\Sigma \subset \mathbb{R}^{n+1}$, $\nu_F = \Phi(\nu)$ is the anisotropic normal and the anisotropic mean curvature H_F is given by the trace of $d\nu_F$ with respect to the induced metric g. Thus

$$H_F = \mathrm{tr}_g((\nabla^2 F + F\sigma) \circ h),$$

where h is the second fundamental form.

The critical point for $\mathcal{E}(\Sigma)$ is given by a hypersurface with constant anisotropic mean curvature and anisotropic ω_0-capillary boundary condition, which means

$$\langle \nu_F, -E_{n+1}\rangle = \omega_0 \text{ on } \partial\Sigma.$$

We prove the following Alexandrov-Wente's theorem in the anisotropic setting.

Theorem 3.3 (Jia-Wang-Xia-Zhang [8])**.** *Any compact embedded constant anisotropic mean curvature, anisotropic capillary hypersurface in a half-space must be a truncated Wulff shape, up to scaling and translation.*

Note that the anisotropic case cannot be proved by Alexandrov's moving plane method, due to the lack of symmetry of F or Wulff shape. We mention that Alexandrov's theorem for closed hypersurfaces in the anisotropic setting has been proved by He-Li-Ma-Ge [5]. Theorem 3.3 follows from the following Heintze-Karcher inequality, whose proof is again based on Heintze-Karcher's idea, with a modified anisotropic normal map.

Theorem 3.4 (Jia-Wang-Xia-Zhang [8])**.** *Let $\Sigma \subset \overline{\mathbb{R}^{n+1}_+}$ be a compact, embedded hypersurface with boundary lying on $\partial\mathbb{R}^{n+1}_+$ such that $H_F > 0$ and*

$$\langle \nu_F(x), -E_{n+1} \rangle = \omega(x) \leq \omega_0, \qquad \text{for any } x \in \partial\Sigma.$$

Then it holds

$$\int_\Sigma \frac{F(\nu) + \omega_0 \left\langle \nu, E^F_{n+1} \right\rangle}{H_F} dA \geq \frac{n+1}{n} \mathrm{Vol}(\Omega).$$

Equality holds if and only if Σ is an ω_0-capillary Wulff shape, up to scaling and translation.

There are several corresponding results for hypersurfaces with boundary in a wedge, in a ball, in a convex cone, or in general convex domains. We refer the interest reader to [7, 8, 9, 16].

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