A SURVEY ON BERNSTEIN-TYPE THEOREM FOR CONSTANT MEAN CURVATURE SURFACES IN ISOTROPIC 3-SPACE

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1. Introduction

The purpose of this paper is to announce the results of the joint work [1] of the author with S-D. Yang and W. Lee at Korea University, while surveying related research. The content of this paper is as follows: while surveying Bernstein-type theorems and related results for surfaces in Euclidean space and in Lorentz-Minkowski space, we introduce analogous global results for constant mean curvature surfaces in the three-dimensional isotropic space $\mathbb{I}^3$, which were established in [1].

Minimal surfaces in Euclidean 3-space $\mathbb{E}^3 = (\mathbb{R}^3, dt^2 + dx^2 + dy^2)$ and (spacelike) maximal surfaces in Lorentz-Minkowski 3-space $\mathbb{L}^3 = (\mathbb{R}^3, -dt^2 + dx^2 + dy^2)$ are closely related each other. In fact, if we introduce the following c-**deformation** $\{X_c\}_{c\in\mathbb{R}}$:

$$X_c(z) = \operatorname{Re}\int \left(2G, 1 - cG^2, \; -i(1 + cG^2)\right) F dz, \quad c \in \mathbb{R}. \tag{1}$$

Here, the pair of complex-valued functions (G, Fdz) is called the **Weierstrass data**, where F is a holomorphic function and G is a meromorphic function. The formula (1) gives Weierstrass representation formulas as follows:

- When $c = 1$, X_1 is a minimal surface in $\mathbb{E}^3$. See [24].
- When $c = -1$, X_{-1} is a maximal surface in $\mathbb{L}^3$. See [17].
- When $c = 0$, $X_0 = \operatorname{Re}\int (2G, 1, -i) F dz = (X_1 + X_{-1})/2$ is a zero mean curvature surface in the **isotropic 3-space** $\mathbb{I}^3 = (\mathbb{R}^3, dx^2 + dy^2)$. See [21].

For values of c other than the specific ones mentioned above, the surface X_c also has zero mean curvature in the following sense.

Proposition 1.1 ([2])**.** *If FG^2 is holomorphic and $c|G|^2 \not\equiv -1$, then X_c is a zero mean curvature surface with positive definite induced metric in $\mathbb{R}^3(c) := (\mathbb{R}^3, cdt^2 + dx^2 + dy^2)$.*

Example 1.2 (Enneper-type surfaces)**.** If take the Weierstrass data $(G, Fdz) = (z,\ dz)$, $z = x + iy \in \mathbb{C}$, then we

$$X_c(x + iy) = \left(x^2 - y^2, x - \frac{c}{3}x^3 + cxy^2, y + cx^2y - \frac{c}{3}y^3\right)$$

The surfaces X_1 and X_{-1} are called Enneper's minimal surface in $\mathbb{E}^3$ and Enneper's maximal surface in $\mathbb{L}^3$, respectively. When $c = 0$, the **hyperbolic paraboloid** $X_0 = (x^2 - y^2, x, y)$ appears exactly in the middle of X_1 and X_{-1}. In particular, it is an entire graph in $\mathbb{I}^3$, that is, a graph of a function defined on the whole plane $\mathbb{R}^2$. See Figure 1.

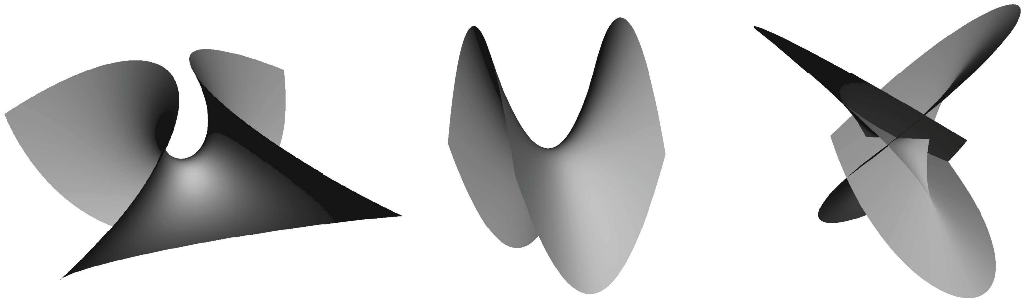

FIGURE 1. Enneper's minimal surface X_0 in $\mathbb{E}^3$ (left) and its deformations X_0 in $\mathbb{I}^3$ (center) and X_{-1} in $\mathbb{L}^3$ (right).

In this way, surfaces in the isotropic space $\mathbb{I}^3$ naturally arise in close connection with surfaces in $\mathbb{E}^3$ and $\mathbb{L}^3$. In this paper, we introduce the Bernstein-type theorems for constant mean curvature surfaces in $\mathbb{I}^3$ established in [1], while comparing them with the corresponding results in $\mathbb{E}^3$ and $\mathbb{L}^3$.

2. Bernstein-type theorems in $\mathbb{E}^3$ and $\mathbb{L}^3$

In this section, we summarize some of the known results, including Bernstein-type theorems for entire graphs and results on complete zero mean curvature and constant mean curvature surfaces in Euclidean space and Lorentz-Minkowski space.

2.1. The case of zero mean curvature $H = 0$.

The following Bernstein theorem was originally proved by Bernstein [4] in 1915-1917.

Fact 2.1 ([4]). *Any entire minimal graph in $\mathbb{E}^3$ is a plane.*

Bernstein's problem in Euclidean n-space $\mathbb{E}^n$ was also solved affirmatively when $n \leq 8$ by Fleming [13], De Giorgi [11], Almgren [3] and Simons [20], and solved negatively when $n \geq 9$ by Bombieri-Giorgi-Giusti [5].

A similar result was also proved for maximal surfaces in $\mathbb{L}^3$ as follows.

Fact 2.2 ([6]). *Any entire maximal graph in $\mathbb{L}^3$ is a plane.*

While there are many complete minimal surfaces in $\mathbb{E}^n$, the following result for complete maximal surfaces in $\mathbb{L}^n$ was proved by Calabi [6] and Cheng-Yau [7]:

Fact 2.3 ([6, 7]). *Any complete maximal hypersurface in Lorentz-Minkowski n-space $\mathbb{L}^n$ is a spacelike hyperplane for arbitrary $n \geq 3$.*

2.2. The case of constant mean curvature $H \neq 0$.

For constant mean curvature $H(\neq 0)$ surfaces, Heinz [14] proved the following estimation of H.

Fact 2.4 ([14]). *Let $t = f(x, y)$ be the graph in $\mathbb{E}^3$ of a C^2-function f defined over the disk $x^2 + y^2 < R^2$. If the mean curvature H satisfies the inequality $|H| \geq c$ for some $c > 0$, then $c \leq \dfrac{1}{R}$.*

By fixing the constant c to be $c = |H|$ and letting the constant R tend to $+\infty$, we obtain the following Bernstein-type theorem.

Corollary 2.5. *Let $t = f(x,y)$ be an entire constant mean curvature H graph in $\mathbb{E}^3$. Then H must be zero. In particular, the surface must be a plane.*

Heinz-type inequality was generalized to hypersurfaces in $\mathbb{E}^n$ by Chern [8] and Flanders [12].

On the other hand, it is known, due to Treibergs [22], that there exist plenty of entire graphs with constant mean curvature in Minkowski space.

Fact 2.6 ([22])**.** *There are many entire spacelike constant mean curvature $H \neq 0$ graphs in $\mathbb{L}^n$ other than the totally umbilic hyperboloid $-x_0^2+x_1^2+\cdots+x_{n-1}^2 = -\frac{1}{H^2}$, $x_0 > 0$.*

For example, the following hyperbolic cylinder provides an example of an entire constant mean curvature graph in $\mathbb{L}^3$.

$$-x_0^2 + x_1^2 = -\frac{1}{4H^2}, \quad x_0 > 0. \tag{2}$$

By Fact 2.6, a Heinz-type inequality, which induces a vanishing theorem of mean curvature, does not hold without any assumption. However, a Heinz-type mean curvature estimation under a gradient bound is recently proved by Honda-Kawakami-Koiso-Tori [15].

3. Results in $\mathbb{I}^3$

Based on the background from the previous section, in this section we present a Bernstein-type theorem for surfaces in $\mathbb{I}^3$ established in [1].

3.1. **Geometry in $\mathbb{I}^3$.**

The isotropic 3-space $\mathbb{I}^3$ can be identified with the lightlike hyperplane

$$\{(x_0, x_1, x_2, x_3) \in \mathbb{L}^4 : x_0 - x_3 = 0\}$$

of the Lorentzian 4-space $\mathbb{L}^4$ with the metric

$$\langle (x_0, x_1, x_2, x_3), (y_0, y_1, y_2, y_3) \rangle := -x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3.$$

The map

$$\mathbb{R}^3 \to \mathbb{L}^4, \qquad (l, x, y) \mapsto (l, x, y, l)$$

provides a global coordinate chart for the lightlike hyperplane, through which we identify $\mathbb{I}^3$ with $\mathbb{R}^3$ equipped with the coordinates (l, x, y), the pullback metric

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2. \tag{3}$$

For any spacelike immersion $X\colon \Sigma \to \mathbb{I}^3$, there is a unique map $G\colon \Sigma \to \mathbb{L}^4$ with

$$\langle G, G \rangle = \langle G, dX \rangle = \langle G, \mathfrak{p} \rangle - 1 = 0, \qquad \text{where} \quad \mathfrak{p} := (1, 0, 0, 1).$$

The map G is called the **lightlike Gauss map** of X. On an isothermal coordinate system (u, v) such that

$$\mathrm{d}s^2 = e^{2\sigma}(\mathrm{d}u^2 + \mathrm{d}v^2) = e^{2\sigma}\,\mathrm{d}z\,\mathrm{d}\bar{z}, \qquad \text{where} \quad z := u + iv$$

for some function $\sigma\colon \Sigma \to \mathbb{R}$, we have

$$H = 2e^{-2\sigma}\langle G, X_{z\bar{z}} \rangle, \qquad Q\,\mathrm{d}z^2 = \langle G, X_{zz} \rangle\,\mathrm{d}z^2, \qquad K = H^2 - 4Q\bar{Q}e^{-4\sigma}$$

for the mean curvature, the Hopf differential, and the Gaussian curvature of X, respectively. Note that $H^2 - K \geq 0$ and the Gauss equation is written as

$$\sigma_{z\bar{z}} = 0.$$

This implies that $\mathrm{d}s^2$ is a flat metric and hence the Gaussian curvature K is not intrinsic.

For the graph of $l = f(x,y)$, H and K are written as

$$H = \frac{1}{2}(f_{xx} + f_{yy}), \qquad K = f_{xx}f_{yy} - f_{xy}^2. \tag{4}$$

We remark that the graph of a smooth function is always spacelike.

3.2. **Weierstrass-type formula for CMC surfaces in $\mathbb{I}^3$.** Let us recall the Weierstrass-type representation formula for constant mean curvature (CMC, for short) H surfaces in $\mathbb{I}^3$ proved by Strubecker [21] for $H = 0$ and Cho-Lee-Lee-Yang [10] for non-zero H.

Fact 3.1 ([21, 10])**.** *Any constant mean curvature H immersion X can locally be represented as*

$$X(z) = \Re \int \left(\overline{h_1} + h_2, 1, -i\right)\omega, \quad h_1 := H\int \omega, \tag{5}$$

where h_2 is a holomorphic function, $\omega = \hat{\omega}dz$ is a nowhere vanishing holomorphic 1-form. We call the pair (h_2, ω) is the **Weierstrass data** *of X.*

3.3. **Bernstein-type theorem in $\mathbb{I}^3$.** The following property concerning the value distribution of the Gauss curvature holds for complete constant mean curvature surfaces in $\mathbb{I}^3$.

Theorem 3.2 ([1])**.** *If a connected complete spacelike constant mean curvature H surface in $\mathbb{I}^3$ has non-constant Gaussian curvature K, then K must take all values less than H^2.*

In particular, if the Gaussian curvature K of a connected complete spacelike constant mean curvature H surface has an exceptional value less than H^2, then K is constant and the surface is either one of the following.

(1) *When $H = 0$, the surface is either a plane or a rectangular hyperbolic paraboloid.*
(2) *When $H \neq 0$, the surface is either a cylinder, an elliptic paraboloid, or a non-rectangular hyperbolic paraboloid.*

Here, the image of the graph $z = \alpha x^2 + \beta y^2$ by any isometry in $\mathbb{I}^3$ is called a **hyperbolic paraboloid** if $\alpha\beta < 0$, a **rectangular hyperbolic paraboloid** if $\alpha = -\beta \neq 0$, an **elliptic paraboloid** if $\alpha\beta > 0$ or a **circular paraboloid** if $\alpha = \beta \neq 0$.

Example 3.3 (Enneper-type surface and its deformation)**.** Let us consider Weierstrass data $(h_2, \omega) = (z, dz)$ of Enneper's minimal surface of order 2. Then the surface (5) is computed as

$$\begin{aligned} X(z) &= \Re \int (H\overline{z} + z, 1, -i)\, dz \\ &= \left(H\frac{x^2+y^2}{2} + \frac{x^2-y^2}{2}, x, y \right) \\ &= \left(\frac{H^2+1}{2}x^2 + \frac{H^2-1}{2}y^2, x, y \right). \end{aligned}$$

Therefore, the surface X is a hyperbolic paraboloid when $0 \leq H < 1$, a cylinder when $H = 1$ or an elliptic paraboloid when $H > 1$.

Moreover, as proved by Sato [19], completeness of a surface in $\mathbb{I}^3$ induces the fact that the surface is an entire graph.

Fact 3.4 ([19, Theorem 5.1]). *If a connected immersed surface $S \subset \mathbb{I}^3$ is complete, then the projection*

$$\begin{array}{cccc} \pi\colon & S \subset \mathbb{I}^3 & \longrightarrow & \mathbb{R}^2 \\ & \cup\!\!\!| & & \cup\!\!\!| \\ & (l, x, y) & \longmapsto & (x, y) \end{array}$$

gives an isometry from the surface $S \subset \mathbb{I}^3$ onto $\mathbb{R}^2 = \pi(S)$ with the metric dx^2+dy^2. In particular, the surface S is an entire graph of the form $\ell = f(x, y)$ for some smooth function f defined on $\mathbb{R}^2$.

Finally, by using Theorem 3.2 and Fact 3.4, we obtain Bernstein-type theorems as follows.

Corollary 3.5 ([1], Bernstein-type theorem for $H \equiv 0$). *If an entire zero mean curvature graph $\ell = f(x, y)$ has bounded Gaussian curvature K, then K must be constant and the surface is either a plane or a rectangular hyperbolic paraboloid.*

Corollary 3.6 ([1], Bernstein-type theorem for CMC $H \neq 0$). *If an entire non-zero constant mean curvature H graph $\ell = f(x, y)$ has bounded Gaussian curvature K, then K must be constant and the surface is a cylinder, an elliptic paraboloid, or a non-rectangular hyperbolic paraboloid.*

Regarding the above corollaries, we give some remarks.

- If we assume the boundedness of f instead of that of K, the conclusion is that f must be constant, i.e., the graph of f is a horizontal plane.
- Since any spacelike surface always satisfies the relation $K \leq H^2$, boundedness of K means $C_0 \leq K \leq H^2$ for some constant C_0.

3.4. **Comparison with the case of $\mathbb{L}^3$.** Although no complete non-planar constant mean curvature surface in Euclidean space is an entire graph, the situation is different in Lorentz-Minkowski space. At the end of this paper, we briefly present some results in $\mathbb{L}^3$ related to those introduced in the previous subsection for $\mathbb{I}^3$.

The following result was proved by Cheng-Yau [7], Choi-Treibergs [9] and Wan [23].

Fact 3.7 ([7, 9, 23])**.** *Any complete spacelike surface in* $\mathbb{L}^3$ *with constant mean curvature* H *is an entire graph of the spacelike* x_1x_2*-plane and has non-positive Gaussian curvature* K*.*

Let κ_i $(i = 1, 2)$ be the principal curvatures. Since the mean and Gaussian curvatures H and K of a spacelike surface in $\mathbb{L}^3$ satisfies

$$H^2 + K = \left(\frac{\kappa_1 + \kappa_2}{2}\right)^2 + (-\kappa_1\kappa_2) = \left(\frac{\kappa_1 - \kappa_2}{2}\right)^2 \geq 0,$$

we can see boundedness of K of complete spacelike surfaces in $\mathbb{L}^3$:

$$-H^2 \leq K \leq 0.$$

Here, on a spacelike surface in $\mathbb{L}^3$, a point with $K = -H^2$ corresponds to an umbilic point, and the existence and non-existence of such points has a significant influence on the global geometry of spacelike complete constant mean curvature H surfaces in $\mathbb{L}^3$. In particular, the following has been shown by T.K. Milnor [18] and Yamada [25] (see also Kawakami-Satake [16] for further details).

Fact 3.8 ([18, 25])**.** *If a complete spacelike surface in* $\mathbb{L}^3$ *with constant mean curvature* $H \neq 0$ *satisfies* $H^2 + K \geq \varepsilon$ *for some* $\varepsilon > 0$*, then the surface is the hyperbolic cylinder* (2)*.*

Fact 3.8 asserts that for any complete spacelike constant mean curvature H surface in $\mathbb{L}^3$ which is not totally umbilic and not the hyperbolic cylinder, K takes values arbitrary close to $-H^2$.

On the other hand, Theorem 3.2 asserts that the image of Gaussian curvature K of a complete constant mean curvature H surface is either

$$\{\text{single point}\}, \quad (-\infty, H^2) \quad \text{or} \quad (-\infty, H^2].$$

These correspond to the cases of

- constant (or bounded) Gaussian curvature,
- unbounded Gaussian curvature without umbilic points, and
- unbounded Gaussian curvature with an umbilic point,

respectively. In other words, it follows that the Gaussian curvature K of every surface, other than those listed in (1) and (2) of Theorem 3.2, necessarily attains all values less than H^2.

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