

DUAL OPTIMIZATION PROBLEMS CONCERNING FIRST EIGENVALUE AND SMOOTH MAPS

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ABSTRACT. In this note, we report our recent work on a dual pair of optimization problems: one concerning smooth maps from a compact manifold into a Hilbert space, and the other concerning the first eigenvalue of the Bakry-Émery Laplacian. We also report recent progress on equivariant versions of these optimization problems.

INTRODUCTION

Let M be a compact manifold equipped with a volume element and a Riemannian metric. In the preprint [5], we formulate and study a dual pair of optimization problems, one maximizing the variance of a smooth map from the manifold into a Hilbert space and the other maximizing the first (nonzero) eigenvalue of the Bakry-Émery Laplacian.

In this note, we first review the general framework and some results (Section 1). We then explain in detail that these problems can be solved explicitly for all flat metrics on the two-dimensional torus (Section 2). Finally, we discuss the equivariant versions of our optimization problems and speculate that the equivariant minimal embedding of the hyperbolic space into a Hilbert sphere, introduced by Besson-Courtois-Gallot [2], would be an inflated equivariant map.

1. DUAL OPTIMIZATION PROBLEMS

Let M be a compact manifold. We assume our M is connected and has no boundary. We equip M with a smooth volume element $d\mu$, which we assume has unit total mass for simplicity, and a Riemannian metric h . A standard choice of volume element is $d\mu = d\mu_h/\text{Vol}(h)$.

Let \mathcal{H} be a Hilbert space. We denote the inner product and norm of \mathcal{H} by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. We first introduce the following *variance maximization problem*:

Problem 1.1. Over all C^∞ -maps satisfying the constraints

$$(1) \quad \varphi^* h_{\mathcal{H}} \leq h \quad \text{and} \quad \int_M \varphi \, d\mu = 0,$$

maximize the variance of φ :

$$\text{var}(\varphi) := \int_M \|\varphi\|^2 \, d\mu.$$

Here, $h_{\mathcal{H}}$ denotes the Riemannian metric of \mathcal{H} , extending the inner product $\langle \cdot, \cdot \rangle$ by parallel translations. Note that the latter constraint in (1) is not restrictive, since it can be achieved by composing a translation in \mathcal{H} to any given map. Set

$$\text{Var}(d\mu, h) := \sup \text{var}(\varphi),$$

where sup is taken over all C^∞ -maps φ satisfying (1).

Definition 1. We call a solution map to Problem 1.1 an *inflated map*.

An inflated map, if it exists, is a globally most expanding map among all locally shrinking maps.

It turns out that Problem 1.1 has a ‘dual’ problem. We start by describing Problem 1.1 in a min-max style. Let \mathcal{F}_0 denote the set of all C^∞ -maps φ from M into \mathcal{H} with average equal to the zero vector. Let \mathcal{M}^* denote the set of all positive semidefinite metrics g^* on the cotangent bundle T^*M . Define a function $L: \mathcal{F}_0 \times \mathcal{M}^* \rightarrow \mathbb{R}$ by

$$L(\varphi, g^*) := \int_M \|\varphi\|^2 \, d\mu + \int_M (g^*, h - \varphi^* h_{\mathcal{H}}) \, d\mu,$$

where (\cdot, \cdot) denotes the natural pairing of contravariant and covariant 2-tensors. The function L is called the Lagrange function associated to Problem 1.1. In fact, g^* will play the role of Lagrange multiplier.

Fix $\varphi \in \mathcal{F}_0$. Since φ satisfies the former constraint in (1) if and only if

$$\int_M (g^*, h - \varphi^* h_{\mathcal{H}}) \, d\mu \geq 0$$

holds for all $g^* \in \mathcal{M}^*$, we have

$$\inf_{g^* \in \mathcal{M}^*} L(\varphi, g^*) = \begin{cases} \int_M \|\varphi\|^2 d\mu & \text{if } \varphi \text{ satisfies the former constraint in (1),} \\ -\infty & \text{otherwise.} \end{cases}$$

Therefore, Problem 1.1 is identical to the problem finding

$$\sup_{\varphi \in \mathcal{F}_0} \inf_{g^* \in \mathcal{M}^*} L(\varphi, g^*).$$

We shall identify the problem finding

$$\inf_{g^* \in \mathcal{M}^*} \sup_{\varphi \in \mathcal{F}_0} L(\varphi, g^*),$$

which is the Lagrangian dual of Problem 1.1. Note that the Lagrange function L is rearranged as

$$L(\varphi, g^*) = \int_M (g^*, h) d\mu + \left(- \int_M \|d\varphi\|_{g^*}^2 d\mu + \int_M \|\varphi\|^2 d\mu \right),$$

where we have used the equality $(g^*, \varphi^* h_{\mathcal{H}}) = \|d\varphi\|_{g^*}^2$.

To proceed, we introduce the following quantity: For $g^* \in \mathcal{M}^*$, set

$$\lambda_1(d\mu, g^*) := \inf \frac{\int_M |du|_{g^*}^2 d\mu}{\int_M u^2 d\mu},$$

where inf is taken over all nonzero C^1 -functions u satisfying $\int_M u d\mu = 0$. The quantity $\lambda_1(d\mu, g^*)$ is nothing but the first nonzero eigenvalue of the (positive) Bakry-Émery Laplacian $-\Delta_{(d\mu, g^*)}$. When g^* is positive definite, it is defined by

$$-\Delta_{(d\mu, g^*)} u = -\Delta_g u + g^*(df, du), \quad u \in C^\infty(M),$$

where Δ_g is the Riemannian Laplacian with respect to g and f is determined by the relation $d\mu_g = e^f d\mu$. (See [5] for the definition when g^* is only positive-semidefinite.) Note that the function $g^* \in \mathcal{M}^* \mapsto \lambda_1(d\mu, g^*) \in \mathbb{R}$ is concave.

We return to identifying the dual of Problem 1.1. Fix $g^* \in \mathcal{M}^*$. Since $\lambda_1(d\mu, g^*) \geq 1$ if and only if

$$- \int_M \|d\varphi\|_{g^*}^2 d\mu + \int_M \|\varphi\|^2 d\mu \leq 0, \quad \forall \varphi \in \mathcal{F}_0,$$

we obtain

$$\sup_{\varphi \in \mathcal{F}_0} L(\varphi, g^*) = \begin{cases} \int_M (g^*, h) d\mu & \text{if } \lambda_1(d\mu, g^*) \geq 1, \\ \infty & \text{otherwise.} \end{cases}$$

Therefore, the problem finding $\inf_{g^* \in \mathcal{M}^*} \sup_{\varphi \in \mathcal{F}_0} L(\varphi, g^*)$ coincides with the following

Problem 1.2. *Over all $g^* \in \mathcal{M}^*$ satisfying the constraint*

$$\lambda_1(d\mu, g^*) \geq 1,$$

minimize

$$\int_M (g^*, h) d\mu.$$

This problem is equivalent to the following *first-eigenvalue maximization problem*:

Problem 1.3. *Over all $g^* \in \mathcal{M}^*$, maximize*

$$\lambda_1(d\mu, g^*) / \int_M (g^*, h) d\mu.$$

Set

$$\Lambda_1(d\mu, h) := \sup_{g^* \in \mathcal{M}^*} \lambda_1(d\mu, g^*) / \int_M (g^*, h) d\mu.$$

By the obvious inequality

$$\sup_{\varphi \in \mathcal{F}} \inf_{g^* \in \mathcal{M}^*} L(\varphi, g^*) \leq \inf_{g^* \in \mathcal{M}^*} \sup_{\varphi \in \mathcal{F}} L(\varphi, g^*),$$

we obtain the *weak-duality inequality*:

$$(2) \quad \text{Var}(d\mu, h) \leq \frac{1}{\Lambda_1(d\mu, h)}.$$

While the idea of duality is useful for identifying the dual of Problem 1.1, the inequality (2) can be proved directly without referring to the duality. In fact, we have the following

Proposition 1.4. *Let M be a compact manifold equipped with a smooth volume element $d\mu$ and a Riemannian metric h . Then, for any $\varphi \in \mathcal{F}_0$ satisfying (1) and $g^* \in \mathcal{M}^*$, we have*

$$(3) \quad \int_M \|\varphi\|^2 d\mu \leq \int_M (g^*, h) d\mu / \lambda_1(d\mu, g^*).$$

Equality holds in (3) if and only if the map φ consists of functions minimizing the Rayleigh quotient RQ over all functions with mean value zero and

$$(g^*, h - \varphi^* h_{\mathcal{H}}) \equiv 0$$

holds. If g^ is positive definite at every point of M , then these equality conditions are rephrased as follows: the map φ consists of first eigenfunctions of $-\Delta_{(d\mu, g^*)}$ and φ is an isometric immersion with respect to h .*

Note that in the last statement of the proposition, the image of φ lies in a finite dimensional subspace of \mathcal{H} since the multiplicity of the first eigenvalue is finite.

Problem 1.2, which is equivalent to Problem 1.3, is to minimize the linear functional $g^* \mapsto \int_M (g^*, h) d\mu$ over the convex set $\{g^* \in \mathcal{M}^* \mid \lambda_1(d\mu, g^*) \geq 1\}$. This convexity, combined with an averaging argument, implies the following

Proposition 1.5. *Let M be a compact manifold equipped with a smooth volume element $d\mu$ and a Riemannian metric h . Let G be the automorphism group of $(M, d\mu, h)$, that is, the set of all diffeomorphisms of M which preserve $d\mu$ and h . If there exists a solution to Problem 1.3, then there also exists a G -invariant solution to Problem 1.3.*

The following theorem asserts that if a solution to Problem 1.3 exists and is positive definite everywhere, then a map consisting of first eigenfunctions of the corresponding Bakry-Émery Laplacian gives a solution to Problem 1.1. It is an analogue of the Nadirashvili minimal surface theorem [4].

Theorem 1.6. *Suppose that a positive definite metric g^* on T^*M is a solution to Problem 1.3 associated with $(d\mu, h)$. Then there exist first eigenfunctions u_1, \dots, u_N of $-\Delta_{(d\mu, g^*)}$ such that the map $\varphi = (u_1, \dots, u_N): M \rightarrow \mathbb{R}^N$ is an isometric immersion with respect to the metric h . Therefore, φ is an inflated map with respect to $(d\mu, h)$.*

2. EXAMPLES

We discuss examples for which our optimization problems can be solved explicitly. We begin with the cases that Problem 1.3 associated with $(d\mu_h/\text{Vol}(h), h)$ is *self-solvable*, that is, that the dual metric h^* gives a solution to the problem. Note that, under $d\mu = d\mu_h/\text{Vol}(h)$ and $g^* = h^*$, the equality condition for (3) reduces to the following: the map φ consists of first eigenfunctions of $-\Delta_h$ and is an isometric immersion with respect to h . Then by the Takahashi theorem [7], this condition is equivalent to the one that φ is an isometric minimal immersion into a sphere of radius $\sqrt{n/\lambda_1(h)}$, where $n = \dim M$, by first eigenfunctions. Therefore, we have

Proposition 2.1. *The metric $g^* = h^*$ provides a solution to Problem 1.3 associated with $(d\mu_h/\text{Vol}(h), h)$ and a map φ is an inflated map with respect to $(d\mu_h/\text{Vol}(h), h)$ if and only if φ is an isometric minimal immersion of (M, h) into a round sphere by first eigenfunctions of $-\Delta_h$.*

By this proposition and Theorem 1.6, we can characterize the metrics h for which Problem 1.3 associated with $(d\mu_h/\text{Vol}(h), h)$ is self-solvable.

Corollary 2.2. *Let (M, h) be a compact Riemannian manifold. Then $g^* = h^*$ gives a solution to Problem 1.3 associated with $(d\mu_h/\text{Vol}(h), h)$ if and only if (M, h) admits an isometric minimal immersion into a sphere (of some radius) by first eigenfunctions of $-\Delta_h$.*

Many examples of compact Riemannian manifolds that admit an isometric minimal immersion into a sphere by first eigenfunctions are known. These include all compact isotropy irreducible Riemannian homogeneous spaces [7]. Yau [9] conjectured that the inclusion map of any compact embedded minimal hypersurface in a sphere was such an immersion.

Other than compact Riemannian manifolds admitting an isometric minimal immersion into a sphere by first eigenfunctions, we can solve the problems (with $d\mu = d\mu_h/\text{Vol}(h)$) for all flat metrics on the 2-torus and for all left-invariant metrics on $\text{SU}(2)$; The latter class contains the so-called Berger metrics on the 3-sphere.

We discuss the case of flat metrics on the 2-torus in detail, as they provide the simplest nontrivial (that is, Problem 1.3 is not self-solvable) examples. It is well-known that any flat metric on the 2-torus is isometric to one of those induced on $\mathbb{R}^2/\mathbb{Z}(1,0) \oplus \mathbb{Z}(a,b)$, the quotient space of the Euclidean plane by the lattice $\mathbb{Z}(1,0) \oplus \mathbb{Z}(a,b)$, with $0 \leq a \leq 1/2$, $b \geq \sqrt{1-a^2}$. In order to work with common coordinates, we pull back these metrics to $\mathbb{R}^2/\mathbb{Z}^2$ by the linear isomorphism fixing $(1,0)$ and sending $(0,1)$ to (a,b) and obtain $h_{a,b} = dx^2 + 2adxdy + (a^2 + b^2)dy^2$. These metrics have the common normalized volume element $d\mu = dxdy$. We will solve Problems 1.1 and 1.3 associated with $(d\mu, h_{a,b})$. It is well-known (cf. [1]) that the first nonzero eigenvalue $\lambda_1(h_{a,b})$ of $-\Delta_{h_{a,b}}$ is given by $\lambda_1(h_{a,b}) = \frac{4\pi^2}{b^2}$ with multiplicity

$$\begin{cases} 6, & \text{if } (a,b) = (1/2, \sqrt{3}/2), \\ 4, & \text{if } b = \sqrt{1-a^2} \text{ and } a \neq 1/2, \\ 2, & \text{if } b > \sqrt{1-a^2}. \end{cases}$$

As a working hypothesis, we suppose that there would exist a solution $g^* \in \mathcal{M}^*$ to Problem 1.3 and g^* be positive definite. Then by Theorem 1.6, there would also exist an inflated isometric immersion with respect to h , consisting of first eigenfunctions of $-\Delta_{(d\mu, g^*)}$. Since flat metrics on the 2-torus, which is an abelian Lie group, are left-invariant, by Proposition 1.5, there would also exist a left-invariant, hence flat, metric that gives an optimal solution to Problem 1.3. Therefore, we search for a flat metric whose dual metric minimizes the objective functional

$$\Phi(g^*) := \int_{T^2} (g^*, h_{a,b}) d\mu / \lambda_1(d\mu, g^*).$$

Indeed, for $g^* = h_{u,v}^*$, we have

$$\Phi(h_{u,v}^*) = \frac{(u-a)^2 + v^2 + b^2}{v^2} / \frac{4\pi^2}{v^2} = \frac{(u-a)^2 + v^2 + b^2}{4\pi^2}.$$

In the range $0 \leq u \leq 1/2$, $v \geq \sqrt{1-u^2}$, this attains minimum when

$$\begin{cases} u = 1/2, v = \sqrt{3}/2 & \text{if } 0 < a \leq 1/2, \\ 0 \leq u \leq 1/2, v = \sqrt{1-u^2} & \text{if } a = 0. \end{cases}$$

Thus, we obtain solutions of Problem 1.3 as

$$g^* = \begin{cases} h_{1/2, \sqrt{3}/2}^* & \text{if } 0 < a \leq 1/2, \\ h_{u, \sqrt{1-u^2}}^*, 0 \leq u \leq 1/2 & \text{if } a = 0. \end{cases}$$

We will identify the corresponding inflated maps. We begin with the case $0 < a \leq 1/2$. A basis for the first eigenspace of $-\Delta_{h_{1/2, \sqrt{3}/2}}$ is given by

$$\exp(2\pi ix), \quad \exp(2\pi iy), \quad \exp(2\pi i(x+y)).$$

We look for inflated maps, assuming they have the following form:

$$\begin{aligned} \psi_{p,q,r}: (x, y) \bmod \mathbb{Z}^2 \in \mathbb{R}^2/\mathbb{Z}^2 &\mapsto \\ \frac{1}{\sqrt{8\pi^2}} (p \exp(2\pi ix), q \exp(2\pi iy), r \exp(2\pi i(x+y))) &\in \mathbb{C}^3, \end{aligned}$$

where we may suppose $p, q, r \geq 0$. The map $\psi_{p,q,r}$ satisfies $\psi_{p,q,r}^* g_{\mathbb{C}^3} = h_{a,b}$ if and only if

$$p = \sqrt{2(1-a)}, \quad q = \sqrt{2(a^2 + b^2 - a)}, \quad r = \sqrt{2a}.$$

Let $\varphi_{a,b}$ denote the map $\psi_{p,q,r}$ with these choices of p, q, r . It is an inflated map with respect to $(d\mu, h_{a,b})$. Observe that the maps $\varphi_{a,b}$ vary smoothly with the parameters a, b and degenerate to maps into \mathbb{R}^4 as $a \rightarrow 0$, that is, as the lattice approaches to a rectangular one.

Next we treat the case $a = 0$. Note that the functions

$$\exp(2\pi ix), \quad \exp(2\pi iy)$$

form a common basis for the first eigenspace of $-\Delta_{h_{u,\sqrt{1-u^2}}}$, $0 \leq u \leq 1/2$. By proceeding as in the previous case, we obtain the same inflated maps $\varphi_{0,b}$.

We summarize what we have observed as

Proposition 2.3. *Let h be any flat metric on T^2 , and let $d\mu$ be its normalized volume element. ($d\mu$ is common to all flat metrics on T^2 .) Then there exists an isometric embedding $\varphi: (T^2, h) \rightarrow \mathbb{R}^6$ by first eigenfunctions of the flat metric $h_{1/2,\sqrt{3}/2}$. In particular, the map φ is an inflated map with respect to $(d\mu, h)$, and the metric $g^* = h_{1/2,\sqrt{3}/2}^*$ is a solution to Problem 1.3 associated with $(d\mu, h)$.*

If h is a rectangular flat metric, that is, a flat metric corresponding to a rectangular lattice, then the image of φ lies in \mathbb{R}^4 and the metrics $h_{u,\sqrt{1-u^2}}^$, $0 \leq u \leq 1/2$, provide solutions to Problems 1.3.*

Remark 1. If we search for an optimal solution to Problem 1.3 among left-invariant metrics, the metrics $h_{u,\sqrt{1-u^2}}^*$, $0 \leq u \leq 1/2$, are the only possibilities. Indeed, the corresponding inflated map must be an isometric immersion, and flat tori do not admit an isometric immersion into \mathbb{R}^3 . Therefore, the first eigenvalue of the solution metric should have multiplicity at least 4.

Remark 2. The metrics $h_{u,\sqrt{1-u^2}}^*$, $0 \leq u \leq 1/2$, which are solutions to Problems 1.3 associated with a rectangular flat metric $h_{0,b}$, are expressed as

$$h_{u,\sqrt{1-u^2}}^* = \frac{1}{1-u^2} \partial_x^2 - \frac{2u}{1-u^2} \partial_x \partial_y + \frac{1}{1-u^2} \partial_y^2.$$

Note that Problem 1.3 associated with $(d\mu, h_{0,b})$ is equivalent to maximizing the concave functional $g^* \mapsto \lambda_1(d\mu, g^*)$ over the convex set

$$\mathcal{C} := \left\{ g^* \in \mathcal{M}^* \mid \int (g^*, h_{0,b}) d\mu = 1 \right\}.$$

Therefore, the set of maximizing metrics in \mathcal{C} should be a convex subset of \mathcal{C} . Rescaled so that $\int (g^*, h_{0,b}) d\mu = 1$, the above metrics become

$$\frac{1}{1+b^2} \partial_x^2 - \frac{2u}{1+b^2} \partial_x \partial_y + \frac{1}{1+b^2} \partial_y^2.$$

Thus, we obtain a segment of metrics joining (likewise rescaled) $h_{0,1}^*$ and $h_{1/2,\sqrt{3}/2}^*$, as expected.

Remark 3. We can also solve Problems 1.1, 1.3 associated with the pair $(d\mu, h)$ for all left-invariant metrics h on $SU(2)$. Here, $d\mu$ is the unique left-invariant normalized volume element on $SU(2)$, and is the common normalized volume element of all h . This class of metrics contains an interesting subfamily, that is, the so-called Berger metrics h_t , $0 < t < \infty$. These are obtained by rescaling the standard metric of S^3 by a factor of t^2 in the direction of fibers of the Hopf fibration $S^3 \rightarrow S^2$.

In his seminal work [8], Urakawa observed that the multiplicity of the first eigenvalue of h_t becomes 7 at $t = 1/\sqrt{6}$, where two different eigenspaces meet. The solutions of the primal and dual problems associated with the pair $(d\mu, h_t)$ exhibit different behavior depending on whether $t \leq 1$ or $t \geq 1$. For $t \leq 1$, the Berger metric $h_{1/\sqrt{6}}$ gives a common solution to Problem 1.3, and maps consisting of first eigenfunctions of the Laplacian with respect to $h_{1/\sqrt{6}}$ give inflated isometric embeddings into \mathbb{R}^7 . On the other hand, for $t \geq 1$, optimal solutions to Problem 1.3 cannot be found among left-invariant Riemannian metrics. In fact, the Carnot-Carathéodory metric $h_\infty = \lim_{t \rightarrow \infty} h_t$ gives a common solution to these problems. We also obtain inflated maps as maps consisting of first eigenfunctions of h_∞ , which is nothing but the standard inclusion map of S^3 into \mathbb{R}^4 , and is therefore not isometric with respect to h_t (unless $t = 1$).

3. EQUIVARIANT SETTING

In this section, we discuss the equivariant versions of our optimization problems.

We begin by reviewing the twisted Laplacian and its spectrum bottom, following Sunada [6]. Let (M, h) be a compact Riemannian manifold, and let $\pi: X \rightarrow M$ be the universal covering of M with covering transformation group Γ . Let \mathcal{H} be a separable Hilbert space, and let $\rho: \Gamma \rightarrow O(\mathcal{H})$ be a homomorphism, where $O(\mathcal{H})$ is the orthogonal group of \mathcal{H} . Then we

can associate a vector bundle E_ρ in the following way: E_ρ is the quotient space of $X \times \mathcal{H}$ by the action of Γ given by $\gamma(x, v) = (\gamma x, \rho(\gamma)v)$. If we denote by $[x, v]$ the Γ -orbit containing (x, v) , then the correspondence $[x, v] \mapsto \pi(x)$ gives a bundle projection $E_\rho \rightarrow M$. The vector bundle E_ρ has a fiber metric $\langle \cdot, \cdot \rangle$ defined by $\langle [x, u], [x, v] \rangle = \langle u, v \rangle_{\mathcal{H}}$. The bundle E_ρ is called the flat vector bundle associated to ρ . A section s of E_ρ can be identified with a map $\varphi: X \rightarrow \mathcal{H}$ satisfying the ρ -equivariance

$$\varphi(\gamma x) = \rho(\gamma)\varphi(x), \quad \forall x \in X, \quad \forall \gamma \in \Gamma,$$

through the relation

$$s(\pi(x)) = [x, \varphi(x)].$$

We denote by $C^\infty(E_\rho)$ (resp. $L^2(E_\rho)$) the space of C^∞ -sections (resp. square integrable sections) of E_ρ . The exterior derivative d can be extended to

$$d_\rho: C^\infty(E_\rho) \rightarrow C^\infty(T^*M \otimes E_\rho).$$

We denote by δ_ρ the formal adjoint of d_ρ , and define the twisted Laplacian Δ_ρ by

$$\Delta_\rho = \delta_\rho d_\rho: C^\infty(E_\rho) \rightarrow C^\infty(E_\rho).$$

It is easy to see that if a smooth map $\varphi: X \rightarrow \mathcal{H}$ corresponds to a section s of E_ρ , then $\Delta_X \varphi$ corresponds to $\Delta_\rho s$. It can be verified that Δ_ρ extends uniquely to a positive self-adjoint operator on $L^2(E_\rho)$. The bottom of the spectrum of Δ_ρ is given by

$$\lambda_0(\rho) = \lambda_0(h; \rho) = \inf_{s \in C^\infty(E_\rho) \setminus \{0\}} \frac{\int_M \|d_\rho s\|^2}{\int_M \|s\|^2}.$$

We have the following facts:

- If ρ is trivial, then Δ_M and Δ_ρ are unitarily equivalent.
- If ρ is the right regular representation ρ_r of Γ on $L^2(\Gamma)$, then Δ_X and Δ_ρ are unitarily equivalent.

We now formulate an analogue of the dual pair of Problems 1.1 and 1.3. We reserve the above setting, and suppose that (M, h) be also equipped with a volume element $d\mu$ of unit volume. Consider the following *second-moment maximization problem*:

Problem 3.1. Over all ρ -equivariant C^∞ -maps $\varphi: X \rightarrow \mathcal{H}$ satisfying the constraint

$$\varphi^* h_{\mathcal{H}} \leq h,$$

maximize the quantity

$$\text{mom}_2(\varphi) := \int_M \|\varphi\|^2 d\mu,$$

where $\|\cdot\|$ is the norm of \mathcal{H} .

A solution map to Problem 3.1 will be called an *inflated equivariant map*.

Searching for the dual of Problem 3.1, we are led to consider

$$\lambda_0(d\mu, g^*; \rho) = \inf \frac{\int_M \|d\varphi\|_{g^*}^2 d\mu}{\int_M \|\varphi\|^2 d\mu}, \quad g^* \in \mathcal{M}^*,$$

where the infimum is taken over all nonzero ρ -equivariant C^∞ -maps $\varphi: X \rightarrow \mathcal{H}$. The quantity $\lambda_0(d\mu, g^*; \rho)$ coincides with the bottom of the spectrum of the twisted Bakry-Émery Laplacian $-\Delta_{(d\mu, g^*; \rho)}$, defined analogously to the usual twisted Laplacian introduced above. Then as the dual of Problem 3.1, we obtain the following problem:

Over all $g^* \in \mathcal{M}^*$ satisfying the constraint

$$\lambda_0(d\mu, g^*; \rho) \geq 1,$$

minimize

$$\int_M (g^*, h) d\mu.$$

This problem can be rephrased to give the *spectrum-bottom maximization problem*:

Problem 3.2. Over all $g^* \in \mathcal{M}^*$, maximize

$$\lambda_0(d\mu, g^*; \rho) / \int_M (g^*, h) d\mu.$$

We obtain the inequality between objective functionals:

$$\text{mom}_2(\varphi) \leq \int_M (g^*, h) d\mu / \lambda_0(d\mu, g^*; \rho).$$

If equality holds for φ and positive definite g^* , then φ attains $\lambda_0(d\mu, g^*; \rho)$ and is an isometric immersion with respect to h .

Note that if ρ is a trivial representation, then $\text{Mom}_2(d\mu, h; \rho) = \infty$ and $\Lambda_0(d\mu, h; \rho) = 0$. On the other hand, if the Kazhdan distance $\delta(\rho, \mathbf{1})$ between ρ and the trivial representation $\mathbf{1}$ is positive and $d\mu = d\mu_h/\text{Vol}(h)$, then $\Lambda_0(d\mu, h; \rho) > 0$, since $\lambda_0(d\mu, h^*; \rho) = \lambda_0(h; \rho) > 0$ by Sunada's result [6]. It follows that $\text{Mom}_2(d\mu, h; \rho) < \infty$.

A potentially interesting example with regard to the above problems is the isometric minimal embedding of the hyperbolic space into a Hilbert sphere, introduced by Besson-Courtois-Gallot [2].¹ Let M be a compact hyperbolic manifold of dimension $n \geq 3$, and let X be the universal covering of M with Γ denoting the covering transformation group. We choose a base point $o \in X$. Then the boundary at infinity ∂X of the hyperbolic space X is identified with the sphere S^{n-1} . Let $d\theta$ denote the standard normalized volume element of $\partial X \cong S^{n-1}$. We shall verify that Γ acts isometrically on the Hilbert space $\mathcal{H} = L^2(\partial X, d\theta)$. Let $p(x, \theta)$ denote the Poisson kernel of X , normalized so that $p(o, \theta) \equiv 1$. Let G denote the isometry group of X , which acts on ∂X . The action of G on \mathcal{H} can be defined by

$$(g(f))(\theta) = f(g^{-1}\theta)\sqrt{p(go, \theta)}.$$

This may be regarded as the action of G on the measures $f(\theta)^2 d\theta$. Namely,

$$g(f(\theta)^2 d\theta) = (g(f))(\theta)^2 d\theta.$$

Then it is easy to verify that

$$\int_{\partial X} g(f(\theta)^2 d\theta) = \int_{\partial X} f(\theta)^2 d\theta.$$

Thus, the action of G on \mathcal{H} is isometric. In particular, Γ acts on \mathcal{H} isometrically. This gives an orthogonal representation $\rho: \Gamma \rightarrow O(\mathcal{H})$. Let S^∞ denote the unit sphere in \mathcal{H} with center at the origin. Thus,

¹Ciao Li [3] asked how this embedding is relevant to our work. This question motivates the content of this section.

for $f(\theta) \in \mathcal{S}$, $f(\theta)^2 d\theta$ is a probability measure on S^{n-1} . We consider ρ -equivariant C^∞ -maps from X into \mathcal{H} . If we write $\varphi(x)(\theta) = \varphi(x, \theta)$, then the ρ -equivariance may be expressed as

$$\varphi(\gamma x, \theta) = \varphi(x, \gamma^{-1}\theta) \sqrt{p(\gamma o, \theta)}.$$

The map $\varphi_0(x) = \sqrt{p(x, \cdot)}$ is an example of such a map. In fact, the map φ_0 is G -equivariant, and the image of φ_0 is the orbit of the constant function equal to 1 under the action of G . In particular, the image of φ_0 lies in S^∞ . The pull-back metric $\varphi_0^* h_{\mathcal{H}}$ is given by

$$\varphi_0^* h_{\mathcal{H}} = \frac{(n-1)^2}{4n} h,$$

where h denotes the hyperbolic metric of X . Thus, the map φ_0 is a homothetic embedding. Besson et al. verified that the map is calibrated by a certain n -form on S^∞ , and in particular, it is a minimal embedding into S^∞ . In order to make φ_0 an isometric embedding, we multiply it by a constant; we re-set

$$\varphi_0(x) = \frac{2\sqrt{n}}{n-1} \sqrt{p(x, \cdot)},$$

so that $\varphi_0(X)$ lies in $S^\infty(2\sqrt{n}/(n-1))$, the sphere in \mathcal{H} of radius $2\sqrt{n}/(n-1)$.

We speculate that the following statement would be true:

The (new) φ_0 is an inflated equivariant map with respect to $(d\mu, h)$, where $d\mu = d\mu_h/\text{Vol}(h)$, and the metric h^ is an optimal solution to Problem 3.2.*

In fact, it is clear that the second moment of φ_0 equals to $\frac{4n}{(n-1)^2}$. Since $\int_M (h^*, h) d\mu = n$, this implies

$$\lambda_0(d\mu, h^*; \rho) = \lambda_0(h; \rho) \leq \frac{(n-1)^2}{4}.$$

Therefore, in order to verify the above statement, we must show that

$$\lambda_0(h; \rho) \geq \frac{(n-1)^2}{4}.$$

Note that φ_0 satisfies

$$-\Delta_X \varphi_0 = \frac{(n-1)^2}{4} \varphi_0.$$

Thus, the section of E_ρ corresponding to φ_0 is an eigensection of the twisted Laplacian $-\Delta_\rho$ with eigenvalue $(n-1)^2/4$. The question is whether this eigenvalue coincides with $\lambda_0(h; \rho)$.

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