Gap Method to Duality

— initial point vs both points —

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Abstract

This paper considers a minimization problem with fixed *initial* point and another minimization problem with fixed *both* initial and terminal points. We discuss a quadratic duality for these minimization problems through gap method.

1 Introduction

Recently, Iwamoto, Kimura, Fujita and Kira have shown that a new duality holds for paired optimization problems [11–28]. For a historical background of dynamic optimization and its related fields, see Bellman and others [1–7, 29], [8, 10, 31, 32].

This paper discusses two models — initial point and both points — through *gap functions*. In section 2, we consider a pair of minimization problem (P_1) and maximization problem (D_1). The problem (P_1) has a fixed *initial* point. We give another pair of (P_2) and (D_2), which is derived from the pair of (P_1) and (D_1). In section 3, we consider a pair of minimization problem (P'_1) and maximization problem (D'_1). The problem (P'_1) has both the *initial* and the *terminal* points. As in section 2, we derive another pair of (P'_2) and (D'_2) from the pair of (P'_1) and (D'_1). We show that the minimization and maximization problems for these two pairs are dual to each other through gap functions.

2 Fixed Initial Point

2.1 Pair 1

We consider a pair of minimization problem of n-variable $x = (x_1, x_2, \ldots, x_n)$ (primal)

$$
(\mathrm{P}_1) \qquad \begin{array}{lll} \text{minimize} & \displaystyle\sum_{k=1}^{n} \left[(x_{k-1} - x_k)^2 + x_k^2 \right] & \\ \text{subject to} & \text{(i)} & x \in R^n \\ & \text{(ii)} & x_0 = c \end{array}
$$

and maximization problem of n-variable $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ (dual)

$$(\mathrm{D}_1) \qquad \begin{array}{ll} \text{Maximize} & 2c\mu_1 - \left\{ \sum_{k=1}^{n-1} \left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] + 2\mu_n^2 \right\} \\ \text{subject to} & \text{(i)} \;\; \mu \in R^n. \end{array}$$

Theorem 1 *[25] The primal* (P_1) *attains a minimum*

$$m = f(\hat{x}) = c(c - \hat{x}_1) = \frac{F_{2n}}{F_{2n+1}} c^2$$

at a point

$$\begin{aligned} \hat{x} &= (\; \hat{x}_1, \;\; \hat{x}_2, \;\; \ldots, \;\; \hat{x}_k, \;\; \ldots, \;\; \hat{x}_{n-1}, \;\; \hat{x}_n \;) \\ &= \frac{c}{F_{2n+1}} (\; F_{2n-1}, \;\; F_{2n-3}, \;\; \ldots, \;\; F_{2n-2k+1}, \;\; \ldots, \;\; F_3, \;\; F_1 \;). \end{aligned}$$

The dual (D_1) *attains a maximum*

$$M = g(\mu^*) = c\mu_1^* = \frac{F_{2n}}{F_{2n+1}} c^2$$

at a point

$$\begin{aligned} \mu^* &= (\; \mu_1^*, \;\; \mu_2^*, \;\; \ldots, \;\; \mu_k^*, \;\; \ldots, \;\; \mu_{n-1}^*, \;\; \mu_n^* \;) \\ &= \frac{c}{F_{2n+1}} (\; F_{2n}, \;\; F_{2n-2}, \;\; \ldots, \;\; F_{2n-2k+2}, \;\; \ldots, \;\; F_4, \;\; F_2 \;). \end{aligned}$$

We remark that *Fibonacci sequence* $\{F_n\}$ is defined as the solution to the second-order linear difference equation

$$x_{n+2} - x_{n+1} - x_n = 0, \qquad x_1 = 1, \; x_0 = 0. \tag{1}$$

It is shown that (P_1) and (D_1) are dual to each other [25].

2.2 Pair 2

Next, we consider a pair of minimization problem of n-variable $y = (y_1, y_2, \ldots, y_n)$ (primal)

$$(\mathrm{P}_2) \qquad \begin{array}{ll} \text{minimize} & \sum_{k=1}^{n} \left(y_k^2 + Y_k^2\right) \\ \text{subject to} & \text{(i)} \;\; y \in R^n \end{array}$$

and maximization problem of n-variable $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ (dual)

$$(\mathrm{D}_2) \qquad \begin{array}{ll} \text{Maximize} & 2c\mu_1 - \left\{ \sum_{k=1}^{n-1} \left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] + 2\mu_n^2 \right\} \\ \text{subject to} & \text{(i)} \;\; \mu \in R^n \end{array}$$

where

$$Y_k \ = \ c - \sum_{l=1}^{k} y_l \qquad 1 \le k \le n. \tag{2}$$

Theorem 2 *[25] The primal* (P_2) *attains a minimum*

$$m \ = \ F(\hat{y}) \ = \ c\hat{y}_1 = \frac{F_{2n}}{F_{2n+1}}c^2$$

at a point

$$\begin{aligned} \hat{y} &= \left(\ \hat{y}_1, \quad \hat{y}_2, \quad \ldots, \quad \hat{y}_k, \quad \ldots, \quad \hat{y}_{n-1}, \quad \hat{y}_n \ \right) \\ &= \frac{c}{F_{2n+1}} \left(\ F_{2n}, \quad F_{2n-2}, \quad \ldots, \quad F_{2n-2k+2}, \quad \ldots, \quad F_4, \quad F_2 \ \right). \end{aligned}$$

The dual (D_2) *attains a maximum*

$$M \ = \ g(\mu^*) \ = \ c\mu_1^* \ = \ \frac{F_{2n}}{F_{2n+1}}c^2$$

at a point

$$\begin{aligned} \mu^* &= \left(\ \mu_1^*, \quad \mu_2^*, \quad \ldots, \quad \mu_k^*, \quad \ldots, \quad \mu_{n-1}^*, \quad \mu_n^* \ \right) \\ &= \frac{c}{F_{2n+1}} \left(\ F_{2n}, \quad F_{2n-2}, \quad \ldots, \quad F_{2n-2k+2}, \quad \ldots, \quad F_4, \quad F_2 \ \right). \end{aligned}$$

It is shown that (P_2) and (D_2) are dual to each other [25].

3 Fixed Both Points

3.1 Pair 1

We consider a pair of minimization problem of n-variable $x = (x_1, x_2, \ldots, x_n)$ (primal)

$$(\mathrm{P}_1') \quad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{k=1}^{n-1} \left[(x_{k-1} - x_k)^2 + x_k^2 \right] + (x_{n-1} - x_n)^2 + (x_n - x_{n+1})^2 \\ \text{subject to} & \text{(i)} \ \ x \in R^n \\ & \text{(ii)} \ \ x_0 = c, \quad x_{n+1} = d \end{array}$$

and maximization problem of n-variable $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ (dual)

$$(\mathrm{D}_1') \quad \begin{array}{ll} \text{Maximize} & 2c\mu_1 - \left\{ \displaystyle\sum_{k=1}^{n-1} \left[\mu_k^2 + (\mu_k - \mu_{k+1})^2 \right] + 2\mu_n^2 + 2d\mu_n \right\} \\ \text{subject to} & \text{(i)} \ \ \mu \in R^n. \end{array}$$

Let $f,\, g : R^n \to R^1$ be the respective objective functions of primal (P_1') and dual (D_1'):

$$f(x) \;=\; (c - x_1)^2 + x_1^2 + \sum_{k=2}^{n-1}\left[(x_{k-1} - x_k)^2 + x_k^2\right] + (x_{n-1} - x_n)^2 + (x_n - d)^2$$

$$g(\mu) \;=\; 2c\mu_1 - \sum_{k=1}^{n-1}\left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] - 2\mu_n^2 - 2d\mu_n.$$

Then we have an inequality

$$f(x) \;\geq\; g(\mu).$$

The sign of equality holds iff $2n$ equalities

$$(\mathrm{EC}_1) \qquad \begin{array}{rclrcll} c - x_1 &=& \mu_1, & x_1 &=& \mu_1 - \mu_2 & \\ x_{k-1} - x_k &=& \mu_k, & x_k &=& \mu_k - \mu_{k+1} & 2 \leq k \leq n-1 \\ x_{n-1} - x_n &=& \mu_n, & x_n - d &=& \mu_n & \end{array}$$

holds. (EC_1) is called an *equality condition* between (P_1') and (D_1').

Now let us define an *inner product* $i : R^n \times R^n \to R^1$ and a *gap function* $h : R^n \times R^n \to R^1$ as follows.

$$\begin{aligned} i(x,\mu) \;&=\; \sum_{k=1}^{n-1}\left[\, (x_{k-1} - x_k)\mu_k + x_k(\mu_k - \mu_{k+1})\,\right] \\ &\qquad\qquad + (x_{n-1} - x_n)\mu_n + (x_n - x_{n+1})\mu_n \qquad\qquad (x_0 = c,\;\; x_{n+1} = d) \\ h(x,\mu) \;&=\; \sum_{k=1}^{n-1}\left[\, (x_{k-1} - x_k - \mu_k)^2 + \{x_k - (\mu_k - \mu_{k+1})\}^2\,\right] \\ &\qquad\qquad + (x_{n-1} - x_n - \mu_n)^2 + (x_n - x_{n+1} - \mu_n)^2. \end{aligned}$$

Lemma 1

$$i(x,\mu) \;=\; c\mu_1 - d\mu_n \quad \textit{on} \;\; R^n \times R^n.$$

Lemma 2

$$\text{(i)} \qquad f(x) - g(\mu) \;=\; h(x,\mu) \;\geq\; 0 \quad \textit{on} \;\; R^n \times R^n$$

$$\text{(ii)} \qquad h(x,\mu) = 0 \;\iff\; (x,\mu) \;\textit{satisfies}\; (\mathrm{EC}_1).$$

Theorem 3 (i) *It holds that*

$$f(x) \;\geq\; g(\mu) \quad \textit{on} \;\; R^n \times R^n.$$

(ii) *It holds that*

$$f(x) \;=\; g(\mu) \;\iff\; (x,\mu) \;\textit{satisfies}\; (\mathrm{EC}_1).$$

Then (P_1') *attains a minimum* $f(x)$*, while* (D_1') *attains a maximum* $g(\mu)$.

Hence a solution (x,μ) to (EC_1) yields a minimum point x for (P_1') and a maximum point μ for (D_1').

3.2 Equality condition 1

Now let us analyze the equality condition (EC_1). The equality condition (EC_1) yields a pair of linear systems of n-equation on n-varaible:

$$(\mathrm{EQ}_x) \qquad \begin{aligned} 3x_1 - x_2 &= c \\ -x_{k-1} + 3x_k - x_{k+1} &= 0 \qquad 2 \le k \le n-1 \\ -x_{n-1} + 2x_n &= d, \end{aligned}$$

$$(\mathrm{EQ}_\mu) \qquad \begin{aligned} 2\mu_1 - \mu_2 &= c \\ -\mu_{k-1} + 3\mu_k - \mu_{k+1} &= 0 \qquad 2 \le k \le n-1 \\ -\mu_{n-1} + 3\mu_n &= -d. \end{aligned}$$

Lemma 3 *The system* (EQ_x) *has a unique solution*

$$\hat{x} = \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \dots \\ \hat{x}_{n-2} \\ \hat{x}_{n-1} \\ \hat{x}_n \end{pmatrix} = \frac{1}{F_{2n+1}} \begin{pmatrix} F_{2n-1}c + F_2 d \\ F_{2n-3}c + F_4 d \\ F_{2n-5}c + F_6 d \\ \dots \\ F_5 c + F_{2n-4} d \\ F_3 c + F_{2n-2} d \\ F_1 c + F_{2n} d \end{pmatrix}. \tag{3}$$

The system (EQ_μ) *has a unique solution*

$$\mu^* = \begin{pmatrix} \mu_1^* \\ \mu_2^* \\ \mu_3^* \\ \dots \\ \mu_{n-2}^* \\ \mu_{n-1}^* \\ \mu_n^* \end{pmatrix} = \frac{1}{F_{2n+1}} \begin{pmatrix} F_{2n}c - F_1 d \\ F_{2n-2}c - F_3 d \\ F_{2n-4}c - F_5 d \\ \dots \\ F_6 c - F_{2n-5} d \\ F_4 c - F_{2n-3} d \\ F_2 c - F_{2n-1} d \end{pmatrix}. \tag{4}$$

.

Hence the condition (EC_1) *has a unique solution* $(\hat{x}\,, \mu^*)$.

Theorem 4 *The primal* (P_1') *attains a minimum*

$$m = f(\hat{x}) = c(c - \hat{x}_1) - d(\hat{x}_n - d) = \frac{1}{F_{2n+1}}(F_{2n}c^2 - 2cd + F_{2n-1}d^2)$$

at the point $\hat{x}$.

The dual (D_1') *attains a maximum*

$$M = g(\mu^*) = c\mu_1^* - d\mu_n^* = \frac{1}{F_{2n+1}}(F_{2n}c^2 - 2cd + F_{2n-1}d^2)$$

at the point μ^*.

Hence the gap function $h = h(x, \lambda)$ *attains the zero minimum at* $(\hat{x}\,, \mu^*)$.

3.3 Pair 2

We consider a pair of minimization problem of n-variable $y = (y_1, y_2, \ldots, y_n)$ (primal)

$$(\mathrm{P}_2') \quad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{k=1}^{n-1}\left(y_k^2 + Y_k^2\right) + y_n^2 + \left(Y_n - d\right)^2 \\ \text{subject to} & \text{(i)} \;\; y \in R^n \end{array}$$

and maximization problem of n-variable $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ (dual)

$$(\mathrm{D}_2') \quad \begin{array}{ll} \text{Maximize} & 2c\mu_1 - \left\{ \displaystyle\sum_{k=1}^{n-1} \left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] + 2\mu_n^2 + 2d\mu_n \right\} \\ \text{subject to} & \text{(i)} \;\; \mu \in R^n \end{array}$$

where

$$Y_k = c - \sum_{l=1}^{k} y_l \qquad 1 \leq k \leq n. \tag{5}$$

Let F, $g : R^n \to R^1$ be the respective objective functions of primal (P_2') and dual (D_2'):

$$\begin{aligned} F(y) &= \sum_{k=1}^{n-1}\left(y_k^2 + Y_k^2\right) + y_n^2 + \left(Y_n - d\right)^2 \\ g(\mu) &= 2c\mu_1 - \left\{ \sum_{k=1}^{n-1} \left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] + 2\mu_n^2 + 2d\mu_n \right\}. \end{aligned}$$

Then we have an inequality

$$F(y) \geq g(\mu).$$

The sign of equality holds iff $2n$ equalities

$$(\mathrm{EC}_2) \quad \begin{array}{lll} y_1 = \mu_1, & Y_1 = \mu_1 - \mu_2 & \\ y_k = \mu_k, & Y_k = \mu_k - \mu_{k+1} & 2 \leq k \leq n-1 \\ y_n = \mu_n, & Y_n - d = \mu_n & \end{array}$$

holds. (EC_2) is called an *equality condition* between (P_2') and (D_2').

Now let us define an *inner product* $i : R^n \times R^n \to R^1$ and a *gap function* $h : R^n \times R^n \to R^1$ as follows.

$$i(y,\mu) = \sum_{k=1}^{n-1} \left[y_k\mu_k + Y_k(\mu_k - \mu_{k+1}) \right] + y_n\mu_n + (Y_n - d)\mu_n$$

$$h(y,\mu) = \sum_{k=1}^{n-1} \left\{ (y_k-\mu_k)^2 + [Y_k - (\mu_k - \mu_{k+1})]^2 \right\} + (y_n-\mu_n)^2 + \left\{ (Y_n - d) - \mu_n \right\}^2.$$

We note that

$$Y_k = c - Z_k\,;\; Z_k = \sum_{l=1}^{k} y_l \quad 1 \le k \le n.$$

Lemma 4

$$i(y,\mu) = c\mu_1 - d\mu_n \quad \textit{on } R^n \times R^n.$$

Lemma 5

$$\text{(i)} \qquad F(y) - g(\mu) = h(y,\mu) \ge 0 \quad \textit{on } R^n \times R^n$$

$$\text{(ii)} \qquad h(y,\mu) = 0 \iff (y,\mu) \textit{ satisfies } (\mathrm{EC}_2).$$

Theorem 5 (i) *It holds that*

$$F(y) \ge g(\mu) \quad \textit{on } R^n \times R^n.$$

(ii) *It holds that*

$$F(y) = g(\mu) \iff (y,\mu) \textit{ satisfies } (\mathrm{EC}_2).$$

Then (P_2') *attains a minimum* $F(y)$, *while* (D_2') *attains a maximum* $g(\mu)$.

Hence a solution (y,μ) to (EC_2) yields a minimum point x for (P_2') and a maximum point μ for (D_2').

3.4 Equality condition 2

Now let us analyze the equality condition (EC_2). The equality condition (EC_2) yields a pair of linear systems of n-equation on n-varaible:

$$(\mathrm{EQ}_y) \qquad \begin{aligned} 2y_1 - y_2 &= c \\ -y_{k-1} + 3y_k - y_{k+1} &= 0 \qquad 2 \le k \le n-1 \\ -y_{n-1} + 3y_n &= -d, \end{aligned}$$

$$(\mathrm{EQ}_\mu) \qquad \begin{aligned} 2\mu_1 - \mu_2 &= c \\ -\mu_{k-1} + 3\mu_k - \mu_{k+1} &= 0 \qquad 2 \le k \le n-1 \\ -\mu_{n-1} + 3\mu_n &= -d. \end{aligned}$$

Needless to say, both the systems are identical.

Lemma 6 *The system* (EQ_y) *has a unique solution*

$$\hat{y} = \begin{pmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \hat{y}_3 \\ \dots \\ \hat{y}_{n-2} \\ \hat{y}_{n-1} \\ \hat{y}_n \end{pmatrix} = \frac{1}{F_{2n+1}} \begin{pmatrix} F_{2n}c - F_1 d \\ F_{2n-2}c - F_3 d \\ F_{2n-4}c - F_5 d \\ \dots \\ F_6 c - F_{2n-5} d \\ F_4 c - F_{2n-3} d \\ F_2 c - F_{2n-1} d \end{pmatrix}. \tag{6}$$

The system (EQ_μ) *has a unique solution*

$$\mu^* = \begin{pmatrix} \mu_1^* \\ \mu_2^* \\ \mu_3^* \\ \dots \\ \mu_{n-2}^* \\ \mu_{n-1}^* \\ \mu_n^* \end{pmatrix} = \frac{1}{F_{2n+1}} \begin{pmatrix} F_{2n}c - F_1 d \\ F_{2n-2}c - F_3 d \\ F_{2n-4}c - F_5 d \\ \dots \\ F_6 c - F_{2n-5} d \\ F_4 c - F_{2n-3} d \\ F_2 c - F_{2n-1} d \end{pmatrix}. \tag{7}$$

.

Hence the condition (EC_2) *has a unique solution* $(\hat{y}\,, \mu^*)$.

Theorem 6 *The primal* (P_2') *attains a minimum*

$$\begin{aligned} m = f(\hat{y}) &= c\hat{y}_1 - d\hat{y}_n = c\hat{y}_1 - d\big(c - \hat{Y}_n - d\big) \\ &= \frac{1}{F_{2n+1}}(F_{2n}c^2 - 2cd + F_{2n-1}d^2) \end{aligned}$$

at the point $\hat{y}$.

The dual (D_2') *attains a maximum*

$$M = g(\mu^*) = c\mu_1^* - d\mu_n^* = \frac{1}{F_{2n+1}}(F_{2n}c^2 - 2cd + F_{2n-1}d^2)$$

at the point μ^*.

Hence the gap function $h = h(y, \mu)$ *attains the zero minimum at* $(\hat{y}\,, \mu^*)$.

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