The stable Bernstein problem for minimal hypersurfaces

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1. Introduction

A two-sided immersion $M^n \to \mathbf{R}^{n+1}$ is minimal if its mean curvature vanishes. Equivalently, M is a critical point of the area functional. A minimal immersion is stable if

$$\int_M |A_M|^2\varphi^2 \leq \int_M |\nabla\varphi|^2,$$

for all $\varphi \in C_c^\infty(M)$, where A_M is the second fundamental form of the immersion. Here stability means that the second variation of the area functional is nonnegative. We are interested in the following question, known as the stable Bernstein problem.

Question 1. *Is a complete, two-sided, stable minimal immersion $M^n \to \mathbf{R}^{n+1}$ an affine hyperplane?*

It is well known that the validity of the stable Bernstein property in $\mathbf{R}^{n+1}$ is equivalent to *a priori* interior curvature estimates for stable minimal hypersurfaces, as shown in [9, Corollary 2.5]:

Corollary 2. *Let (X^{n+1}, g) be a complete Riemannian manifold with bounded sectional curvature $|\sec_g| \leq K$. Then any compact, two-sided, stable minimal immersion $M^n \to (X^{n+1}, g)$ satisfies*

$$|A_M|(x)\min\{1, d_M(x, \partial M)\} \leq C(K).$$

Consequently, one has compactness of the space of stable minimal hypersurfaces (and more generally of minimal hypersurfaces with finite Morse index) in X. This makes the stable Bernstein property a central topic in the theory and applications of minimal surfaces.

In this article, we will discuss the geometric ideas behind the recent progress on Question 1:

Theorem 3. *For $2 \leq n \leq 5$, a complete, two-sided, stable minimal immersion $M^n \to \mathbf{R}^{n+1}$ is an affine hyperplane.*

The case $n = 2$ in Theorem 3 was proved by do Carmo–Peng [10], Fischer-Colbrie–Schoen [11], and Pogorelov [15] independently around 1980; the case $n = 3$ was proved recently by Chodosh and the author [6] (subsequently, two alternative solutions were obtained in [7] and [4]). Last year, the case

$n = 4$ was proved by the author together with Chodosh, Minter, and Stryker [8]; a refinement of the approaches used in the case $n = 4$ was obtained by Mazet [14] to settle the case $n = 5$. The problem in $\mathbf{R}^7$ ($n = 6$) remains open, but the conclusion holds under the additional assumption that M has extrinsic Euclidean volume growth, by work of Schoen–Simon and Simons [17] (for embeddings) and Bellettini [1] (for immersions). For $n \geq 7$, there are non-flat area-minimizing hypersurfaces; see, for example, the classical examples of Bombieri–De Giorgi–Guisti [2].

2. Intrinsic volume growth

The key step in the proof of Theorem 3 (when $n \geq 3$) is to establish *a priori* control of the intrinsic volume growth of a stable minimal hypersurface, namely an estimate of the form

$$\sup_{\rho>0} \frac{|B_M(p,\rho)|}{\rho^n} < \infty. \tag{2.1}$$

Here $B_M(p,\rho)$ stands for the geodesic ball of radius ρ centered at p on M. To see the relevance of (2.1), recall that the classical result of Schoen–Simon–Yau [18] shows that whenever $2 \leq n \leq 5$, a two-sided stable minimal immersion $M^n \to \mathbf{R}^{n+1}$ satisfying (2.1) is an affine hyperplane. When $n = 2$, Pogorelov [15] obtained such an estimate:

Theorem 4 (Pogorelov)**.** *Suppose $M^2 \to \mathbf{R}^3$ is a simply connected stable minimal immersion. Then for any $p \in M$ and $\rho > 0$, it holds that*

$$|B_M(p,\rho)| \leq \frac{4}{3}\pi\rho^2.$$

The additional assumption that M is simply connected does not narrow the scope of its application, because two-sided stability passes to the universal cover of the immersion (cf. [11]). Theorem 4 is obtained by plugging a suitable choice of radial test function into the stability inequality and the Gauss–Bonnet formula. In [7], [8], and [14], the authors extended Theorem 4 to higher dimensions.

Theorem 5 ([7],[8],[14])**.** *Let $3 \leq n \leq 5$ and suppose $M^n \to \mathbf{R}^{n+1}$ is a simply connected stable minimal immersion. Then for any $p \in M$ and $\rho > 0$, we have*

$$\mathcal{H}^n(B_M(p,\rho)) \leq c_n\rho^n,$$

where the constants c_n can be chosen explicitly:

(1) $c_3 = \left(\frac{32\pi}{3}\right)^{\frac{3}{2}} \frac{e^{\frac{30\pi}{\sqrt{3}}}}{6\sqrt{\pi}}$.
(2) $c_4 = 8\pi^2 e^{44\pi}$.
(3) $c_5 = \mathrm{Vol}(\mathbb{B}^5)\left(\frac{800}{43}\right)^{\frac{5}{2}}\left(2e^{100\pi}\right)^5$, *here* $\mathbb{B}^5$ *is the unit ball in* $\mathbf{R}^5$.

In the sequel, we explain the idea behind Theorem 5.

3. Conformal change and macroscopic geometry

Without loss of generality, suppose $0 \in \mathbf{R}^{n+1}$ lies in the image of the immersion $F : M^n \to \mathbf{R}^{n+1}$. Let g denote the induced metric on M and consider the conformally changed metric $\tilde{g} = r^{-2}g$, where r is the Euclidean distance to 0. This conformal change was first considered by Gulliver–Lawson in their study of isolated singularities in stable minimal hypersurfaces [13]. To get a first glance at the effect of this conformal change, note that when M is an affine hyperplane, $\tilde{g}$ gives the standard round cylinder $S^{n-1}(1)\times\mathbf{R}$.

A key observation, first made by Chodosh and the author in [7], is that Theorem 5 follows if we know that $(M \setminus F^{-1}(\{0\}), \tilde{g})$ behaves like an one-dimensional metric space in a suitable macroscopic sense. To make this precise, let us first consider the case $n = 3$. Here we use a crucial observation due to Gulliver and Lawson: $\tilde{g}$ has uniformly positive scalar curvature in a weak spectral sense, namely,

$$-\tilde{\Delta} + \frac{1}{2}\left(\tilde{R} - \frac{n(n-2)}{2}\right) \geq 0. \tag{3.1}$$

Here $\tilde{\Delta}$ and $\tilde{R}$ are the Laplace–Beltrami operator and the scalar curvature of $\tilde{g}$, respectively. In the past few years, there have been important new developments on scalar curvature for 3-manifolds. In particular, it is known that a 3-manifold N with uniformly positive scalar curvature (even in the weak spectral sense) has *macroscopic dimension one* in several concrete senses; for example, N admits evenly spaced separating surfaces with uniformly bounded area [5, 7].

Proposition 6. *Suppose* (N^3, h) *is simply connected and satisfies* $\lambda_1(-\Delta + \frac{1}{2}R) \geq 1$. *Then there exists an exhaustion of* N *by precompact open sets* $\{\Omega_k\}$, $\Omega_k \subset \Omega_{k+1}$, *such that:*

(1) $\Omega_{k+1} \subset N_{10\pi}\Omega_k$;
(2) *each connected component of* $\partial\Omega_k$ *has diameter bounded by* 4π *and area bounded by* 4π.

Such an exhaustion can be constructed by using the μ-bubble localization method introduced by Gromov [12]. Using Proposition 6, one may prove Theorem 5 by keeping careful track of the conformal change and applying the isoperimetric inequality for minimal hypersurfaces in $\mathbf{R}^4$.

This type of argument faces immediate difficulties in higher dimensions. Most notably, in dimensions at least 4, one cannot expect a uniform positive lower bound for scalar curvature to force one-dimensionality – for example, the manifold $S^2 \times \mathbf{R}^2$ has $R > 1$ and is macroscopically two-dimensional. A crucial new idea here is to replace the scalar curvature with a stronger curvature condition, namely the *bi-Ricci curvature*. The bi-Ricci curvature of two orthonormal vectors $v, w \in T_pM$ is defined as

$$\mathrm{BiRic}(v, w) = \mathrm{Ric}(v, v) + \mathrm{Ric}(w, w) - R(v, w, w, v),$$

where R is the curvature tensor. This notion of curvature was introduced by Shen–Ye [16]. In 3 dimensions, the bi-Ricci curvature is half of the scalar curvature. In [8], we prove that the conformally deformed metric $\tilde{g}$ has uniformly positive bi-Ricci curvature in the weak spectral sense when $n = 4$:

$$-\tilde{\Delta} + (\tilde{\lambda}_{\mathrm{BiRic}} - 1) \geq 0, \tag{3.2}$$

where $\tilde{\lambda}_{\mathrm{BiRic}}(p)$ is the smallest bi-Ricci curvature of $\tilde{g}$ at p.

The next step is to establish one-dimensionality under (3.2). We proceed with the same strategy of slicing by μ-bubbles. This gives an exhaustion by regions with μ-bubble boundary Σ satisfying

$$-\Delta^{\Sigma} + \frac{3}{4}(\lambda^{\Sigma}_{\mathrm{Ric}} - \frac{1}{2}) \geq 0. \tag{3.3}$$

In other words, this μ-bubble Σ has a positive Ricci curvature lower bound in the weak spectral sense. In a final crucial step, we extend the Bishop volume comparison theorem to 3-manifolds with positive lower Ricci curvature bounds in the spectral sense.

Theorem 7. *Suppose* (Σ^3, γ) *is a connected closed* 3*-manifold such that*

$$\lambda_1(-\Delta + \alpha^{-1}(\lambda_{\mathrm{Ric}} - 2)) \geq 0,$$

for some constant $\alpha \in (0, 2]$*. Then*

$$\mathrm{diam}(\Sigma, \gamma) \leq \pi$$

and

$$\mathrm{Vol}(\Sigma, \gamma) \leq 2\pi^2.$$

We prove Theorem 7 by extending Bray's proof of the Bishop volume comparison theorem and exploiting convexity properties of a certain weighted isoperimetric profile.

Finally, let us briefly discuss Mazet's extension of this approach when $n = 5$. In this case, a similar bound to (3.2) no longer holds. Instead, he proved that a suitable combination in the form of $\mathrm{BiRic} + \alpha \mathrm{Ric}$ has a uniform positive lower bound in the spectral sense. This condition allowed him to carry out the μ-bubble construction with a similar spectral Ricci bound in the form of (3.3). The rest of the argument is the same.

4. Discussions and further questions

The solution to the stable Bernstein problem relies on recent improvements in our understanding of curvature conditions. In [3], Brendle, Hirsch, and Johne investigated the notion of C_k curvature of a Riemannian metric in n dimensions, interpolating between the Ricci curvature C_1 and the scalar curvature C_{n-1}. Their key result states that these curvatures satisfy a dimension descent property (for suitable choices of k and n): if (M^n, g) has positive C_k curvature in the spectral sense, then a two-sided stable minimal hypersurface Σ^{n-1} has positive C_{k-1} curvature in the spectral sense. The case $k = n-1$ corresponds to the classical Schoen–Yau minimal hypersurface descent, and the case $k = 2$ gives the descent from the bi-Ricci curvature (C_2) to the Ricci curvature.

Recall that Gromov has the following conjecture relating scalar curvature to the macroscopic geometry of a manifold.

Conjecture 8 (Gromov)**.** *There is a constant $c = c(n)$ such that the following holds. Suppose (M^n, g) satisfies $R_g \geq 1$. Then the Urysohn $(n-2)$-width of M is bounded above by c.*

Here a metric space has Urysohn k-width bounded by c if it admits a continuous map to a k-dimensional simplicial complex such that the preimage of any point has diameter bounded by c. Of course, under the curvature conditions, one may also formulate variants of this conjecture by replacing 'diameter' with 'volume' and other geometric quantities.

We note that Gromov's conjecture fits well with the dimension descent property of curvature conditions. In fact, it is tempting to conjecture that (at least for suitably small n, k) a manifold (M^n, g) with C_k curvature larger than 1 has Urysohn $(k-1)$-width bounded by some constant $c(n, k)$. For example, when $k = 1$ this is the classical Myers' theorem on the diameter upper bound under Ric > 1. The recent solution of the Bernstein problem relies on our understanding of the case when $k = 2$. The case when $k \geq 3$ is likely to lead to new discoveries in geometry and topology. However, it seems very delicate and is wide open.

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