STABLE MINIMAL HYPERSURFACES VIA CONFORMAL METHOD AND APPLICATIONS

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Abstract. In this note we present the results on stable minimal hypersurfaces that we discussed at the conference *Extrinsic Geometric Analysis*, held at RIMA, Kyoto, in June 2025.

1. Introduction

1.1. **Minimal surfaces in $\mathbb{R}^3$.** A *minimal* surface $M^2 \subset \mathbb{R}^3$ is a critical point of the area functional $\mathcal{A}_t$, for all compactly supported variations, i.e.

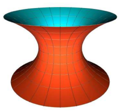
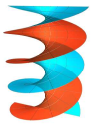
$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{A}_t = 0\,.$$

Equivalently:

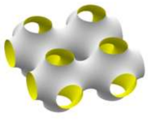
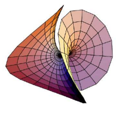
- M^2 is minimal $\iff$ the mean curvature $H \equiv 0$;
- M^2 is minimal $\iff$ M^2 can be expressed, **locally**, as the graph $\Gamma(u)$, where u solves the *minimal surfaces equation*:

$$\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0\,.$$

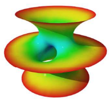
Classical examples: planes, catenoids, helicoids.

19th century examples (golden age): Schwarz minimal surfaces, Enneper surface, Henneberg surface, Bour's minimal surface, Neovius surface.

Modern examples: Gyroid, Costa's minimal surface, Chen-Gackstatter surface.



1.2. The Bernstein problem.

Theorem 1.1 (Bernstein, 1914). *Let $u \in C^2(\mathbb{R}^2)$ be a solution of*

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \quad \text{in } \mathbb{R}^2 .$$

Then u is an affine function, i.e.

$$u(x,y) = \alpha x + \beta y + \gamma ,$$

for some $\alpha, \beta, \gamma \in \mathbb{R}$. Equivalently, an entire (i.e. defined on the whole plane $\mathbb{R}^2$) minimal graph in $\mathbb{R}^3$ is necessarily a plane.

The proof is based on a Liouville-type theorem for elliptic (not uniformly elliptic) operators, which holds true only in dimension 2.

Bernstein problem in higher dimension: an entire minimal graph in $\mathbb{R}^{n+1}$ is necessarily a hyperplane? It resisted for a half-century and was solved thanks to the combined effort of:

- *Fleming (1965)*: new proof in the case $n = 2$;
- *De Giorgi (1965)*: $n = 3$;
- *Almgren (1966)*: $n = 4$;
- *Simons (1968)*: $n \leq 7$;

Their proofs are not based on Liouville-type theorems but on tools in Geometric Measure Theory.

- *Bombieri-De Giorgi-Giusti (1969)*: for $n \geq 8$, there are minimal entire graphs that are not hyperplanes.
- PDE proof for $3 \leq n \leq 7$? **Open**.

1.3. A natural extension.

Remark 1.2. *Two remarks:*

- *A minimal graph is area-minimizing: i.e. it is not only a critical point of the area functional, but also a minimum.*
  This is not true for general minimal hypersurfaces (that are not graphic).
- *Area-minimizing implies stability, that is*

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \mathcal{A}_t \geq 0 ,$$

  for all compactly supported variations.

Stable Bernstein problem: if $M^n \hookrightarrow \mathbb{R}^{n+1}$ is a complete, orientable, immersed, stable minimal hypersurface, does M have to be necessarily a hyperplane?

2. The stable Bernstein problem

- True if $n = 2$: *do Carmo-Peng (1979)*, *Fischer-Colbrie-Schoen (1980)*, *Pogorelov (1981)*;
- False if $n \geq 7$: if $n \geq 8$, *Bombieri-De Giorgi-Giusti (1969)* constructed non-flat, orientable, complete, stable, minimal hypersurfaces (graphs) in $\mathbb{R}^{n+1}$, $n \geq 8$. Moreover, if $n \geq 7$, there exist non-flat area minimizing (not graphs) smooth hypersurfaces constructed by *Bombieri-De Giorgi-Giusti (1969)* (*Hardt-Simon (1985)*).
- *Schoen-Simon-Yau (1975)*: true if $n \leq 5$ and **Euclidean volume growth** of geodesic balls:

$$\operatorname{Vol}(B_R) \leq C R^n .$$

- This result was extended up to dimension $n = 6$ by *Bellettini (2023)* under an *extrinsic* volume growth condition.
- True if $n = 3$: *Chodosh-Li (2021,2022)* and *C.-Mastrolia-Roncoroni (2022)*.
- True if $n = 4$ *Chodosh-Li-Minter-Striker (2024)* and if $n = 5$ *Mazet (2024)*.
- $n = 6$? Open.

Theorem 2.1 (Chodosh-Li, 2021, 2022 and C.-Mastrolia-Roncoroni, 2022)**.** *A complete, orientable, immersed, stable minimal hypersurface $M^3 \hookrightarrow \mathbb{R}^4$ is a hyperplane.*

Stable minimal hypersurface: M is *minimal* iff the mean curvature $H \equiv 0$; in this case we have that M is *stable* if and only if

$$\int_M |A|^2\varphi^2 \leq \int_M |\nabla\varphi|^2\,, \qquad \text{for all } \varphi \in C_0^\infty(M)$$

where $A = A_M$ denotes the second fundamental form of M. Hyperplanes have $A \equiv 0$.

Remark 2.2. *Stability means non-negativity of the second variation, or, equivalently, non-positivity of the* Jacobi operator

$$-\left(\Delta + |A|^2\right) \geq 0\,.$$

Before going into the proof, I will give you a 3-slides proof of *Schoen-Simon-Yau (1975)* result on stable minimal hypersurfaces $M^n \hookrightarrow \mathbb{R}^{n+1}$, $n \leq 5$, satisfying the Euclidean volume growth condition:

$$\mathrm{Vol}(B_R) \leq CR^n\,.$$

Remark 2.3. *By* Chodosh-Li*,* Chodosh-Li-Minter-Striker *and* Mazet*, the Euclidean volume growth condition holds.*

2.1. **Proof of *Schoen-Simon-Yau (1975) I.*** Take a complete, orientable, immersed, stable minimal hypersurface $M^n \hookrightarrow \mathbb{R}^{n+1}$. We have

$$\int_M |A|^2\varphi^2 \leq \int_M |\nabla\varphi|^2\,, \qquad \text{for all } \varphi \in C_0^\infty(M).$$

We test it with $\varphi = |A|^{1+q}\psi$, $q \geq 0$, with $\psi \in C_0^\infty(M)$, obtaining

$$\int_M |A|^{4+2q}\psi^2 \leq [(1+q)^2 + \varepsilon]\int_M |A|^{2q}|\nabla|A||^2\psi^2 + \frac{1+q}{\varepsilon}\int_M |A|^{2+2q}|\nabla\psi|^2,$$

for every $\varepsilon > 0$, where we used Young's inequality. On the other hand, using the Codazzi equation

$$\nabla_k A_{ij} = \nabla_j A_{ik}$$

and Gauss equation, we get the well known **Simons' identity** for minimal hypersurfaces in $\mathbb{R}^{n+1}$

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 - |A|^4.$$

Moreover, since A is a Codazzi tensor, we have the improved **Kato's inequality**

$$|\nabla A|^2 \geq \frac{n+2}{n}|\nabla|A||^2.$$

Combining these, we obtain

$$|A|\Delta|A| + |A|^4 \geq \frac{2}{n}|\nabla|A||^2.$$

Multiplying it by $|A|^{2q}\psi^2$ and integrating by parts, we get

$$\left(\frac{2}{n} + 1 + 2q - \varepsilon\right)\int_M |A|^{2q}|\nabla|A||^2\psi^2 \leq \int_M |A|^{4+2q}\psi^2 + \frac{1}{\varepsilon}\int_M |A|^{2+2q}|\nabla\psi|^2$$

for every $\varepsilon > 0$, where we used again Young's inequality. Since $q \geq 0$, for $\varepsilon > 0$ sufficiently small, combining these two estimates we obtain

$$\left\{1 - [(1+q)^2 + \varepsilon]\left(\frac{2}{n} + 1 + 2q - \varepsilon\right)^{-1}\right\} \int_M |A|^{4+2q}\psi^2 \leq C \int_M |A|^{2+2q}|\nabla\psi|^2.$$

Let $q := \frac{p-4}{2}$. For $\varepsilon > 0$ small enough, we have

$$1 - [(1+q)^2 + \varepsilon]\left(\frac{2}{n} + 1 + 2q - \varepsilon\right)^{-1} > 0$$

if $p \in [4, 4 + \sqrt{8/n}]$ and we finally obtain

$$\int_M |A|^p\psi^2 \leq C \int_M |A|^{p-2}|\nabla\psi|^2 \qquad \forall \psi \in C_0^\infty(M).$$

Taking $\psi = \phi^{p/2}$, by Holder's inequality, we get

$$\int_M |A|^p\phi^p \leq C \int_M |\nabla\phi|^p \qquad \forall \phi \in C_0^\infty(M)$$

for all $p \in [4, 4 + \sqrt{8/n}]$.

$$\int_M |A|^p\phi^p \leq C \int_M |\nabla\phi|^p \qquad \forall \phi \in C_0^\infty(M)$$

for all $p \in [4, 4 + \sqrt{8/n}]$. In particular, if $n \leq 5$, we can take $p = 5 + \delta$, for some $\delta > 0$ small. Let $x_0 \in M^n$, and let r denotes the distance function from x_0. We choose

$$\phi := \eta(r)\,,$$

where $0 \leq \eta \leq 1$, $\eta = 1$ on $[0, R]$, $\eta = 0$ on $[2R, \infty)$ and $|\eta'| \leq \frac{C}{R}$, for some $C > 0$, $\forall R > 0$. Plugging in the previous estimate, $\forall R > 0$, we obtain

$$\int_M |A|^{5+\delta}\eta^{5+\delta} \leq C \int_{B_{2R}\setminus B_R} |\nabla\eta|^{5+\delta} \leq \frac{C}{R^{5+\delta}}\mathrm{Vol}(B_{2R}) \leq C\,R^{n-5-\delta} \leq \frac{C}{R^\delta}$$

where we used the **Euclidean volume growth** assumption. Since $\delta > 0$, letting $R \to \infty$, we get

$$|A| \equiv 0 \quad \text{on } M^n,$$

and this concludes the proof. ☐

Remark 2.4. *By Gauss' equation, we know that the Ricci curvature of a minimal hypersurfaces $M^n \hookrightarrow \mathbb{R}^{n+1}$ satisfies:*

$$Ric_g = -A^2 \leq 0.$$

2.2. **Proof of the stable Bernstein problem in $\mathbb{R}^4$.** Idea of our proof:

- Step 1: we construct $\widetilde{g}$ conformal to g such that
$$\mathrm{Ric}^{2,f}_{\widetilde{g}} := \mathrm{Ric}_{\widetilde{g}} + \nabla^2_{\widetilde{g}} f - \frac{1}{2} df \otimes df \geq 0\,.$$
- Step 2: we prove that $\widetilde{g}$ is complete.
- Step 3: we obtain a weighted Bishop-Gromov volume estimate:
$$\mathrm{Vol}_f\left(B^{\widetilde{g}}_R\right) := \int_{B^{\widetilde{g}}_R} e^{-f}\,dV_{\widetilde{g}} \leq CR^{3+2} = CR^5\,. \tag{2.1}$$
- Step 4: we prove a weighted integral estimate as in SSY
- Step 5: we use a cutoff function $\phi = \eta(\widetilde{r})$ to conclude that
$$|A| \equiv 0 \quad \text{on } M^3.$$

Step 1: the conformal change of the metric. It is well-known that the stability of $M^n \hookrightarrow \mathbb{R}^{n+1}$ implies the existence of $0 < u \in C^\infty(M)$ satisfying

$$-\Delta_g u = |A|_g^2 u \quad \text{in } M,$$

where g denotes the induced metric on M. Let $k > 0$ and consider the conformal metric

$$\widetilde{g} = u^{2k} g\,.$$

Lemma 2.5. *Let $f = k(n-2)\log u$. Then the Ricci tensor of the metric $\widetilde{g}$ satisfies*

$$\mathrm{Ric}_{\widetilde{g}} + \nabla^2_{\widetilde{g}} f - \frac{1-k(n-2)}{k(n-2)^2} df \otimes df \geq \left(k - \frac{n-1}{n}\right) |A|_g^2 g\,.$$

In particular, if $n = 3$ and $k = \frac{2}{3}$, then the 2-Bakry-Emery-Ricci tensor satisfies

$$\mathrm{Ric}^{2,f}_{\widetilde{g}} := \mathrm{Ric}_{\widetilde{g}} + \nabla^2_{\widetilde{g}} f - \frac{1}{2} df \otimes df \geq 0\,.$$

Sketch. since $f = k(n-2)\log u$, we have

$$df = k(n-2)\frac{du}{u} \qquad \text{and} \qquad \nabla^2_g f = k(n-2)\left(\frac{\nabla^2_g u}{u} - \frac{du \otimes du}{u^2}\right),$$

which implies

$$\Delta_g f = k(n-2)\left(\frac{\Delta_g u}{u} - \frac{|\nabla_g u|_g^2}{u^2}\right).$$

On the other hand, from the standard formulas for a conformal change of the metric $\widetilde{g} = e^{2\varphi} g$, $0 < \varphi \in C^\infty(M)$, we get

$$\mathrm{Ric}_{\widetilde{g}} = \mathrm{Ric}_g - (n-2)\left(\nabla^2\varphi - d\varphi \otimes d\varphi\right) - \left[\Delta_g \varphi + (n-2)|\nabla_g \varphi|_g^2\right] g\,,$$

and

$$\nabla^2_{\widetilde{g}} f = \nabla^2_g f - (df \otimes d\varphi) + g(\nabla_g f, \nabla_g \varphi) g\,.$$

In our case $\varphi = k \log u$ and u solves $-\Delta_g u = |A|_g^2 u$, thus

$$\mathrm{Ric}_{\widetilde{g}} + \nabla^2_{\widetilde{g}} f = \mathrm{Ric}_g - \frac{df \otimes df}{n-2} + k|A|_g^2 g + \frac{|\nabla_g f|_g^2}{k(n-2)} g\,.$$

From Cauchy-Schwarz inequality we have

$$|\nabla_g f|_g^2 g \geq df \otimes df$$

and from Gauss equation in the minimal case we know that

$$\mathrm{Ric}_g = -A^2.$$

Moreover, since A is traceless (M is minimal), we also have the inequality

$$A^2 \leq \frac{n-1}{n}|A|_g^2 g\,.$$

Substituting in

$$\mathrm{Ric}_{\widetilde{g}} + \nabla^2_{\widetilde{g}} f = \mathrm{Ric}_g - \frac{df \otimes df}{n-2} + k|A|_g^2 g + \frac{|\nabla_g f|_g^2}{k(n-2)} g\,,$$

we conclude

$$\mathrm{Ric}_{\widetilde{g}} + \nabla^2_{\widetilde{g}} f - \frac{1-k(n-2)}{k(n-2)^2} df \otimes df \geq \left(k - \frac{n-1}{n}\right) |A|_g^2 g\,.$$

□

Step 2: completeness of $\widetilde{g}$. From now on, we take

$$n = 3\,, \quad k = \frac{2}{3}\,,$$

and $0 < u \in C^\infty(M)$ solution of

$$-\Delta_g u = |A|_g^2 u \quad \text{in } M.$$

Lemma 2.6. *The metric $\widetilde{g} = u^{\frac{4}{3}} g$ is complete.*

Sketch. as shown in *Fischer-Colbrie (1985)*, one can construct a minimizing geodesic in the metric $\widetilde{g} = u^{2k} g$, $\gamma = \gamma(s)$, where s is the $g-$arclength. By construction, the completeness of $\widetilde{g}$ is equivalent to prove that γ has infinite $\widetilde{g}-$length, i.e.

$$\int_\gamma d\widetilde{s} = \int_0^{+\infty} u^k(\gamma(s))\, ds = +\infty\,.$$

Since γ is minimizing, by the second variation formula

$$\int_0^\infty \left[(n-1)(\varphi_{\widetilde{s}})^2 - \widetilde{R}_{11}\varphi^2\right] d\widetilde{s} \geq 0\,, \quad \forall\, \varphi \in C_0^\infty(0,+\infty)$$

where $\widetilde{R}_{11}$ denotes the $\widetilde{g}-$Ricci curvature in the direction $\gamma_{\widetilde{s}}$. Using the formula for the conformal change of the Ricci curvature, some integration by parts (cfr. *Elbert-Nelli-Rosenberg (2007)*) and controlling the $|A|^2$ terms, we get

$$(n-1)\int_0^{+\infty} (\varphi_s)^2 u^{-k}\, ds \geq 2k(n-2)\int_0^{+\infty} \varphi\varphi_s u^{-k-1} u_s\, ds$$
$$+ k\left[1 - k(n-2)\right]\int_0^{+\infty} \varphi^2 u^{-k-2}(u_s)^2\, ds\,,$$

for $n \geq 3$, for every $\varphi \in C_0^\infty(0,+\infty)$ and for every $k \geq \frac{n-1}{n}$.
Now we choose $\varphi = u^k\psi$, with $\psi \in C_0^\infty(0,+\infty)$, we apply Young inequality, we manipulate and we obtain, assuming $k < 1$

$$0 \leq \left[\frac{k(t-1)^2}{1-k} - 2t + (n-1)\right]\int_0^{+\infty} u^k(\psi_s)^2\, ds - 2t\int_0^{+\infty} u^k\psi\psi_{ss}\, ds,$$

for every $t > 1$. If $n = 3$, $k = \frac{n-1}{n} = \frac{2}{3}$ and $t = \frac{3}{2}$ we have that

$$\frac{k(t-1)^2}{1-k} - 2t + (n-1) < 0\,,$$

hence,

$$0 \leq -\int_0^{+\infty} u^{\frac{2}{3}}(\psi_s)^2\, ds - 6\int_0^{+\infty} u^{\frac{2}{3}}\psi\psi_{ss}\, ds\,, \quad \forall\, \psi \in C_0^\infty(0,+\infty).$$

Hence,

$$0 \leq -\int_0^{+\infty} u^{\frac{2}{3}}(\psi_s)^2\, ds - 6\int_0^{+\infty} u^{\frac{2}{3}}\psi\psi_{ss}\, ds\,, \quad \forall\, \psi \in C_0^\infty(0,+\infty).$$

Finally, we choose $\psi = s\eta$, where $\eta = 1$ in $[0,R]$, $\eta = 0$ in $[2R,\infty)$, $0 \leq \eta \leq 1$ such that

$$|\eta_s| \leq \frac{C}{R} \quad \text{and} \quad |\eta_{ss}| \leq \frac{C}{R^2}\,.$$

Then

$$\int_0^R u^{\frac{2}{3}}\, ds \leq \int_0^\infty u^{\frac{2}{3}}\eta^2\, ds \leq C\int_R^\infty u^{\frac{2}{3}}\, ds\,,$$

and we conclude that

$$\int_0^{+\infty} u^{\frac{2}{3}}\, ds = +\infty\,,$$

i.e. the metric $\widetilde{g} = u^{\frac{4}{3}} g$ is complete. □

Step 3: weighted volume estimate. The two previous lemmas imply that the metric $\widetilde{g} = u^{\frac{4}{3}} g$ is complete and has non-negative 2-Bakry-Emery-Ricci curvature. Hence, by well-known comparison results e.g. in *Qian (1997)*, *Lott (2003)*, *Bakry-Qian (2005)*, *Wei-Wylie (2009)*, we obtain a weighted Bishop-Gromov volume estimate for a geodesic ball $B_R^{\widetilde{g}}(x_0)$.

Corollary 2.7. *Let $x_0 \in M^3$. Then, for every $R > 0$, there exists $C > 0$ such that*

$$\mathrm{Vol}_f\left(B_R^{\widetilde{g}}(x_0)\right) := \int_{B_R^{\widetilde{g}}(x_0)} e^{-f}\, dV_{\widetilde{g}} \leq CR^{3+2} = CR^5\,,$$

where $f = \frac{2}{3}\log u$. Equivalently, in terms of u and the volume form of g,

$$\int_{B_R^{\widetilde{g}}(x_0)} u^{\frac{4}{3}}\, dV_g \leq CR^5\,.$$

We have seen that Schoen-Simon-Yau proved the following estimate ($\delta > 0$ small)

$$\int_M |A|^{5+\delta}\phi^{5+\delta}\, dV_g \leq C\int_M |\nabla\phi|^{5+\delta}\, dV_g \qquad \forall \phi \in C_0^\infty(M).$$

Lemma 2.8. *For every $\delta > 0$ small enough, there exists $C > 0$ such that*

$$\int_M |A|^{5+\delta} u^{-2-\frac{2\delta}{3}}\phi^{5+\delta}\, dV_g \leq C\int_M u^{-2-\frac{2\delta}{3}}|\nabla\phi|^{5+\delta}\, dV_g \quad \forall\, \phi \in C_0^\infty(M).$$

Sketch. from *Schoen-Simon-Yau (1975)* proof

$$\int_M |A|^p\psi^2 \leq C\int_M |A|^{p-2}|\nabla\psi|^2 \qquad \forall\, \psi \in C_0^\infty(M),$$

for every $p \in [4, 4 + \sqrt{8/n}]$ and for some $C > 0$. We test this inequality with $\psi = u^\alpha\phi$, $\phi \in C_0^\infty(M)$, with u the positive solution to

$$-\Delta u = |A|^2 u$$

and $\alpha < 0$. Using this equation and Simons' identity, after *some* estimates, we get the result. □

Step 4: final estimate. Let $x_0 \in M^3$, and let $\widetilde{r}$ denotes the distance function from x_0 with respect to the metric $\widetilde{g} = u^{\frac{4}{3}} g$. By completeness, the closure of $\widetilde{g}$-balls are compact. We choose

$$\phi := \eta(\widetilde{r})\,,$$

where $0 \leq \eta \leq 1$, $\eta = 1$ on $[0, R]$, $\eta = 0$ on $[2R, \infty)$ and $|\eta'| \leq \frac{C}{R}$, for some $C > 0$, $\forall R > 0$. From the weighted integral estimate, we have ($|\widetilde{\nabla}\widetilde{r}|_{\widetilde{g}} = 1$)

$$\begin{aligned}\int_M |A|^{5+\delta} u^{-2-\frac{2\delta}{3}} \phi^{5+\delta} \, dV_g &\leq C \int_M u^{-2-\frac{2\delta}{3}} |\nabla \phi|_g^{5+\delta} \, dV_g \\ &= C \int_M u^{-2-\frac{2\delta}{3}+\frac{2(5+\delta)}{3}} |\widetilde{\nabla} \phi|_{\widetilde{g}}^{5+\delta} \, dV_g \\ &\leq \frac{C}{R^{5+\delta}} \int_{B^{\widetilde{g}}_{2R}(x_0)} u^{\frac{4}{3}} \, dV_g \\ &\leq \frac{C}{R^\delta} \, .\end{aligned}$$

Since $\delta > 0$, letting $R \to \infty$, we get

$$|A| \equiv 0 \quad \text{on } M^3,$$

and this concludes the proof of the stable Bernstein theorem in $\mathbb{R}^4$. □

3. Applications

The conformal technique that we used has been recently applied in different contexts. In particular we will focus on these applications

(1) Bonnet-Myers type result *[C.-Roncoroni (2023)].*
(2) Splitting theorem under spectral Ricci curvature bounds *[C.-Mari-Mastrolia-Roncoroni (2024)].*
(3) Classification of critical metrics to the quadratic curvature functional

$$\mathfrak{S}^2(g) = \int R_g^2 dV_g \, ,$$

R_g being the scalar curvature *[C.-Mastrolia-Monticelli (2023)].*

3.1. **A Bonnet-Myers type result I. Bonnet-Myers's theorem**: (M^n, g), $n \geq 3$, complete

$$\mathrm{Ric} \geq (n-1)\lambda g, \ \lambda > 0 \quad \Longrightarrow \quad \mathrm{diam}(M^n, g) \leq \frac{\pi}{\sqrt{\lambda}}. \tag{3.1}$$

In particular M^n is compact and has finite fundamental group. Generalization:

$$\mathrm{Ric}^{m,f} := \mathrm{Ric} + \nabla^2 f - \frac{1}{m} df \otimes df \geq (n+m-1)\lambda g, \ \lambda > 0 \quad \Longrightarrow \quad \text{(2) holds.}$$

Letting $f = -\log u$, the previous assumption reads

$$\mathrm{Ric} \geq \frac{\nabla^2 u}{u} + \left(1 + \frac{1}{m}\right) \frac{du \otimes du}{u^2} + (n+m-1)\lambda g \, .$$

It is well known that, if the Ricci (or the m-Bakry-Emery Ricci) tensor is *not uniformly positive*, the closeness of the manifold is not guaranteed. However, using a conformal method, we show that this is not the case if the potential u is a positive supersolution to a suitable elliptic PDE.

Theorem 3.1 (C.-Roncoroni (2023)). *Let (M^n, g), $n \geq 3$, be a complete Riemannian manifold such that*

$$\mathrm{Ric} \geq \alpha \frac{\nabla^2 u}{u} + \beta \frac{du \otimes du}{u^2} + \mathcal{Q} \quad \text{in } M \, ,$$

where $\alpha, \beta \in \mathbb{R}$, $\mathcal{Q}$ is a symmetric two tensor and $u \in C^\infty(M)$ satisfies

$$u > 0, \quad -\Delta u \geq V u + \gamma \frac{|\nabla u|^2}{u} \quad \text{in } M \, ,$$

where $\gamma \in \mathbb{R}$, $V \in C^\infty(M)$. *Assume that, there exists* $k \geq 0$ *such that*

$$\mathcal{Q} + kV\, g \geq (n-1)\lambda\, g, \qquad k\,(\gamma + 1 - \alpha) \geq 0, \qquad \alpha + \beta + k(\gamma+1) - (n-1)\frac{k^2}{4} > 0$$

for some $\lambda > 0$. *Then* M^n *is compact, has finite fundamental group and its diameter satisfies*

$$\mathrm{diam}(M^n, g) \leq \pi \sqrt{\frac{1}{\lambda}\left(1 + \frac{[2\alpha - k(n-3)]^2}{4(n-1)\left[\alpha + \beta + k(\gamma+1) - (n-1)\frac{k^2}{4}\right]}\right)}.$$

- By taking $u \equiv$ const and $V \equiv 0$ (or $\alpha = \beta = \gamma = k = 0$ and $\mathcal{Q} = (n-1)\lambda g$), this result recovers the classical Bonnet-Myers's theorem.
- By taking $\alpha = \beta = \gamma = 0$, $0 < k < \frac{4}{n-1}$ and $\mathcal{Q} = (n-1)\lambda g - kV\, g$, the assumptions can be written has

  $$\mathrm{Ric} \geq (n-1)\lambda\, g - kV\, g,\ -(\Delta + V) \geq 0 \iff -k\Delta + \lambda_{\mathrm{Ric}} \geq (n-1)\lambda.$$

  See also a recent result by *Antonelli-Xu (2024)*. The range for k is sharp.
- By taking u the stability function, $V = |A|^2$, $\alpha = \beta = \gamma = 0$, $k = \frac{n-1}{n}$ and $\mathcal{Q} = -A^2$ we can reprove the following

  Corollary 3.2 (C.-Mastrolia-Roncoroni, 2022). *If* (X^{n+1}, h) *is a closed* $(n+1)$*-dimensional,* $n \leq 5$, *manifold with non-negative sectional curvature and positive Ricci curvature, then there is no complete, orientable, immersed,* stable *minimal hypersurface* $M^n \hookrightarrow (X^{n+1}, h)$.

  In particular, there are no complete, orientable, stable minimal hypersurfaces of the round spheres $M^n \hookrightarrow (\mathbb{S}^{n+1}, g_{\mathrm{std}})$, provided $n \leq 5$.
  If $n > 5$? **Open**

3.2. **A Splitting type result. Cheeger-Gromoll's splitting theorem**: (M^n, g), $n \geq 3$, complete

$$\mathrm{Ric} \geq 0 \quad \Longrightarrow \quad \textit{either } M \textit{ has only one end or } M = \mathbb{R} \times N,\ N \text{ compact with } \mathrm{Ric}_N \geq 0.$$

Theorem 3.3 (C.-Mari-Mastrolia-Roncoroni, 2024). *Let* (M^n, g), $n \geq 3$, *be a complete Riemannian manifold and assume that there exists* $V \in C^\infty(M)$ *such that*

$$\mathrm{Ric} \geq -kV\, g,\ -(\Delta + V) \geq 0 \iff -k\Delta + \lambda_{\mathrm{Ric}} \geq 0$$

for some $0 < k < \frac{4}{n-1}$. *Then, either* M *has only one end or* $V \equiv 0$ *and* $M = \mathbb{R} \times N$, *where* N *is compact with* $\mathrm{Ric}_N \geq 0$.

- The result was also proved by *Antonelli-Pozzetta-Xu (2024)* using a different method. The range for k is sharp.
- Exploiting the limiting case in the above, we were able to prove an optimal characterization of the *catenoid* as the only complete, oriented, immersed, $\frac{1}{3}$-stable minimal hypersurface $M^3 \hookrightarrow \mathbb{R}^4$ with more than one end.

  $$(\delta\text{-stability}) \qquad \delta \int_M |A|^2 \varphi^2 \leq \int_M |\nabla \varphi|^2, \quad \text{for all } \varphi \in C_0^\infty(M)$$

  As a consequence, using *Hong-H.Li-Wang (2024)* we showed

Theorem 3.4 (C.-Mari-Mastrolia-Roncoroni, 2024). *A complete, orientable, properly immersed,* δ*-stable,* $\delta > \frac{1}{3}$, *minimal hypersurface* $M^3 \hookrightarrow \mathbb{R}^4$ *is a hyperplane.*

3.3. **Critical metrics of $\mathfrak{S}^2$.** Let M^n, $n \geq 3$, be a smooth manifold and consider the quadratic curvature functional

$$\mathfrak{S}^2(g) = \int_M R_g^2 dV_g \,.$$

Euler-Lagrange equations for critical metrics (variations with compact support):

$$\begin{cases} R\,\mathrm{Ric} - \nabla^2 R \,=\, \frac{3}{4(n-1)} R^2\, g \\ \Delta R \,=\, \frac{n-4}{4(n-1)} R^2 \end{cases}$$

Known results:

- $n \geq 3$, $R > 0 \implies$ no solutions, unless $n = 4$, g Einstein [*C. (2014)*];
- $n = 3$, $R \in L^2(M^3)$ (finite energy) $\implies R \equiv 0$ [*C.-Mastrolia-Monticelli (2021)*];
- $n = 4$, $R \in L^2(M^4) \implies R \equiv$ *const.*, thus $R \equiv 0$ or $R \equiv C \neq 0$ and g Einstein;
- $n \geq 5$, $R \in L^2(M^n)$, $R \geq -C \implies R \equiv 0$ [*C.-Mastrolia-Monticelli (2021)*].

By using a conformal method, we were able to prove the following:

Theorem 3.5 (C.-Mastrolia-Monticelli, 2023)**.** *Let (M^n, g), $n \geq 10$, be a complete critical metric of the functional $\mathfrak{S}^2$ with finite energy, i.e. $R_g \in L^2(M^n)$. Then (M^n, g) is scalar flat, and thus a global minimum of the functional $\mathfrak{S}^2$.*

Let (M^n, g), $n \geq 5$, be a critical metric of $\mathfrak{S}^2$ with finite energy, i.e. $R_g \in L^2(M^n)$:

$$R\,\mathrm{Ric} - \nabla^2 R \,=\, \frac{3}{4(n-1)} R^2\, g, \qquad \Delta R \,=\, \frac{n-4}{4(n-1)} R^2 \,.$$

- Either $R \equiv 0$ or $R < 0$ on M^n (Yau's argument and strong max. princ.).
- By contradiction, assume $R < 0$, then the conformal metric $\widetilde{g} = |R|^{\frac{6}{n-4}} g$ satisfies

  $$\mathrm{Ric}_{\widetilde{g}} + \nabla^2_{\widetilde{g}} f - \frac{n-10}{4(n-1)} df \otimes df = 0$$

  with $f = \frac{2(n-1)}{n-4} \log |R|$. A Quasi-Einstein metric.
- If $n \geq 10$, we prove that the metric $\widetilde{g} = |R|^{\frac{6}{n-4}} g$ is complete.
- In particular, the scalar curvature of $\widetilde{g}$ is nonnegative, $R_{\widetilde{g}} \geq 0$ [*Chen (2009), Wang (2011)*]. This is equivalent to the gradient estimate on the scalar curvature R

  $$|\nabla R|^2 \leq \frac{(n-4)^2}{6(n-1)(n+2)} |R|^3.$$

- In particular $|R|(x) \geq C(1 + \mathrm{d}(x, O))^{-2}$, and via integral estimates and the energy condition we reach the contradiction $R \equiv 0$. □
- $5 \leq n \leq 9$? **Open**. Without the finite energy assumption? **Open**.

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