Stochastic Process Modeling of an Infectious Disease in a Finite Population

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Abstract

A birth-and-death process model with a finite state space is proposed and studied for an infectious disease spreading with several solution techniques such as a planar partial differential equation solved by Lagrange's method, direct inversion of Laplace transforms of the forward Kolmogorov equations, Syski's spectral decomposition for general continuous-time Markov processes, and Karlin-McGregor's symmetrization of birth-and-death processes along with simple numerical examples.

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1 Introduction

So-called SIR models have been used as major mathematical tools for quantitative analysis and prediction of the recent world-wide spread of COVID-19 without much success. The SIR-type models are based on a set of simultaneous nonlinear ordinary differential equations in several non-probabilistic variables. In principle, it would be very difficult for theoretically deterministic approaches to account for the large statistical variation in the number of infected people observed in different cities and countries with similar conditions around the world. The deterministic models could, at most, only yield average values and explain the time-varying characteristics of the pandemic in the past.

The spreading of an infectious disease is a probabilistic phenomenon because the contact with infectious persons does not necessarily cause the infection. It cannot be either certain that the vaccination suppresses infection with 100 percent of success. Therefore, we should not depend on deterministic models for important prediction.

Stochastic process models have been exploited for population biology and epidemics mostly assuming an infinite population. In the present paper, we focus on the analysis of a finite population model. A real case of the infection in a finite population is the COVID-19 outbreak among passengers and crew on Diamond Princess cruise ship in Yokohama Port, Japan, in early 2020 [7]. We believe that exploration of stochastic techniques for epidemic models will contribute to academia as well as to the society in general with significant impact.

2 Kolmogorov Equations for the Birth-and-Death Process

Our basic model for the spreading of an infectious disease is a continuous-time birth-and-death process of a stochastic variable $I(t)$, the number of infected persons at time t (≥ 0) in a discrete state space between 0 and N inclusive. We assume the three positive, constant parameters as follows:

λ (birth rate) : Mean number of persons who get infected from others per unit time,

μ (removal rate) : Mean number of persons who die or recover per unit time,

ν (immigration rate) : Mean number of persons who become infectious per unit time.

We study the probability $P_n(t) \triangleq P\{I(t) = n \mid P(0) = I_0\}, 0 \leq n \leq N$. For a short interval $\Delta t \ll t$, we consider the events occurring during an infinitesimal interval $(t, t+\Delta t]$. We get the forward Chapman-Kolmogorov equations for the process $\{I(t); t \geq 0\}$ as

$$\begin{aligned} P_0(t+\Delta t) &= (1 - N\nu\Delta t)P_0(t) + \mu\Delta t P_1(t) + o(\Delta t), & (1) \\ P_n(t+\Delta t) &= \{1 - [n(\lambda+\mu) + (N-n)\nu]\Delta t\}P_n(t) + (n+1)\mu\Delta t P_{n+1}(t) \\ &+ \{(n-1)\lambda + (N-n+1)\nu\}\Delta t P_{n-1}(t) + o(\Delta t) \quad 1 \leq n \leq N-1, & (2) \\ P_N(t+\Delta t) &= (1 - N\mu\Delta t)P_N(t) + \{(N-1)\lambda + \nu\}\Delta t P_{N-1}(t) + o(\Delta t). & (3) \end{aligned}$$

By moving some terms, dividing both sides by Δt, and making $\Delta t \to 0$, we obtain the following set of simultaneous differential-difference equations for $\{P_n(t); 0 \leq n \leq N\}$:

$$\begin{aligned} \frac{dP_0(t)}{dt} &= -N\nu P_0(t) + \mu P_1(t), & (4) \\ \frac{dP_n(t)}{dt} &= -\{n(\lambda+\mu) + (N-n)\nu\}P_n(t) + (n+1)\mu P_{n+1}(t) \\ &+ \{(n-1)\lambda + (N-n+1)\nu\}P_{n-1}(t) \qquad 1 \leq n \leq N-1, & (5) \\ \frac{dP_N(t)}{dt} &= -N\mu P_N(t) + \{(N-1)\lambda + \nu\}P_{N-1}(t). & (6) \end{aligned}$$

In addition, we impose the normalization condition: $\sum_{n=0}^{N} P_n(t) = 1$.

Let us introduce the Probability Generating Function (PGF) for $I(t)$ by

$$G(z,t) \triangleq E\left[z^{I(t)}\right] = \sum_{n=0}^{N} P_n(t) z^n \qquad ; \qquad \frac{\partial G(z,t)}{\partial z} = \sum_{n=1}^{N} n P_n(t) z^{n-1}. \tag{7}$$

Then, we get the following planar partial differential equation:

$$\frac{\partial G(z,t)}{\partial t} - (z-1)\{(\lambda-\nu)z - \mu\}\frac{\partial G(z,t)}{\partial z} = N\nu(z-1)G(z,t) + N\lambda P_N(t) z^N. \tag{8}$$

So far, we have been unable to solve this equation analytically due to the last term on the right-hand side. Therefore, in the next section, we show the numerical solution using the "DSolve" function of Mathematica.

3 Numerical Solution by Mathematica DSolve Function

In Table 1, we tabulate the probabilities $\{P_n(t); 0 \le n \le 8\}$, the mean $E[I(t)]$, and the variance $\mathrm{Var}[I(t)]$ by assuming $\lambda = 0.5$. The numerical values are obtained by the solution of the set of simultaneous ordinary differential equations given in eqs. (4)–(6) by Mathematica "DSolve" function.

Table 1: Probabilities $\{P_n(t); 0 \le n \le 8\}$, the mean $E[I(t)]$, and the variance $\mathrm{Var}[I(t)]$. Parameters: $N = 8, I_0 = 5, \lambda = 0.5, \nu = 0.2, \mu = 0.1$.

t	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_3(t)$	$P_4(t)$	$P_5(t)$	$P_6(t)$	$P_7(t)$	$P_8(t)$	$E[I(t)]$	$\mathrm{Var}[I(t)]$
0	0	0	0	0	0	1.000000	0	0	0	5.00000	0
1	≈ 0	0.000037	0.000646	0.006009	0.031521	0.093509	0.161557	0.225604	0.481116	7.01049	1.413950
2	≈ 0	0.000059	0.000514	0.002631	0.009265	0.024756	0.061208	0.181213	0.718915	7.56685	0.698757
3	≈ 0	0.000038	0.000239	0.001004	0.003402	0.011135	0.040970	0.172656	0.770553	7.69165	0.431854
4	≈ 0	0.000019	0.000105	0.000450	0.001831	0.007923	0.036509	0.170672	0.782489	7.72219	0.356205
5	≈ 0	0.000009	0.000054	0.000274	0.001392	0.007097	0.035420	0.170236	0.785516	7.73030	0.334643
6	≈ 0	0.000001	0.000036	0.000219	0.001265	0.006871	0.035133	0.170130	0.786340	7.73258	0.328290
7	≈ 0	0.000003	0.000030	0.000202	0.001227	0.006806	0.035053	0.107103	0.786576	7.73325	0.326360
8	≈ 0	0.000003	0.000027	0.000196	0.001216	0.006787	0.035029	0.170095	0.786646	7.73345	0.325759
9	≈ 0	0.000003	0.000027	0.000195	0.001212	0.006781	0.035022	0.170093	0.786667	7.73352	0.325568
10	≈ 0	0.000003	0.000027	0.000193	0.001211	0.006779	0.035020	0.170093	0.786674	7.73354	0.325507

4 Special Case of No Internal Infection

In this section, we consider a special case $\lambda = 0$, which means that the internal infection never occurs by means of perfect management of infected persons. In such a case, we are to solve the partial differential equation [1]

$$\frac{\partial G(z,t)}{\partial t} + (z-1)(\nu z+\mu)\frac{\partial G(z,t)}{\partial z} = N\nu(z-1)G(z,t). \tag{9}$$

Then we can use well-known Lagrange's method. The corresponding set of auxiliary differential equations defining the normal at a point on the solution surface is given by

$$\frac{dt}{1} = \frac{dz}{(z-1)(\nu z+\mu)} = \frac{dG(z,t)}{N\nu(z-1)G(z,t)}. \tag{10}$$

Following a standard procedure, we obtain the PGF for $I(t)$ as follows:

$$G(z,t) = [q(t)z + 1 - q(t)]^{I_0}[r(t)z + 1 - r(t)]^{N-I_0}, \tag{11}$$

which is a polynomial in z of degree N, where

$$q(t) \triangleq \frac{\nu + \mu e^{-(\nu+\mu)t}}{\nu+\mu} \quad ; \quad r(t) \triangleq \frac{\nu\left(1 - e^{-(\nu+\mu)t}\right)}{\nu+\mu}. \tag{12}$$

Therefore, the PGF $G(z,t)$ is the product of the PGF's for two statistically independent processes, each being the binomial (or Bernoulli) distribution.

4.1 Mean and Variance

Since the two binomial distributions are independent, the mean of $I(t)$ is given by the sum of the means for each distribution. Similarly, the variance of $I(t)$ is given by the sum of the variances for each distribution.

$$E[I(t)] = I_0 e^{-(\nu+\mu)t} + \frac{N\nu}{\nu+\mu}\left(1 - e^{-(\nu+\mu)t}\right), \tag{13}$$

$$\mathrm{Var}[I(t)] = I_0\frac{\nu-\mu}{\nu+\mu}e^{-(\nu+\mu)t}\left(1 - e^{-(\nu+\mu)t}\right) + \frac{N\nu^2}{(\nu+\mu)^2}\left(1 - e^{-2(\mu+\nu)t}\right). \tag{14}$$

4.2 Probability Mass Function

The probability mass function (PMF) of $I(t)$ is given by the convolution of two binomial distributions (Bailey [1, p.64], Kobayashi and Ren [5], Syski [6, p.138]).

$$P_0(t) = [1-q(t)]^{I_0}[1-r(t)]^{N-I_0} = \left[\frac{\mu\left(1 - e^{-(\nu+\mu)t}\right)}{\nu+\mu}\right]^{I_0}\left[\frac{\mu + \nu e^{-(\nu+\mu)t}}{\nu+\mu}\right]^{N-I_0}, \tag{15}$$

[1] Solution to the equivalent partial differential equation is shown in [5] and [6, p.138].

$$
\begin{aligned}
P_n(t) &= \sum_{i=\max\{0,I_0-N+n\}}^{\min\{n,I_0\}} \binom{I_0}{i} [q(t)]^i [1-q(t)]^{I_0-i} \\
&\qquad \times \binom{N-I_0}{n-i} [r(t)]^{n-i} [1-r(t)]^{N-I_0-(n-i)} \quad 1 \le n \le N-1, \quad (16)
\end{aligned}
$$

$$
P_N(t) = [q(t)]^{I_0}[r(t)]^{N-I_0} = \left[\frac{\nu + \mu e^{-(\nu+\mu)t}}{\nu+\mu}\right]^{I_0} \left[\frac{\nu\left(1-e^{-(\nu+\mu)t}\right)}{\nu+\mu}\right]^{N-I_0}. \quad (17)
$$

We note that $P_0(t)$ is the probability that there are no infected persons and that $P_N(t)$ is the probability that all persons are infected at time t, respectively.

4.3 Steady State

In the steady state at time $t \to \infty$, we have

$$
q(\infty) \triangleq \lim_{t\to\infty} q(t) = \frac{\nu}{\nu+\mu} \quad ; \quad r(\infty) \triangleq \lim_{t\to\infty} r(t) = \frac{\nu}{\nu+\mu} = q(\infty). \quad (18)
$$

Therefore, we get the following statistics.

$$
G(z,\infty) \triangleq \lim_{t\to\infty} G(z,t) = \left(\frac{\nu z+\mu}{\nu+\mu}\right)^N \quad \text{(binomial distribution)}. \quad (19)
$$

$$
P_n = \lim_{t\to\infty} P_n(t) = \binom{N}{n} \left(\frac{\nu}{\nu+\mu}\right)^n \left(\frac{\mu}{\nu+\mu}\right)^{N-n} \quad 0 \le n \le N. \quad (20)
$$

$$
E[I(\infty)] = \frac{N\nu}{\nu+\mu} \quad ; \quad \mathrm{Var}[I(\infty)] = \frac{N\nu\mu}{(\nu+\mu)^2}. \quad (21)
$$

In Table 2, we assume $\lambda = 0$. The numerical values have been obtained from the exact analytical solution given above.

5 Inversion of Laplace Transforms

We first try inversion of the Laplace transforms of the set of simultaneous differential equations for $\{P_n(t); 0 \le n \le N\}$ shown in eqs. (4)–(6), which are written as a set of algebraic equations in terms of Laplace transforms $\{P_n^*(s); 0 \le n \le N\}$ via

$$
P_n^*(s) \triangleq \int_0^\infty e^{-st} P_n(t)dt \quad ; \quad \int_0^\infty e^{-st} \frac{dP_n(t)}{dt} dt = sP_n^*(s) - P_n(0) \qquad 0 \le n \le N \quad (22)
$$

as follows:

$$
\frac{d\boldsymbol{P}(t)}{dt} = \boldsymbol{P}(t)\mathcal{Q} \quad ; \quad s\boldsymbol{P}^*(s) - \boldsymbol{P}(0) = \boldsymbol{P}^*(s)\mathcal{Q}, \quad (23)
$$

Table 2: Probabilities $\{P_n(t); 0 \leq n \leq 8\}$, the mean $E[I(t)]$, and the variance $\mathrm{Var}[I(t)]$. Parameters: $N = 8, I_0 = 5, \lambda = 0, \nu = 0.2, \mu = 0.1$.

t	$P_0(t)$	$P_1(t)$	$P_2(t)$	$P_3(t)$	$P_4(t)$	$P_5(t)$	$P_6(t)$	$P_7(t)$	$P_8(t)$	$E[I(t)]$	$\mathrm{Var}[I(t)]$
0	0	0	0	0	0	1.000000	0	0	0	5.00000	0
1	0.000003	0.000146	0.003137	0.034146	0.190938	0.471277	0.248359	0.048711	0.003283	5.08639	0.82345
2	0.000026	0.000777	0.009367	0.058665	0.199849	0.351167	0.273425	0.098606	0.012047	5.15040	1.26983
3	0.000067	0.001487	0.013744	0.067947	0.192502	0.311302	0.272751	0.119631	0.020569	5.19781	1.51072
4	0.000105	0.001994	0.016073	0.070965	0.186091	0.293925	0.270874	0.133123	0.026851	5.23294	1.63989
5	0.000130	0.002287	0.017153	0.071635	0.181596	0.285142	0.270077	0.140935	0.031045	5.25896	1.70853
6	0.000145	0.002435	0.017580	0.071445	0.178493	0.280345	0.270042	0.145762	0.033752	5.27823	1.74453
7	0.000153	0.002498	0.017692	0.070979	0.176325	0.277600	0.270369	0.148894	0.035491	5.29251	1.76306
8	0.000156	0.002517	0.017664	0.070468	0.174789	0.275974	0.270818	0.150995	0.036618	5.30309	1.77231
9	0.000157	0.002516	0.017583	0.070003	0.173689	0.274985	0.271269	0.152439	0.037359	5.31093	1.77671
10	0.000157	0.002507	0.017489	0.069612	0.172893	0.274367	0.271669	0.153451	0.037855	5.31674	1.77863
∞	0.000152	0.000305	0.000610	0.001219	0.024390	0.004877	0.009755	0.019509	0.039018	5.33333	1.77778

which can be modified to

$$\boldsymbol{P}^*(s)\,(s\mathcal{I} - \mathcal{Q}) = \boldsymbol{P}(0) \quad ; \quad \boldsymbol{P}^*(s) = \boldsymbol{P}(0)\,(s\mathcal{I} - \mathcal{Q})^{-1}\,, \tag{24}$$

where $\boldsymbol{P}^*(s)$ is a row vector for $\{P_n^*(s); 0 \leq n \leq N\}$, $\boldsymbol{P}(0)$ is a row vector for the initial condition of a row vector $\boldsymbol{P}(t)$ for $\{P_n(t); 0 \leq n \leq N\}$ at $t = 0$, The **infinitesimal generator** $\mathcal{Q}$ is an $(N+1) \times (N+1)$ tridiagonal (non-symmetric) matrix, of which the (m, n)th element is the state transition rate from state n to state m, $0 \leq m, n \leq N$. $\mathcal{I}$ is the $(N+1) \times (N+1)$ identity matrix in which the diagonal elements are 1 and all off-diagonal elements are 0:

$$\boldsymbol{P}^*(s) \triangleq (P_0^*(s), P_1^*(s), \ldots, P_n^*(s), \ldots, P_N^*(s)), \tag{25}$$

$$\boldsymbol{P}(t) \triangleq (P_0(t), P_1(t), \ldots, P_n(t), \ldots, P_N(t)), \tag{26}$$

$$\mathcal{Q} \triangleq \begin{pmatrix} Q_{0,0} & Q_{0,1} & Q_{0,2} & \cdots & Q_{0,n} & \cdots & Q_{0,N-1} & Q_{0,N} \\ Q_{1,0} & Q_{1,1} & Q_{1,2} & \cdots & Q_{1,n} & \cdots & Q_{1,N-1} & Q_{1,N} \\ Q_{2,0} & Q_{2,1} & Q_{2,2} & \cdots & Q_{2,n} & \cdots & Q_{2,N-1} & Q_{2,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ Q_{N-1,0} & Q_{N-1,1} & Q_{N-1,2} & \cdots & Q_{N-1,n} & \cdots & Q_{N-1,N-1} & Q_{N-1,N} \\ Q_{N,0} & Q_{N,1} & Q_{N,2} & \cdots & Q_{N,n} & \cdots & Q_{N,N-1} & Q_{N,N} \end{pmatrix}, \tag{27}$$

where the nonzero elements of $\mathcal{Q}$ are given by

$$Q_{n,n} = -(\lambda_n + \mu_n) = -\{n\lambda + (N-n)\nu + n\mu\}, \quad 1 \leq n \leq N-1,$$
$$Q_{0,0} = -N\nu \quad ; \quad Q_{N,N} = -N\mu,$$
$$Q_{n,n+1} = \lambda_n \triangleq n\lambda + (N-n)\nu, \quad 0 \leq n \leq N-1,$$
$$Q_{n,n-1} = \mu_n \triangleq n\mu, \quad 1 \leq n \leq N.$$

Since $Q_{n,n}+Q_{n,n+1}+Q_{n,n-1}=0$, $1 \le n \le N-1$, and $Q_{0,0}+Q_{1,0}=Q_{N-1,N}+Q_{N,N}=0$, the sum of all elements in every row (row sum) is zero.

The **stationary distribution** $\boldsymbol{\pi}=(\pi_0,\pi_1,\ldots,\pi_N)$ (a row vector) can be obtained either from the **detailed balance equations**

$$\pi_n\lambda_n=\pi_{n+1}\mu_{n+1}, \quad 0 \le n \le N-1, \tag{28}$$

or as the left eigenvector of matrix $\mathcal{Q}$ associated with eigenvalue 0, both along with the usual normalization:

$$\boldsymbol{\pi}\mathcal{Q}=0\boldsymbol{\pi}, \qquad \sum_{n=0}^{N}\pi_n=1. \tag{29}$$

For our example with the same parameters as in Table 1, we have

$$\mathcal{Q} \triangleq \begin{pmatrix} -1.6 & 1.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.1 & -2 & 1.9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & -2.4 & 2.2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & -2.8 & 2.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & -3.2 & 2.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & -3.6 & 3.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.6 & -4 & 3.4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & -4.4 & 3.7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.8 & -0.8 \end{pmatrix}, \tag{30}$$

Through the above-mentioned procedure, we obtain

$$\begin{aligned}
P_0(t) &= 1.73732\times 10^{-7} - 1.92016\times 10^{-7}e^{s_1 t} - 7.6608\times 10^{-6}e^{s_2 t} + 0.0000419876e^{s_3 t} \\
&+ 0.0000216199e^{s_4 t} - 0.000499537e^{s_5 t} + 0.00117554e^{s_6 t} - 0.00113797e^{s_7 t} \\
&+ 0.000406046e^{s_8 t}, \\
P_1(t) &= 2.77972\times 10^{-6} + 9.23127\times 10^{-6}e^{s_1 t} + 0.000254654e^{s_2 t} - 0.000943395e^{s_3 t} \\
&- 0.000304069e^{s_4 t} + 0.0036748e^{s_5 t} - 0.00237707e^{s_6 t} - 0.00239809e^{s_7 t} \\
&+ 0.00208116e^{s_8 t}, \\
P_2(t) &= 0.0000264073 - 0.0002019e^{s_1 t} - 0.0036619e^{s_2 t} + 0.00837795e^{s_3 t} + 0.00135717e^{s_4 t} \\
&- 0.00217077e^{s_5 t} - 0.0117552e^{s_6 t} + 0.00178083e^{s_7 t} + 0.00624636e^{s_8 t}, \\
P_3(t) &= 0.000193654 + 0.00263863e^{s_1 t} + 0.0291974e^{s_2 t} - 0.0344438e^{s_3 t} - 0.000817651e^{s_4 t} \\
&- 0.0237394e^{s_5 t} - 0.0083689e^{s_6 t} + 0.0211877e^{s_7 t} + 0.0141524e^{s_8 t}, \\
P_4(t) &= 0.00121033 - 0.022687e^{s_1 t} - 0.134906e^{s_2 t} + 0.0441058e^{s_3 t} - 0.00704244e^{s_4 t} \\
&- 0.0156198e^{s_5 t} + 0.0437773e^{s_6 t} + 0.0649309e^{s_7 t} + 0.0262311e^{s_8 t}, \\
P_5(t) &= 0.00677788 + 0.132346e^{s_1 t} + 0.319201e^{s_2 t} + 0.115118e^{s_3 t} + 0.00136186e^{s_4 t} \\
&+ 0.0916948e^{s_5 t} + 0.164228e^{s_6 t} + 0.129206e^{s_7 t} + 0.0400664e^{s_8 t}, \\
P_6(t) &= 0.035019 - 0.513411e^{s_1 t} - 0.0748694e^{s_2 t} - 0.253279e^{s_3 t} + 0.034212e^{s_4 t} \\
&+ 0.266118e^{s_5 t} + 0.287783e^{s_6 t} + 0.173058e^{s_7 t} + 0.0453692e^{s_8 t}, \\
P_7(t) &= 0.170092 + 1.1797e^{s_1 t} - 1.31476e^{s_2 t} - 0.56505e^{s_3 t} + 0.0425288e^{s_4 t} \\
&+ 0.22666e^{s_5 t} + 0.17626e^{s_6 t} + 0.0732389e^{s_7 t} + 0.0113342e^{s_8 t}, \\
P_8(t) &= 0.786677 - 0.778394e^{s_1 t} + 1.17956e^{s_2 t} + 0.686073e^{s_3 t} - 0.0713173e^{s_4 t} \\
&- 0.546117e^{s_5 t} - 0.650772e^{s_6 t} - 0.459866e^{s_7 t} - 0.145888e^{s_8 t},
\end{aligned}$$

where $\{s_1 = -6.40755, s_2 = -4.92412, s_3 = -3.84732, s_4 = -3.00643, s_5 = -2.33564,\ s_6 = -1.80221, s_7 = -1.38927, s_8 = -1.08746\}$ are negative eigenvalues of matrix $\mathcal{Q}$. The stationary distribution is given by the left eigenvector: $\boldsymbol{u}_0 = \boldsymbol{\pi} = (1.73732\times10^{-7},\ 2.77972\times 10^{-6},\ 0.0000264073,\ 0.000193654,\ 0.00121033,\ 0.00677788,\ 0.035019,\ 0.170092,\ 0.786677)$.

We can confirm that the expressions

$$\sum_{n=0}^{8} P_n(t) = 1 \quad ; \quad E[I(t)] = \sum_{n=0}^{8} nP_n(t) \quad ; \quad \mathrm{Var}[I(t)] = \sum_{n=0}^{8}(n - E[I(t)])^2 P_n(t) \tag{31}$$

yield the numerical values for each time t shown in Table 1.

6 Syski's Spectral Decomposition

Syski's spectral decomposition method is presented in his classic book [6, Chapter 5]. It can be applied to general continuous-time Markov processes. It was used by Kobayashi and Ren [5] for their transient analysis of a finite number of on-off sources in a statistical multiplexer of communication networks.

Let s_n be the nth eigenvalue of matrix $\mathcal{Q}$, and $\boldsymbol{u}_n$ and $\boldsymbol{v}_n$ be the left (row) and right (column) eigenvectors, respectively, associated with s_n: [2]

$$\boldsymbol{u}_n\mathcal{Q} = s_n\boldsymbol{u}_n, \quad \mathcal{Q}\boldsymbol{v}_n = s_n\boldsymbol{v}_n, \qquad 0 \le n \le N. \tag{32}$$

The left eigenvectors $\{\boldsymbol{u}_n\}$ are calculated from the right eigenvectors $\{\boldsymbol{v}_n\}$ as

$$\left(\boldsymbol{u}_0^T, \boldsymbol{u}_1^T, \ldots, \boldsymbol{u}_N^T\right)^T = (\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_N)^{-1} \tag{33}$$

so that they are **bi-orthogonal** and normalized as

$$\boldsymbol{u}_m\boldsymbol{v}_n = \sum_{k=0}^{N} u_m(k)v_n(k) = \delta_{m,n} = \begin{cases} 1 & m = n \\ 0 & m \ne n \end{cases}, \tag{34}$$

where $u_m(k)$ is the kth element of the mth left eigenvector $\boldsymbol{u}_m$ and $v_n(k)$ is the kth element of the nth right eigenvector $\boldsymbol{v}_n$. We tabulate $s_m, \boldsymbol{v}_m$ and $\boldsymbol{u}_m$ in Tables 3 and 4.

In particular, since $s_0 = 0$, we have $\mathcal{Q}\boldsymbol{v}_0 = 0$ so that we can set $\boldsymbol{v}_0 = (1, 1, \ldots, 1)^T$, because all row sums are zero. On the other hand, since $\boldsymbol{u}_0\mathcal{Q} = 0\boldsymbol{u}_0$, we have the stationary distribution $\boldsymbol{u}_0 = \boldsymbol{\pi}$, so that $\boldsymbol{u}_0\boldsymbol{v}_0 = 1$, i.e., the normalization conditions in eq. (29) and in eq. (34) for $m = n = 0$ are satisfied.

Now, we have the formal solution

$$\frac{d\boldsymbol{P}(t)}{dt} = \boldsymbol{P}(t)\mathcal{Q} \quad \Longrightarrow \quad \boldsymbol{P}(t) = \boldsymbol{P}(0)e^{\mathcal{Q}t}, \tag{35}$$

where the **matrix exponential** is given by

$$e^{\mathcal{Q}t} = \sum_{n=0}^{N} e^{s_n t}\boldsymbol{v}_n\boldsymbol{u}_n = \boldsymbol{v}_0\boldsymbol{u}_0 + \sum_{n=1}^{N} e^{s_n t}\boldsymbol{v}_n\boldsymbol{u}_n. \tag{36}$$

[2] In Mathematica, the right eigenvectors of a matrix Q are obtained by `Eigenvectors`$[Q]$.

If we define the following $n+1$ scalars

$$\gamma_0 = 1 \quad ; \quad \gamma_n \triangleq \boldsymbol{P}(0)\boldsymbol{v}_n = \sum_{k=0}^{N} P_k(0) v_n(k), \qquad 1 \le n \le N, \tag{37}$$

we get the **spectral decomposition** of $\boldsymbol{P}(t)$:

$$\begin{aligned} \boldsymbol{P}(t) &= \boldsymbol{P}(0)\boldsymbol{v}_0\boldsymbol{u}_0 + \sum_{n=1}^{N} e^{s_n t}\boldsymbol{P}(0)\boldsymbol{v}_n\boldsymbol{u}_n = \gamma_0\boldsymbol{u}_0 + \sum_{n=1}^{N} e^{s_n t}\gamma_n\boldsymbol{u}_n \\ &= \boldsymbol{\pi} + \sum_{n=1}^{N} e^{s_n t}\gamma_n\boldsymbol{u}_n, \qquad t \ge 0. \end{aligned} \tag{38}$$

If the initial distribution vector $\boldsymbol{P}(0)$ is such that the only I_0th element is 1, i.e., the process starts with state I_0, the mth element of the row vector $\boldsymbol{P}(t)$ is given by

$$P_m(t) = \pi_m + \sum_{n=1}^{N} e^{s_n t} v_n(I_0) u_n(m), \qquad 0 \le m \le N, \tag{39}$$

where $v_n(I_0)$ is the I_0th element of the nth right eigenvector $\boldsymbol{v}_n$, and $u_n(m)$ is the mth element of the nth left eigenvector $\boldsymbol{u}_n$, respectively, associated with eigenvalue s_n. This calculation agrees with the results given in Section 5.

For example, in order to obtain the coefficient of $e^{s_1 t}$ in the expression for $P_2(t)$, we multiply $v_1(5) = 0.55162$ in Table 3 and $u_1(2) = -0.00036612$ in Table 4. The result is -0.0002019, which agrees precisely with the corresponding term.

7 Karlin-McGregor's Symmetrization

For efficient application of Syski's method, we refer to the Karlin-McGregor's symmetrization for birth-and-death processes, which was originally derived in [3] for general processes and later expressed in a matrix formulation by Keilson [4, Section 3.2] for processes with a finite number of states. The symmetrization follows from the reversibility of the birth-and-death process and the associated detailed balance equations in (28). In terms of the generator matrix $\mathcal{Q}$, the detailed balance equations can be written as

$$\pi_n Q_{n,n+1} = Q_{n+1,n}\pi_{n+1}, \qquad 0 \le n \le N-1. \tag{40}$$

We define the diagonal matrix $D \triangleq \text{diag}(\pi_0, \pi_1, \ldots, \pi_N)$ in terms of the stationary distribution. Then, eq. (28) implies that

$$D\mathcal{Q} = \mathcal{Q}^T D = (D\mathcal{Q})^T, \tag{41}$$

i.e., the matrix DQ is symmetric. We further define the matrix

$$J \triangleq D^{1/2}\mathcal{Q}D^{-1/2}, \tag{42}$$

Table 3: Eigenvalues $\{s_m\}$ and the nth element of the mth right eigenvector $v_m(n)$ of $\mathcal{Q}$. All the elements of $\boldsymbol{v}_0$ are equal (set to 1) because each row sum of matrix $\mathcal{Q}$ is zero.

m	s_m	$v_m(0)$	$v_m(1)$	$v_m(2)$	$v_m(3)$	$v_m(4)$	$v_m(5)$	$v_m(6)$	$v_m(7)$	$v_m(8)$
0	0	1	1	1	1	1	1	1	1	1
1	−6.40755	−0.031223	0.093817	−0.215991	0.384924	−0.529534	0.551622	−0.414174	0.195934	−0.027953
2	−4.92412	−0.170347	0.353908	−0.535702	0.582452	−0.430594	0.181933	−0.008259	−0.029861	0.005792
3	−3.84732	−0.435713	0.611992	−0.572092	0.320729	−0.065712	−0.030627	0.013042	0.005990	−0.001573
4	−3.00643	−0.716733	0.630022	−0.296000	0.024318	−0.033512	0.001157	−0.005627	−0.001440	0.000522
5	−2.33564	−0.907563	0.417275	−0.025946	−0.038693	−0.004073	−0.004270	0.002399	0.000421	−0.000219
6	−1.80221	−0.989961	0.125113	0.065127	0.006323	−0.005292	−0.003545	−0.001202	−0.000152	0.000121
7	−1.38927	−0.991213	−0.130551	0.010205	0.016557	0.008118	0.002885	0.000748	0.000065	−0.000088
8	−1.08746	0.947481	0.303515	0.095907	0.029627	0.008786	0.002396	0.000525	0.000027	−0.000075

Table 4: The nth element of the mth left eigenvector $u_m(n)$ of $\mathcal{Q}$, where $u_0(n) = \pi_n$ (stationary distribution).

m	$u_m(0)$	$u_m(1)$	$u_m(2)$	$u_m(3)$	$u_m(4)$	$u_m(5)$	$u_m(6)$	$u_m(7)$	$u_m(8)$
0	1.74×10^{-7}	2.78×10^{-6}	0.000026	0.000194	0.001210	0.006778	0.035019	0.170092	0.786677
1	3.48×10^{-7}	1.67×10^{-5}	−0.000366	0.004783	−0.04113	0.239923	−0.930729	2.138603	-1.411102
2	-4.21×10^{-5}	0.00139971	−0.020130	0.160484	−0.74152	1.754495	−0.411522	−7.226630	6.483459
3	−0.001370662	0.03080298	−0.273550	1.124630	−1.44011	−3.758730	8.269860	18.449600	−22.401100
4	−1.087457439	0.26275462	−1.172760	0.706554	6.08556	−1.176820	−29.563488	−36.750300	61.627201
5	−1.018682367	0.86058297	−0.508361	−5.559420	−3.65793	21.473559	62.320822	53.080280	−127.892500
6	−0.116984115	0.67054878	3.316013	2.360783	−12.3492	−46.326962	−81.180816	−49.721100	183.562290
7	−0.331609365	−0.8313008	0.617329	7.344756	22.50841	44.789676	59.990771	25.388420	−159.413600
8	0.169438943	0.86844670	2.606961	5.905663	10.94596	16.719304	18.932117	4.729649	−60.877540

where

$$D^{1/2} \triangleq \mathrm{diag}\left(\sqrt{\pi_0}, \sqrt{\pi_1}, \ldots, \sqrt{\pi_N}\right) \quad ; \quad D^{-1/2} \triangleq \mathrm{diag}\left(\frac{1}{\sqrt{\pi_0}}, \frac{1}{\sqrt{\pi_1}}, \ldots, \frac{1}{\sqrt{\pi_N}}\right). \tag{43}$$

We can verify that J is symmetric using eq. (40) as follows:

$$J^T = D^{-1/2}\mathcal{Q}^T D^{1/2} = D^{-1/2}(\mathcal{Q}^T D)D^{-1/2} = D^{-1/2}(DQ)D^{-1/2} = D^{1/2}QD^{-1/2} = J. \tag{44}$$

Since J is symmetric, it has real eigenvalues and an orthonormal basis of (right) eigenvectors $\{\boldsymbol{w}_0, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_N\}$. The left eigenvectors are simply $\{\boldsymbol{w}_0^T, \boldsymbol{w}_1^T, \ldots, \boldsymbol{w}_N^T\}$.

The matrices J and $\mathcal{Q}$ have the same diagonal elements and J retains the tridiagonal form of $\mathcal{Q}$. Noting that

$$\mathcal{Q}\boldsymbol{v}_n = s_n\boldsymbol{v}_n \qquad \Longrightarrow \qquad D^{1/2}\mathcal{Q}\boldsymbol{v}_n = s_n D^{1/2}\boldsymbol{v}_n, \tag{45}$$

and letting $\boldsymbol{w}_n \triangleq D^{1/2}\boldsymbol{v}_n$, we obtain

$$D^{1/2}\mathcal{Q}D^{-1/2}\boldsymbol{w}_n = s_n\boldsymbol{w}_n \qquad \Longrightarrow \qquad J\boldsymbol{w}_n = s_n\boldsymbol{w}_n, \qquad 0 \le n \le N. \tag{46}$$

Thus, J has the same eigenvalues $\{s_0, s_1, s_2, \ldots, s_N\}$ as $\mathcal{Q}$. The right and left eigenvectors of $\mathcal{Q}$ can be obtained directly from the eigenvectors of J as follows:

$$\boldsymbol{v}_n = D^{-1/2}\boldsymbol{w}_n, \quad ; \quad \boldsymbol{u}_n = \boldsymbol{w}_n^T D^{1/2}, \qquad 0 \le n \le N, \tag{47}$$

respectively. Since J is symmetric, numerical computation of the eigenvectors of J is more efficient and stable than computation of the eigenvectors of $\mathcal{Q}$ [2, Chapter 8, pp.408–412]. Therefore, for efficient numerical computation of $\boldsymbol{P}(t)$ in Syski's method, we apply the symmetrization transformation given by (41) and then compute the decomposition of J to obtain the eigenvalues $\{s_n\}$ and the eigenvectors $\{\boldsymbol{w}_n\}$. The spectral decomposition in (37) can then be applied using the relations in (47).

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