On Chern's Conjecture ($g = 4$ and $g = 6$ cases)

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Abstract

Chern's conjecture states that a closed minimal hypersurface in the euclidean sphere is isoparametric if it has constant scalar curvature. When the number g of distinct principal curvatures exceeds three, few satisfactory results are known, and additional assumptions seem necessary. In this work, we impose the Dupin condition, which is far weaker than the isoparametric one, and obtain the following results: A closed proper Dupin hypersurface with constant mean curvature is isoparametric (i) if $g = 3$, (ii) if $g = 4$ and it has constant scalar curvature, or (iii) if $g = 4$ and it has constant Lie curvature, and (iv) if $g = 6$ and it has constant Lie curvatures. These cover all non-trivial cases for closed proper Dupin hypersurfaces. The argument relies on advanced tools from Lie sphere geometry and combines topological and geometric methods, in contrast to earlier algebraic and analytic approaches. [1]

1 Introduction

This is a survey of the paper "Chern's conjecture in the Dupin case" [28]. Since the notion of Dupin hypersurfaces arises naturally in *Lie sphere geometry*–a framework broader than both Riemannian and conformal geometry–we begin with an expository introduction to Lie sphere geometry and then outline the proof of the main result. A cruicial tool is the *Lie curvature*–the cross ratio of four distinct principal curvatures–introduced by the author in [23], which is a Lie invariant quantity.

To explain Chern's conjecture, we start with the basic definition:

Definition. Let $\bar{M}(c) = H^{n+1}, E^{n+1}, S^{n+1}$ be a real space form of constant sectional curvature $c = -1, 0, 1$, respectively. A hypersurface $\varphi : M^n \to \bar{M}(c)$ is called *isoparametric* if it has constant principal curvatures.

The *principal curvatures* $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the *shape operator* $A : TM \ni X \mapsto -\nabla_X n \in TM$, where n is a unit normal vector field

[1]Supported by Grants-in-Aid for Scientific Research C 24K06701 JSPS.

and ∇ denotes the Levi-Civita connection on $\bar{M}(c)$. Let $g = \sharp\{\lambda_1, \ldots, \lambda_n\}$ be the number of distinct principal curvatures. The following quantities are fundamental:

$$H = \sum \lambda_j, \quad R = n(n-1)c + H^2 - \|A\|^2,$$

where H is the *mean curvature* (not devided by n) and R is the *scalar curvature*.

Chern's conjecture (1968, [13], Yau's 105th problem [45]): *Let $\varphi : M \to S^{n+1}$ be a hypersurface satisfying:*
(0) *closed*
(1) *minimal, and*
(2) *has constant scalar curvature (CSC).*
Then M is isoparametric.

Since (1) and (2) are equivalent to

$$(1)\, H = \sum \lambda_j = 0, \quad (2)\, S := \sum \lambda_j^2 \text{ is constant},$$

the conjecture is locally true when $g \leq 2$:
· If $g = 1$, M is totally geodesic.
· If $g = 2$, is a piece of Clifford hypersurface $S^k \times S^{n-k}$.
For $g \geq 3$, however, the situation becomes far more subtle. The known results are summarized as follows (see also [41]):

g	n	closed	local
3	3	True [15] [35]...	Bryant's conjecture
	≥ 4	in progress	in progress [11]
≥ 4	≥ 4	+ conditions, True [43]	

When $g = n = 3$, the affirmative resolution of Chern's conjecture was established through the contributions of Peng-Terng [35], Almeida-Brito [15], Q.M. Cheng-Q. Wan [12], S.P. Chang [10] among others. Bryant's conjecture–the local analogue of Chern's conjecture in the case $g = n = 3$– remains open. Y. Chen-T. Li [11] made further progress in the range $g \leq 3 < 4 \leq n$. For general g, the following is known:

Fact A. (Z.Z. Tang-W.J. Yan, 2023, [43], see also [42]) Let $\varphi : M \to S^{n+1}$ be a hypersurface satisfying $n \geq 4$ and
(0) M is closed,
(1) $R \geq 0$,
(2) $\sum \lambda_j^k$, $k = 1, ..., n-1$ are constant.
Then M is isoparametric. Moreover, if $g = n$ at some point, then $R = 0$.

In the closed case, analytic tools, such as Simon's formula and maximum principle play a crucial role. The closedness assumption also allows the use of more geometric techniques, including topological arguments.

To state our results, we introduce the following definition:

Definition. A hypersurface $\varphi : M \to \bar{M}(c)$ is called a (*proper*) *Dupin* hypersurface if
(i) each principal curvature has constant multiplicity, and
(ii) each λ_j is constant along its curvature distribution $D(\lambda_j)$.

In this article, we omit the word "proper". If only (ii) holds, we call it *weak Dupin.* The Dupin condition is invariant under *Lie contact transformations* (§4), and thus defines a class of hypersurfaces far broader than those arising in Riemannian geometry. In particular, the Dupin condition remains substantially weaker than the isoparametric condition (see §6), especiallly in the case $g \geq 4$. Consequently, proving Chern's conjecture in this setting is still highly nontrivial.

We establish the following:

Theorem 1.1 (M., 2025, [28])**.** *A closed proper Dupin hypersurface with constant mean curvature is isoparametric*

(i) *if* $g = 3$*,*

(ii) *if* $g = 4$ *and it has constant scalar curvature,*

(iii) *if* $g = 4$ *and it has constant Lie curvature,*

(iv) *if* $g = 6$ *and it has constant Lie curvatures.*

2 Isoparametric hypersurfaces

In the early twentieth century, Italian geometric opticians began studying light wave fronts that propagate at a constant speed [18], [40]. This led to the notion of isoparametric hypersurfaces, which consist of parallel hypersurfaces of constant mean curvature (CMC), and, indeed, possess constant principal curvatures.

Classically, it is known that in the Euclidean space E^{n+1} and the hyperbolic space H^{n+1}, the only isoparametric hypersurfaces are umbilic hypersurfaces including totally geodesic ones, and cylinders $S^k \times E^{n-k}$ (respectively $S^k \times H^{n-k}$); see [1].

In contrast, in the sphere

$$S^{n+1} = \{x = (x_1, \ldots, x_{n+1}, x_{n+2}) \in \mathbb{R}^{n+2} \mid \|x\| = 1\}$$

É. Cartan (1937, [1], [2]) discovered examples with $g = 3, 4$. Later, Ozeki-Takeuchi found infinitely many homogeneous and non-homogeneous ones with $g = 4$ (1976, [34]), which were subsequently generalized by Ferus-Karcher-Münzner (1981, [17]) to families arising from representations of all Clifford algebra. There are two more homogeneous ones with $g = 6$.

Fact B. (Münzner, 1980-81, [31]) Isoprametric hypersurfaces in S^{n+1} satisfy the following properties:
1. Topology. Each is an iterated sphere bundles over a sphere.
2. Algebraicity. There exists a homogeneous *Cartan-Münzner polynomial* of degree g on $\mathbb{R}^{n+2}$ such that

$$M = F^{-1}(c) \cap S^{n+1}, \quad c \in (-1, 1).$$

3. Focal submanifolds. The level sets $M_{\pm} = F^{-1}(\pm 1) \cap S^{n+1}$ are two focal submanifolds from which the parallel hypersurfaces in the family emanate.
4. Possible values of g**.** The number of distinct principal curvatures satisfies $g \in \{1, 2, 3, 4, 6\}$.

The classification (Yau's 34th problem [46]) was completed in 2020, [5], [14], [16], [25], [26], [27],[29]. The figures below illustrate *examples* of such hypersurfaces projected onto the plane containing a normal geodesic. The two polygons in each figure represent the two focal submanifolds from which wave fronts emanate. (graphics generated by S. Fujimori,Hiroshima U.).

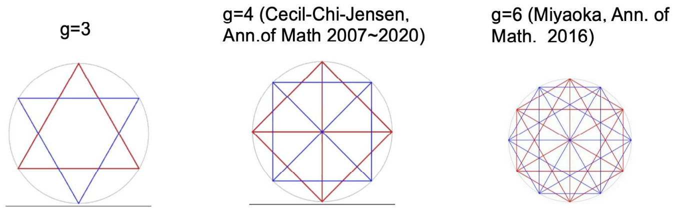


Figure 1: Foal submanifolds of isoparametric hypersurfaces

3 Dupin hypersurfaces

Recall the definition of Dupin hypersurface in §1. The following facts are important in the study of Dupin hypersurfaces:

Fact C. (T. Otsuki, 1970, [32], H. Reckziegel, 1976, [39])

(1) When the multiplicity m of a principal curvature λ is constant, its curvature distribution

$$D(\lambda) = \{X \in TM : AX = \lambda X\}.$$

is integrable. Moreover, if $m > 1$, λ is constant along $D(\lambda)$.

(2) When λ is constant along $D(\lambda)$, its curvature surfaces–the leaves of $D(\lambda)$– are m-dimensional subspheres of the corresponding curvature sphere.

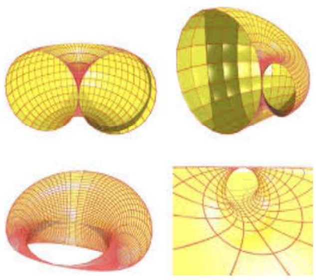


Figure 2: Dupin hypersurfaces in $\mathbb{R}^3$

Example 3.1 : (1) Isoparametric hypersurfaces and their conformal images are Dupin hypersurfaces.

(2) If M is a Dupin hypersurface in $\mathbb{R}^m(=E^m)$, a cylinder over M or a tube around M in $\mathbb{R}^m \oplus \mathbb{R}^k$, is again a (weak) Dupin hypersurface in $\mathbb{R}^{m+k} = E^{m+k}$. By applying stereographic projection, one obtains a weak Dupin hypersurface in S^{m+k} [36]. Thus Dupin hypersurfaces with any number of principal curvatures and arbitrary multiplicities exist locally.

(2) of Fact C implies that a Dupin hypersurface is foliated by spherical leaves. Consequently, such hypersurfaces are most naturally studied within the broader framework of Lie sphere geometry, which encompasses and extends both Riemannian and conformal geometry (§4).

4 Lie sphere geometry

Conformal transformations composed with dilations on $\bar{M}(c)$ are called *Lie contact transformations*. Since real space forms are equivalent under Lie contact transformations, we may restrict our argment to the case of S^{n+1}. Note that these transformations act not on S^{n+1} itself, but on its *unit tangent bundle* T_1S^{n+1}. For a detailed and thorough treatment of Lie sphere geometry, consult [3].

Definition. Let $\mathbb{R}^{n+4}_2$ be endowed with a bilinear form $\langle\ ,\ \rangle_2$ of signature $(+, ..., +, -, -)$. The linear transformation group $\mathrm{O}(n+2, 2)$ preserving $\langle\ ,\ \rangle_2$ is called the *Lie contact transformation group.*

Remark 4.1 : The subgroup $\mathrm{O}(n+2, 1)$ is precisely the *conformal* (*i.e., Moebius*) *transformation group.*

An oriented hypersphere of S^{n+1}, centered at $p \in S^{n+1}$ with oriented radius θ, is expressed by

$$k = (p, \cos\theta, \sin\theta) \in \mathbb{R}^{n+4}_2, \quad -\pi < \theta \leq \pi.$$

This is a null vector in the projective space $\mathbb{R}P^{n+3}$, since $\langle k, k\rangle_2 = 0$.

Example 4.2 : The vector $k = (p, 1, 0)$ represents a *point sphere* p, while $h = (n, 0, 1)$ represents the totally geodesic hypersphere with center at n.

The *space of oriented hyperspheres* is

$$Q^{n+2} = \{[k] = [p, \sin\theta, \cos\theta], \langle k, k\rangle = 0\} \subset \mathbb{R}P^{n+3},$$

and Q^{n+2} is invariant under the action of $\mathrm{O}(n+2, 2)$. For brevity, we write k for the projective class $[k]$.

Definition. Two oriented hyperspheres $k_1, k_2 \in Q^{n+2}$ are said to be in *oriented contact* if they touch at a common point of S^{n+1} with a common normal direction.

Example 4.3 : For $(p, n) \in T_1S^{n+1}$, (i.e., $p \perp n$), the pair $k_1 = (p, 1, 0)$ and $k_2 = (n, 0, 1) \in Q^{n+2}$ has oriented contact, satisfying $\langle k_1, k_2\rangle = 0$.

Lemma 4.4. *Two null vectors $k_1, k_2 \in Q^{n+2}$ are in oriented contact if and only if*

$$\langle k_i, k_j\rangle = 0, \quad i, j \in \{1, 2\}.$$

This condition is preserved under Lie contact transformations.

Proof. For $k_1 = (p, \cos\theta, \sin\theta)$, $k_2 = (q, \cos\varphi, \sin\varphi)$, we compute

$$\langle k_1, k_2 \rangle_2 = \langle p, q \rangle - \cos(\theta - \varphi) = 0.$$

Thus $\langle k_1, k_2 \rangle = 0$ holds precisely when k_1 and k_2 have oriented contact.

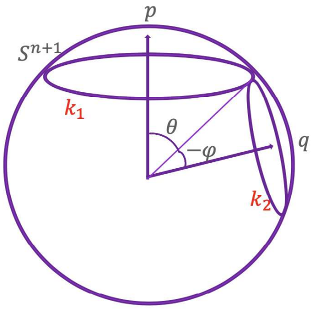


Figure 3: Oriented contact

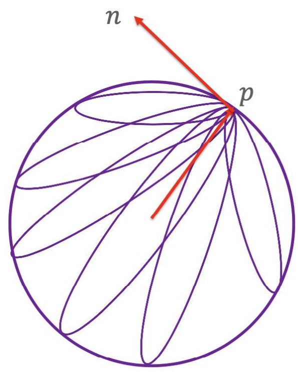


Figure 4: A line of Q

Whenever $\langle k_i, k_j \rangle = 0$ holds, the projective line

$$l = \{[ak_1 + bk_2], a, b \in \mathbb{R}\} \subset Q^{n+2}$$

represents a *one-parameter family of oriented hyperspheres* sharing an oriented contact at a common point $p \in S^{n+1}$. The line l is determined by the *contact point sphere* $k_1 = (p, 1, 0)$ and the *contact direction* $k_2 = (n, 0, 1)$. Therefore, the *space of lines*

$$\Lambda^{2n+1} = \{\text{lines of } Q^{n+2}\}$$

can be identified with the unit tangent bundle

$$\Lambda^{2n+1} \cong T_1 S^{n+1}.$$

The Lie contact transformation group $\mathrm{O}(n+2, 2)$ acts on Q^{n+2} and on Λ^{2n+1}, preserving oriented contact.

5 Lie image

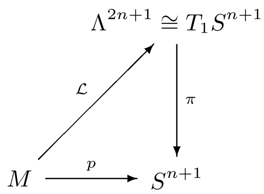
Let $p : M \to S^{n+1}$ be a hypersurface and $n : M \to S^{n+1}$ its unit normal vector field. Define

$$k_1 = (p, 1, 0), \quad k_2 = (n, 0, 1)$$

which determine a line $l = [ak_1 + bk_2]$ in the Lie quadric Q^{n+2}. The map

$$\mathcal{L} : M \ni p \mapsto l \in \Lambda^{2n+1} \cong T_1 S^{n+1}.$$

is precisely the *Legendre map* of M:



Since $A \in$O$(n+2,2)$ acts naturally on T_1S^{n+1}, the composition

$$\pi \circ A\mathcal{L} : M \to S^{n+1}$$

defines a new (*possibly singular*) hypersurface in S^{n+1}.

Definition. We call $\pi \circ A\mathcal{L}(M)$ a *Lie image of* M.

6 Cecil-Ryan conjecture

A global characterization of Dupin hypersurfaces is given by:

Fact D. (Thorbergsson, 1983, [44]) Let M be a *closed* Dupin hypersurface in S^{n+1}. Then:

(i) The topology coincides with that of an an isoparametric hypersurface in S^{n+1}.

(ii) The number g of distinct principal curvatures belongs to $\{1,2,3,4,6\}$.

(iii) M is *taut*: for $x \in S^{n+1} \setminus M$, the spherical distance function

$$d_x : M \to \mathbb{R}, \quad d_x(p) = d(p,x),$$

has the property that d_x^2 is a *perfect Morse function*; that is, it achieves equality in the *Morse inequalities.*

Remark 6.1 : Conversely, taut hypersurfaces are weak Dupin [22],[37].

From Fact C, we immediately obtain:

Fact E. A closed Dupin hypersurface in S^{n+1} with $g = 1,2$ is a *Lie image* of an isoparametric hypersurface.

The following is substantially more difficult.

Theorem 6.2 (M. 1984 [21], see also [19], [20], [6])**.** *A closed Dupin hypersurface in* S^{n+1} *with* $g = 3$ *is a Lie image of an isoparametric hypersurface.*

These results led to the following conjecture.

Cecil-Ryan's conjecture (1985, [8]) *Every closed Dupin hypersurface in* S^{n+1} *is a Lie image of an isoparametric hypersurface.*

However, it turned out to be *false.*

Theorem 6.3 (M.-Ozawa, 1989 [30])**.** *For* $g = 4,6$*, there exist counterexamples.*

Independently, Pinkall-Thorbergsson constructed counterexamples for $g = 4$ [38]. For the way of construction of these counterexamples, see the expository article [4]. Tautness is effectively usesd in Theorem 6.3, based on the argument by T. Ozawa in [33].

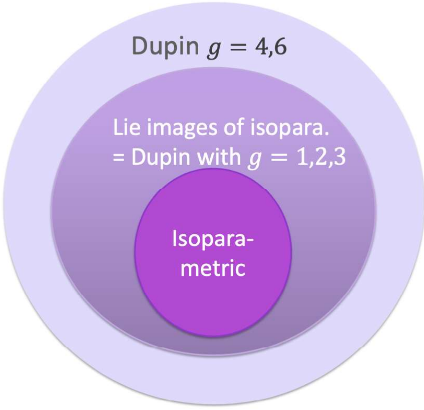


Figure 5: Spaces of closed Dupin and isoparametric hypersurfaces

7 Lie curvature

To distinguish counterexamles, the following invariant plays a crucial role.

Theorem 7.1 (M. 1989 [23])**.** *For a hypersurface in $\bar{M}(c)$, rhe cross ratio of four distinct principal curvatures $\lambda_i, \lambda_j, \lambda_k, \lambda_l$,*

$$\Psi = \frac{(\lambda_i - \lambda_k)(\lambda_j - \lambda_l)}{(\lambda_i - \lambda_l)(\lambda_j - \lambda_k)} \tag{1}$$

is invariant under Lie contact transformations. We call Ψ Lie curvature.

Recall that the cross ratio of four points $z_1, z_2, z_3, z_4 \in \mathbb{C}$ is defined by

$$[z_1, z_2; z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} \in \mathbb{C}.$$

This quantity takes real values when z_1, z_2, z_3, z_4 *are concircular*. Such a geometric configuration naturally appears in the following context [23].

A *curvature sphere* of M is an oriented hypersphere C having an oriented contact with M of contact order at least 2. When $\bar{M}(c) = S^{n+1}$, using the *radius angle* θ of C, the corresponding principal curvature λ is given by

$$\lambda = \cot \theta. \tag{2}$$

Let γ be the normal geodesic at $p \in M$; note that γ is a circle in S^{n+1}. If M has *four distinct principal curvatures* $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, the curvature spheres C_i corresponding to λ_i intersect γ at four points $z_1, z_2, z_3, z_4 \in \gamma$. Then one computes

$$\Psi = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} = \frac{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}{(\lambda_1 - \lambda_4)(\lambda_2 - \lambda_3)}.$$

using $\lambda_i = \cot \theta_i$.

Proposition 7.2 (M (1989) [23]). *Under a* Lie contact transformation $L \in O(n+2,2)$*, each principal curvature* λ *transforms by a* linear fractional map

$$\hat{\lambda} = \frac{a\lambda + b}{c\lambda + d},$$

where $a, b, c, d : M \to \mathbb{R}$ *are determined by* L*. Consequently, the Lie curvature is preserved by* L*.*

The invariance of Lie curvature constitutes a key tool in the investigation of the Cecil-Ryan conjecture. Figure 5 provides a schematic depiction of how the spaces of isoparametric hypersurfaces and closed Dupin hypersurfaces are related. For $g = 4, 6$, even when all the Lie curvatures are constant, this does not guarantee Lie equivalence to an isoparametric hypersurface [23],[24]; see also [7]. Consequently, the Dupin condition remains *substantially weaker*, and establishing Chern's conjecture in the Dupin case for $g \geq 4$ remains highly nontrivial.

For further background and a thorough exposition of hypersurface geometry, we refer the reader to the comprehensive monograph [9].

8 Strategy of the proof of Main theorem

In the proof of Theorem 1.1, we make use of the following properties of the closed Dupin hypersurface M:

(1) Tautness

(2) Constant Mean Curvature (CMC)

(3) Constant Scalar Curvature (CSC) or Constant Lie Curvature (CLC)

The main strategy of the proof is as follows:

- (i) **Existence of a critical point:** Since M is closed, properties (2) and (3) guarantee the existence of a point $p_1 \in M$ where all principal curvatures λ_i are critical.
- (ii) **Intersection with normal geodesic:** The curvature leaves through p_1 intersect the normal geodesic γ passing through p_1 orthogonally at g points p_{2i} $(i = 1, \ldots, g)$.
- (iii) **Formation of a $2g$-gon:** Property (1) implies that $M \cap \gamma$ consists of $2g$ points p_{2i} and p_{2i-1} $(i = 1, \ldots g)$, which together form a $2g$*-gon.*
- (iv) **Parallelism of the $2g$-gon:** Using (2) and (3) again, one shows that this $2g$-gon $M \cap \gamma$ is a *parallel* $2g$*-gon*, that is, a $2g$-gon obtained from a regular $2g$-gon by a parallel transformation.
- (v) **Determination by mean curvature:** This parallel $2g$-gon is uniquely determined by the mean curvature of M.
- (vi) **Relation to principal curvatures:** The spherical distance between adjacent vertices of $M \cap \gamma$ corresponds to the principal curvatures λ_1 and λ_g of M, where $\lambda_1 > \cdots > \lambda_g$.
- (vii) **Equality of extrema:** From (v) and (vi), the $2g$-gon at p_1 and q_1– points where λ_1 attains its maximum and minimum of λ_1, respectively– are isometric. Hence $\max \lambda_1 = \min \lambda_1$ follows, implying that λ_1 is constant on M.

(viii) **Isoparametric conclusion:** Similarly, all λ_i are constant, i.e., M is *isoparametric*.

A lengthy computation is required to establish steps (i) and (iv), using the assumptions (2) and (3). These involve explicit calculations with *Lie contact transformations* and *Lie curvature*, which are particularly intricate when $g = 6$.

9 Remarks

Due to part (ii) of Theorem 1.1, Chern's conjecture is reduced to the following statement:

A closed minimal hypersurface M in S^{n+1} with four principal curvatures is Dupin if M has constant scalar curvature (CSC).

By Fact C, if each principal curvature is constant along its corresponding curvature distribution, then M is Dupin. This leads to the following corollary:

Corollary 9.1. *Let M be a closed hypersurface with g principal curvatures each constant along its corresponding curvature distribution. Then M is isoparametric if*

(i) *if $g = 3$,*

(ii) *if $g = 4$ and M has constant scalar curvature,*

(iii) *if $g = 4$ and M has constant Lie curvature,*

(iv) *if $g = 6$ and M has constant Lie curvatures.*

Remark 9.2 : When $g = 6$, the assumption *CLC* appears to be stronger than that of *CSC*.

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