Multiple Stopping Options *

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1 Introduction

One of the primary aims of this article is to identify and verify optimal stopping times in multiple stopping problems. In discrete time, dynamic programming (Bellman) methods are typically used to derive optimal strategies: in continuous time, the analogous approach leads to a Hamilton–Jacobi–Bellman equation, which often reduces to a free-boundary problem.

However, even with only two stopping opportunities, rigorous verification of the optimal strategy can be mathematically intractable. For instance, in the case of a finite-maturity American put option, it is natural to expect the optimal policy to involve exercising when the underlying price falls below a threshold and, subsequently, exercising again upon crossing a (possibly lower) threshold. To the best of our knowledge, however, this intuition regarding the optimal multiple stopping strategy has not yet been rigorously established. Motivated by this challenge, we study the following discrete time multiple stopping problem:

$$\sup_{0\le\tau_m<\cdots<\tau_1\le N} E\left[\sum_{i=1}^{m} X_{\tau_i}\right], \tag{1.1}$$

where $(\tau_1, \cdots, \tau_m)$ are stopping times and $X = (X_n)_{n\in\mathbb{N}}$ is a sequence of nonnegative random variables.

*This is an abbreviated version of "Multiple Stopping Options on Geometric Random Walk," dated November 7, 2025.

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Section 2.1 provides a concise overview of optimal stopping theory, including a summary of relevant results and the notation needed for completeness. In Section 2.2, we extend the classical martingale system theory developed by Snell [10] to discrete time optimal multiple stopping problems. Under a suitable integrability condition (A1) in Section 2.1, the key structural properties of the single stopping problem carry over naturally to the multiple stopping setting. Consequently, the fundamental properties (a) - (f) in Section 2.1 may be applied directly without re-verifying them for each m-stopping problem. In particular, an optimal stopping time is given by $\tau^{[m]*} = \min\{n \in \mathbb{N} : X_n^{[m]} = V_n^{[m]}\}$, where the Snell envelope $V_n^{[m]}$ satisfies the dynamic programming equation for multiple stopping. This serves as the starting point for a detailed analysis of the value function $V_n^{[m]}(x)$. Our presentation draws on Neveu [6] for martingale system theory, and Lamberton [4] for American options.

In Section 3, we derive optimal multiple stopping times for an American put option on a geometric random walk. When the asset price X_n falls below K, the holder may exercise the right to sell at K, receiving the payoff $(K - X_n)^+$. The seller is then obligated to buy the asset at K, protecting the holder from downside risk. Multiple-exercise American puts on geometric Brownian motion were studied by Carmona and Touzi [2] in the perpetual (infinite-horizon) setting, and Meinshausen and Hambly [5] employed a computational approach. However, in the Black–Scholes market the optimal multiple stopping strategy for the more realistic finite maturity case has yet to be theoretically established, likely due to its inherent complexity. We address this question within the discrete time Cox–Ross–Rubinstein framework and provide a theoretical characterization of the optimal multiple stopping strategy for American put options and related contracts. We assume that the holder may exercise up to m times, but is not obliged to use all opportunities.

For an American put option with m exercise rights on a geometric random walk, we show that there exist nondecreasing boundary sequences $\{b_t^{[i]}\}_{t=0}^N$ for $i = 1, 2, \ldots, m$, and the optimal stopping times $(\tau^{[m]*}, \tau^{[m-1]*}, \ldots, \tau^{[1]*})$ are given by

$$\tau^{[i]*} = \min\{t \in \{0, 1, \ldots, N\} : X_t \leq b_t^{[i]}\}, \qquad i = 1, 2, \ldots, m, \tag{1.2}$$

with the ordering $0 \leq \tau^{[m]*} < \tau^{[m-1]*} < \cdots < \tau^{[1]*} \leq N$, where $[i]$ denotes the number of remaining exercise rights, so that $\tau^{[m]*}$ is the first optimal exercise time, and t indexes the calendar time to maturity N. Each $\tau^{[i]*}$ is characterized by a threshold sequence satisfying

$$0 \leq b_0^{[i]} \leq b_1^{[i]} \leq \cdots \leq b_{N-i}^{[i]} \leq b_{N-i+1}^{[i]} = b_{N-i+2}^{[i]} = \cdots = b_N^{[i]} = K, \tag{1.3}$$

and, for each $t = 0, 1, \ldots, N$,

$$b_t^{[1]} \leq b_t^{[2]} \leq \cdots \leq b_t^{[m]}. \tag{1.4}$$

In Section 3.3, we briefly explore the connection between our discrete time approach and the continuous time free-boundary problem, previously studied in depth by Jacka [3] and Peskir and Shiryaev [8] for the single stopping case under geometric Brownian motion.

In Section 4, we incorporate random expiration risk into the American put, where the expiration occurs exogenously and independently of movements in the underlying asset price. We investigate the optimal multiple exercise strategy by addressing the following questions: Is the strategy equivalent to that of an American put option with a deterministic maturity? If so, what conditions on the distribution of the random maturity are required for such equivalence? To this end, we specifically examine cases where the random maturity follows uniform, geometric, Poisson, and Negative binomial distributions. Section 5 applies our approach to a multiple stopping version of the Russian option, in which the payoff is determined by the running maximum of the asset price at the exercise time. This option was originally proposed by Shepp and Shiryaev [9] and further developed by Peskir and Shiryaev [8] using free-boundary methods.

2 Optimal multiple stopping

This section provides a general theory for the multiple stopping problem. In this abbreviated version, Section 2 is omitted.

3 American put option

3.1 Single stopping

To establish the necessary notation, this section outlines the derivation of the optimal stopping time for the single American put option using the value function $V_n^{[1]}$ and the dynamic programming equation. Let the maturity T be divided into N time steps, with $\Delta t = T/N$ and $t_n = n\Delta t$. The stock price X evolves as

$$X_{n+1} = \begin{cases} uX_n & \text{with probability } p, \\ dX_n & \text{with probability } 1-p, \end{cases}$$

where the parameters are given by $u = e^{\sigma\sqrt{\Delta t}}$, $d = e^{-\sigma\sqrt{\Delta t}}$, $p = (e^{r\Delta t} - d)/(u-d)$. Let $\alpha = e^{-r\Delta t}$ be the one-step discount factor. Since $p \in (0,1)$, it follows that $d < 1/\alpha < u$, ensuring the existence of a unique risk-neutral measure $\mathbb{P}$ such that $\alpha(pu + (1-p)d) = 1$. Under $\mathbb{P}$, the discounted stock price $(\alpha^n X_n)_{n\geq 0}$ is a martingale.

For an American put option with m exercise rights, we assume that multiple exercises cannot occur simultaneously and require a minimum waiting time $\ell = 1$ step between successive exercises. The reward function is $g(x) = (K-x)^+$. Since the random walk satisfies condition (A1), we have $\mathbb{E}[\sup_{n\in\mathcal{N}_0} \alpha^n g(X_n)] < \infty$. From here on, we redefine the time index $n \in \mathcal{N}_0 = \{0, 1, \ldots, N\}$ to represent the *remaining time* to maturity. This introduces a reversed time convention: X_n denotes the stock price when n steps remain until maturity. We say the option is in state $(n, X_n, 1)$ when n steps remain and the holder has one exercise right. The value function $V_n^{[1]}(x)$ is the Snell envelope:

$$V_n^{[1]}(x) = \operatorname*{esssup}_{0\leq\tau\leq n} \mathbb{E}_x \left[\alpha^\tau (K - X_{n-\tau})^+\right], \quad n \in \mathcal{N}_0, \tag{3.1}$$

where $\mathbb{E}_x$ denotes the expectation given $X_n = x$. The dynamic programming equation is $V_0^{[1]}(x) = (K-x)^+$, and for $n \geq 1$

$$V_n^{[1]}(x) = \max{(K-x)^+, \alpha\mathbb{E}_x[V_{n-1}^{[1]}(X_{n-1})]}. \tag{3.2}$$

Theorem 3.1. *For the American put option on a geometric random walk, assume the time index n denotes the* remaining time *to maturity. When the option holder has one exercise right, the optimal exercise policy at time n is of threshold type: exercise if and only if $X_n \leq x_n^{[1]*}$, where $x_n^{[1]*} = \sup\{x \in (0, K] : (K-x)^+ = V_n^{[1]}(x)\}$. Equivalently, $x_n^{[1]*}$ is the unique solution in $(0, K]$ of $(K-x)^+ = V_n^{[1]}(x)$. Moreover, an optimal stopping time is given by*

$$\tau^{[1]*} = \max\{n \in \mathcal{N}_0 : \; X_n \leq x_n^{[1]*}\}.$$

The boundary sequence satisfies that $0 \leq x_N^{[1]*} \leq x_{N-1}^{[1]*} \leq \cdots \leq x_1^{[1]*} \leq x_0^{[1]*} = K$.

By reverting to calendar time, define $b_t^{[1]*} := x_{N-t}^{[1]*}, \; t = 0, 1, \ldots, N$. For notational simplicity, we write $b_t^{[1]}$ instead of $b_t^{[1]*}$. Then the optimal stopping time can be equivalently expressed as $\tau^{[1]*} = \min\{t \in \mathcal{N}_0 : X_t \leq b_t^{[1]}\}$, where the boundary sequence satisfies $0 \leq b_0^{[1]} \leq b_1^{[1]} \leq \cdots \leq b_N^{[1]} = K$.

To prove Theorem 3.1, we introduce the following lemma which outlines the essential properties of the value function $V_n^{[1]}(x)$. Following the standard literature, we use the term "*ncdc function*" to denote a function that is *n*onnegative, *c*ontinuous, *d*ecreasing (nonincreasing), and *c*onvex.

Lemma 3.2. *(i) For each $n \in \mathcal{N}_0$, the mapping $x \mapsto V_n^{[1]}(x)$ is an ncdc function. (ii) For each $x \geq 0$, the sequence $n \mapsto V_n^{[1]}(x)$ is nondecreasing, and $V_0^{[1]}(x) \geq 0$.*

Lemma 3.3. *For each $n \in \mathcal{N}_0$, $\lim_{x \downarrow 0} V_n^{[1]}(x) = K$.*

Lemma 3.4. *For each $n \in \mathcal{N}$, let $x_n^{[1]*}$ be the optimal stopping boundary for the American single stopping put option. The value function $V_n^{[1]}(x)$ satisfies the following condition in terms of its subdifferential $\partial V_n^{[1]}(x)$ at $x = x_n^{[1]*}$ $-1 \in \partial V_n^{[1]}(x_n^{[1]*})$. Equivalently, $\partial_x^- V_n^{[1]}(x_n^{[1]*}) \leq -1 \leq \partial_x^+ V_n^{[1]}(x_n^{[1]*})$.*

Remark 1. Unlike the continuous-time geometric Brownian motion model, the value function $V_n^{[1]}(x)$ in the discrete-time geometric random walk does not generally satisfy the *smooth fit* condition (continuity of the derivative) at the optimal exercise boundary $x_n^{[1]*}$. Instead, the value function exhibits "continuous pasting," where the function itself is continuous, but its derivative may have a discontinuity (a "kink") at the boundary. However, the fundamental properties such as convexity and monotonicity (ncdc) established in Lemma 3.2 remain valid and are sufficient for our analysis.

Proof of Theorem 3.1. For each $n \in \mathcal{N}_0$, the dynamic programming equation (3.2) implies $V_n^{[1]}(x) \geq (K - x)^+$ for all $x \geq 0$. From Lemma 3.2 (i), $V_n^{[1]}(x)$ is a convex and nonincreasing function (ncdc). Combined with Lemma 3.3 ($\lim_{x \to 0} V_n^{[1]}(x) = K$), it follows that there exists a unique threshold $x_n^{[1]*} \in (0, K]$ such that $x_n^{[1]*} = \sup\{x \in (0, K] : (K - x)^+ = V_n^{[1]}(x)\}$. Since $(K - x)^+$ is linear for $x \leq K$ and $V_n^{[1]}(x)$ is convex, the exercise region $\{x : (K - x)^+ = V_n^{[1]}(x)\}$ is an interval $[0, x_n^{[1]*}]$.

Because our time index n represents the *remaining time* to maturity, the optimal exercise policy is to stop at the first instance when the stock price X hits the stopping region. In this backward notation, this corresponds to the largest n (earliest calendar time) satisfying the condition. Thus, an optimal stopping time is given by $\tau^{[1]*} = \max\{n \in \mathcal{N}_0 : X_n \leq x_n^{[1]*}\}$. Furthermore, Lemma 3.2 (ii) establishes that $V_n^{[1]}(x)$ is nondecreasing in n for any fixed x. This implies that as n increases (more time to maturity), the continuation value $\alpha \mathbb{E}_x[V_{n-1}^{[1]}(X_{n-1})]$ increases or stays constant, which in turn makes the exercise region smaller. Therefore, the threshold sequence is nonincreasing in n. This monotonic behavior of the optimal boundary is illustrated in Figures 1 and 2. □

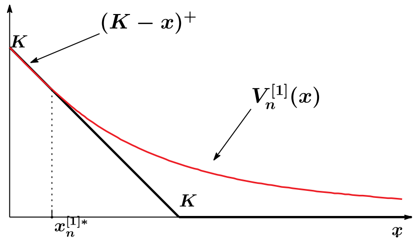

Figure 1: $x_n^{[1]*}$

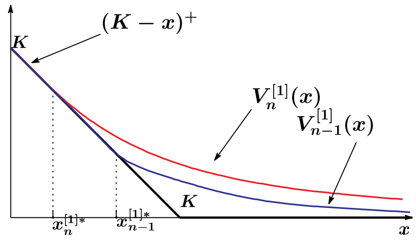

Figure 2: $x_n^{[1]*}$ and $x_{n-1}^{[1]*}$

3.2 Multiple stopping

Suppose that the option holder is permitted to exercise the rights of a put option at most m times. This assumption implies that the holder may exercise only one right at a time. If, instead, the holder were allowed to exercise all m rights simultaneously, the problem would reduce to the simpler case of holding m identical American put options, each with a single exercise opportunity, which would require no separate analysis.

The state of the American put option is characterized by the triplet (n, x, m), where n denotes the remaining time to maturity, $x = X_n$ is the current stock price, and m is the number of remaining exercise rights. Let $V_n^{[m]}(X_n)$ denote the maximum expected reward in state (n, X_n, m). This is defined as the Snell envelope of the reward process $(\alpha^n g^{[m]}(X_n))_{n\in\mathcal{N}}$ for each $m \in \mathcal{N}$,

$$V_n^{[m]}(X_n) = \operatorname*{esssup}_{0\le\tau_m<\cdots<\tau_1\le n} \mathbb{E}_{X_n}\left[\sum_{i=1}^{m} \alpha^{\tau_i}(K - X_{n-\tau_i})^+\right], \quad n \in \mathcal{N}_0, \tag{3.3}$$

where τ_i denotes the stopping time at which the i-th remaining exercise right is used. By Lemma 2.2, we obtain the following dynamic programming equation for the multiple stopping American put option. For each $m \in \mathcal{N}$ and $n \in \mathcal{N}_0$,

$$V_n^{[m]}(x) = \max\{(K-x)^+ + \alpha\mathbb{E}_x[V_{n-1}^{[m-1]}(X_{n-1})], \alpha\mathbb{E}_x[V_{n-1}^{[m]}(X_{n-1})]\}, \tag{3.4}$$

with the initial conditions $V_0^{[m]}(x) = (K-x)^+$ and $V_n^{[0]}(x) = 0$, for all $n \in \mathcal{N}_0$. From Theorem 2.2, we know that the optimal multiple stopping time is characterized by $\tau^{[m]*} = \min\{n : V_n^{[m]}(x) = (K-x)^+ + \alpha\mathbb{E}_x[V_{n-1}^{[m-1]}(X_{n-1})]\}$ for each $m \in \mathcal{N}$. To gain deeper insight into the structure of the optimal stopping times, we analyze the value function $V_n^{[m]}$. We introduce the following functions related to the marginal value of an additional right. Define for each $m \in \mathcal{N}$,

$$\Delta V_n^{[m]}(x) := V_n^{[m]}(x) - V_n^{[m-1]}(x), \quad n \in \mathcal{N}_0, \tag{3.5}$$

$$f_n^{[m]}(x) := \alpha \mathbb{E}_x[\Delta V_{n-1}^{[m]}(X_{n-1})], \quad n \in \mathcal{N}, \quad f_0^{[m]}(x) := 0. \tag{3.6}$$

The optimality equation (3.4) can be rewritten as

$$V_n^{[m]}(x) = \max\{(K-x)^+, f_n^{[m]}(x)\} + \alpha \mathbb{E}_x[V_{n-1}^{[m-1]}(X_{n-1})]. \tag{3.7}$$

Therefore, it is optimal to stop (exercise) at state (n, x, m) if and only if $(K-x)^+ \geq f_n^{[m]}(x)$. The function $f_n^{[m]}(x)$ thus serves as the exercise threshold.

Theorem 3.5. *For the American put option on a geometric random walk, let n be the* remaining time *to maturity. When the option holder has $m \in \mathcal{N}$ exercise rights, the optimal exercise policy at time n is of threshold type: exercise if and only if $X_n \leq x_n^{[m]*}$, where $x_n^{[m]*} = \sup\{x \in (0, K] : (K-x)^+ \geq f_n^{[m]}(x)\}$. Equivalently, $x_n^{[m]*}$ is the unique solution in $(0, K]$ to $(K-x)^+ = f_n^{[m]}(x)$. Moreover, an optimal stopping time is given by*

$$\tau^{[m]*} = \max\{n \in \mathcal{N}_0 : X_n \leq x_n^{[m]*}\}.$$

The boundary sequence satisfies $0 \leq x_N^{[m]*} \leq x_{N-1}^{[m]*} \leq \cdots \leq x_0^{[m]*} = K$, and $0 \leq x_N^{[m]*} \leq x_{N-1}^{[m]*} \leq \cdots \leq x_m^{[m]*} \leq x_{m-1}^{[m]*} = \cdots = x_0^{[m]*} = K$.

By reverting to calendar time, define $b_t^{[m]*} := x_{N-t}^{[m]*}$ for $t = 0, 1, \ldots, N$. For notational simplicity, we write $b_t^{[m]}$ instead of $b_t^{[m]*}$. Then, for each $m \in \mathcal{N}$, the optimal stopping time can be equivalently expressed as $\tau^{[m]*} = \min\{t \in \mathcal{N}_0 : X_t \leq b_t^{[m]}\}$. Moreover, the boundary sequence satisfies $0 \leq b_0^{[m]} \leq b_1^{[m]} \leq \cdots \leq b_{N-m}^{[m]} \leq b_{N-m+1}^{[m]} = \cdots = b_N^{[m]} = K$.

Define the optimal stopping region for an option holder with m stopping chances, denoted by $D^{[m]}$, as follows $D^{[m]} = \{(n, x) \in \mathcal{N}_0 \times (0, K] : x \leq x_n^{[m]*}\}$. Therefore, $\tau^{[m]*}$ is the first hitting time when the process (n, X_n, m) enters $D^{[m]}$.

To prove the properties of the optimal boundaries stated in Theorem 3.5, specifically the ordering of boundaries, we establish the following lemmas regarding the value function $V_n^{[m]}(x)$ and its related functions.

Lemma 3.6. *For each $n \in \mathcal{N}_0$, $m \in \mathcal{N}$, and $k = 0, 1, \ldots, m$, we have $2V_n^{[m]}(x) \geq V_n^{[2m-k]}(x) + V_n^{[k]}(x)$. In particular, if $k = m-1$, then $\Delta V_n^{[m]}(x) \geq \Delta V_n^{[m+1]}(x)$ for all $m = 1, 2, \ldots$. This implies that the marginal value of an additional exercise right is nonincreasing.*

Lemma 3.7. *For each* $n \in \mathcal{N}_0$ *and* $m \in \mathcal{N}$, $\lim_{x \downarrow 0} V_n^{[m]}(x) = (\sum_{i=0}^{\min\{m,n+1\}-1} \alpha^i)K$.

We use the abbreviated term "*ncd function*" to denote a nonnegative, continuous, and decreasing (i.e., nonincreasing) function.

Lemma 3.8. *For each* $n \in \mathcal{N}_0$ *and* $m = 1, 2, \dots, n+1$, *(i) The mapping* $x \longmapsto V_n^{[m]}(x)$ *is an ncdc function. (ii) The mapping* $x \longmapsto \Delta V_n^{[m]}(x)$ *is an ncd function. (iii) The mapping* $x \longmapsto f_n^{[m]}(x)$ *is an ncd function.*

Lemma 3.9. *For each* $n \in \mathcal{N}_0$ *and* $m \in \mathcal{N}$ *satisfying* $m \le n$, $\lim_{x \downarrow 0} f_n^{[m]}(x) = \alpha^m K$. *Note: If* $m > n$, *the limit is* 0.

Lemma 3.10. *For each* $m \in \mathcal{N}$, *(i) The sequence* $n \longmapsto \Delta V_n^{[m]}(x)$ *is nondecreasing, and* $\Delta V_0^{[m]}(x) = 0$. *(ii) The sequence* $n \longmapsto f_n^{[m]}(x)$ *is nondecreasing, and* $f_1^{[m]}(x) = 0$.

Lemma 3.11. *Fix* $n \in \{2, \dots, N\}$ *and* $m \in \{2, \dots, N\}$. *Let* $g(x) = (K - x)^+$. *The equation,* $g(x) = f_n^{[m]}(x)$, *has a unique solution* $x^* \in [0, K]$. *Moreover, if* $f_n^{[m]}(K) > 0$, *then* $x^* \in (0, K)$; *otherwise* $x^* = K$.

Proof of Theorem 3.5. Recall that the optimal strategy at time n is to exercise if and only if $(K - X_n)^+ \ge f_n^{[m]}(X_n)$.
(Existence and uniqueness of the threshold) By Lemma 3.11, for each n and m, the equation $(K - x)^+ = f_n^{[m]}(x)$ has a unique solution in $(0, K]$. Consequently, the set $\{x \in (0, K] : (K - x)^+ \ge f_n^{[m]}(x)\}$ forms an interval $(0, x_n^{[m]*}]$, where the threshold is defined by $x_n^{[m]*} = \sup\{x \in (0, K] : (K - x)^+ \ge f_n^{[m]}(x)\}$. Thus, the optimal exercise region at time n is $\{x : x \le x_n^{[m]*}\}$. Since the time index n represents the remaining time, the first optimal exercise in calendar time corresponds to the *largest* remaining time index at which the condition holds. Therefore, an optimal stopping time is given by $\tau^{[m]*} = \max\{n \in \mathcal{N}_0 : (K - X_n)^+ \ge f_n^{[m]}(X_n)\} = \max\{n \in \mathcal{N}_0 : X_n \le x_n^{[m]*}\}$.
(Monotonicity of the boundary) By Lemma 3.10 (ii), the function $n \mapsto f_n^{[m]}(x)$ is nondecreasing for every x. As n increases, $f_n^{[m]}(x)$ increases, making the exercise condition $(K - x)^+ \ge f_n^{[m]}(x)$ stricter (harder to satisfy). This implies that the exercise region shrinks as n increases. Therefore, the boundary sequence is nonincreasing in n, $x_{n+1}^{[m]*} \le x_n^{[m]*}$ for all $n \in \mathcal{N}_0$.
(Values at small n) From $f_0^{[m]}(x) = 0$ (see (3.6)), we have $x_0^{[m]*} = \sup\{x \in (0, K] : (K - x)^+ \ge 0\} = K$. Moreover, for $n \le m - 1$, the number of remaining exercise rights (m) is greater than or equal to the number of remaining periods ($n + 1$). In this case, it is optimal to exercise at every remaining opportunity provided the option is in-the-money. Hence, the boundary is equal to the strike price K, $x_0^{[m]*} = x_1^{[m]*} = \cdots = x_{m-1}^{[m]*} = K$. □

Corollary 3.1. *The optimal stopping regions are nested with respect to the number of exercise rights* $D^{[1]} \subseteq D^{[2]} \subseteq \cdots \subseteq D^{[m]}$.

This corollary states that for each remaining time $n \in \mathcal{N}_0$ and number of rights m, the boundary sequence satisfies that $x_n^{[1]*} \leq x_n^{[2]*} \leq \cdots \leq x_n^{[m]*}$. By reverting to calendar time, let $b_t^{[m]} := x_{N-t}^{[m]*}$. This inequality is equivalently expressed as $b_t^{[1]} \leq b_t^{[2]} \leq \cdots \leq b_t^{[m]}$. This result aligns with economic intuition: having more exercise rights lowers the opportunity cost (or marginal value) of using one right, thereby making the holder willing to exercise at a higher stock price.

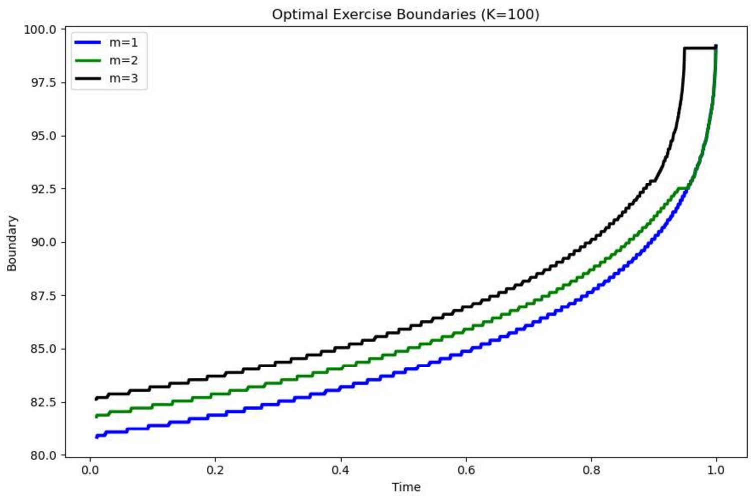

Figure 3: The optimal stopping boundaries are derived via backward induction from the dynamic programming equations for $m = 1, 2, 3$, with maturity $T = 1.0$ and strike $K = 100$, as calculated in Python.

3.3 Connection with free-boundary problem

We briefly highlight the connection with free-boundary problem here. Let $(X_s)_{s \geq 0}$ be a geometric Brownian motion under the risk-neutral measure, satisfying $dX_s = rX_s\,ds + \sigma X_s\,dB_s$, where $r > 0$ is the interest rate, $\sigma > 0$ is the volatility, and $(B_s)_{s \geq 0}$ is a standard Brownian motion. Let $T > 0$ be the maturity and define the payoff function by $g(x) := (K - x)^+$. For the *calendar time* $t \in [0, T]$ and $x > 0$, the value function of the American put option with a single exercise right is given by $V^{[1]}(t, x) = \sup_{0 \leq \tau \leq T-t} \mathbb{E}\left[e^{-r\tau} g(X_{t+\tau}) \mid X_t = x\right]$. Let $\mathcal{L}_X$ denote the infinitesimal generator of $(X_s)_{s\geq 0}$:

$$\mathcal{L}_X = rx\frac{\partial}{\partial x} + \frac{\sigma^2}{2}x^2\frac{\partial^2}{\partial x^2}.$$

It is well known that the optimal stopping time is of threshold type $\tau^{[1]*} = \inf\{s \in [t,T] : X_s \leq b(s)\}$, and that the unknown pair $(V^{[1]}(t,x), b(t))$ solves the associated free-boundary problem described below (see Jacka [3] or Peskir and Shiryaev [8]).

- $\partial_t V^{[1]} + \mathcal{L}_X V^{[1]} - rV^{[1]} = 0$ in the continuation region $x > b(t)$.
- $V^{[1]}(t,x) > g(x)$ in the continuation region $x > b(t)$.
- $V^{[1]}(t,x) = g(x)$ in the stopping region $0 < x \leq b(t)$.
- $V^{[1]}(t,x) = g(x)$ for $x = b(t)$. (value matching)
- $\partial_x V^{[1]}(t,x) = -1$ for $x = b(t)$. (smooth fit)

Furthermore, the following properties are well established. Each corresponds to a lemma or theorem in our discrete-time setting.

- $V^{[1]}(t,x)$ is a nonnegative, continuous, nonincreasing, and convex function of $x \longleftrightarrow$ Lemma 3.2.
- $V^{[1]}(t,x)$ is a nonincreasing function of $t \longleftrightarrow$ Lemma 3.2.
- For the calendar time t, the process $(e^{-rs}V^{[1]}(s,X_s))_{s\in[t,T]}$ is a supermartingale $\longleftrightarrow$ For the remaining time t, the process $(\alpha^n V^{[1]}_{N-n}(X_{N-n}))_{n=0}^N$ is a supermartingale which follows from the dynamic programming equation.
- $b(t)$ is a nondecreasing and continuous function of t with $0 < b(t) < K$ and $b(T-) = K \longleftrightarrow$ Theorem 3.1.

This free-boundary formulation can be approximated by its discrete-time counterpart. In this following elementary derivation, the time variable t denotes *calendar time*. We write $V^{[1]}_n(x) \approx V^{[1]}(t_n, x)$ with $t_n = n\Delta t$ for sufficiently small Δt. Using the standard Taylor expansion for $V^{[1]}(t + \Delta t, xe^{\pm\sigma\sqrt{\Delta t}})$ up to $O(\Delta t)$, and taking expectations under the risk-neutral measure, the discrete drift terms sum up to match the risk-neutral drift rx. Thus, we obtain $\mathbb{E}[V^{[1]}_{n+1}] = V^{[1]} + \Delta t(\partial_t V^{[1]} + rx\partial_x V^{[1]} + \sigma^2/2x^2\partial_{xx}V^{[1]}) + o(\Delta t)$. Using the approximation $e^{-r\Delta t} = 1 - r\Delta t + o(\Delta t)$, the dynamic programming equation implies

$$V^{[1]} = \max\{(K-x)^+,\ V^{[1]} + \Delta t[\partial_t V^{[1]} + \mathcal{L}_X V^{[1]} - rV^{[1]}] + o(\Delta t)\}.$$

Subtracting $V^{[1]}$ from both sides, dividing by Δt, and letting $\Delta t \to 0$, we arrive at the *quasi-variational inequality*

$$\max\{(K-x)^+ - V^{[1]},\ \partial_t V^{[1]} + \mathcal{L}_X V^{[1]} - rV^{[1]}\} = 0, \quad 0 \le t < T.$$

The terminal condition is $V^{[1]}(T,x) = (K-x)^+$. This quasi-variational inequality is equivalent to the free-boundary problem described above.

Applying Itô's formula to $e^{-rt}V^{[1]}(t,X_t)$, and observing that the discounted value process is a martingale up to the optimal stopping time, we recover the condition that $\partial_t V^{[1]} + \mathcal{L}_X V^{[1]} = rV^{[1]}$ holds in the continuation region. This corresponds, in discrete time, to the fact that the stopped process $(\alpha^{n\wedge\tau^{[1]*}} V^{[1]}_{N-(n\wedge\tau^{[1]*})})_{n\in\mathcal{N}_0}$ is a martingale.

This analogy suggests that the optimal multiple stopping problem for a continuous time American put option with finite maturity can be formulated as a sequence of recursive problems, yielding pairs $(V^{[\ell]}(t,x), b^{[\ell]}(t))$ for $\ell = 1,2,\ldots,m$. Sections 3.1 and 3.2 show that the American multiple stopping put option on a geometric random walk exhibits the following properties for each $m \in \mathcal{N}$. The time index n below represents the remaining time to maturity N.

- $V^{[m]}_n(x)$ is a nonnegative, continuous, and nonincreasing (ncd) function of x for each $n \in \mathcal{N}_0$. (Lemma 3.8)
- $V^{[m]}_n(x) > g^{[m]}_n(x)$ in the continuation region. (see (3.4))
- $V^{[m]}_n(x) = g^{[m]}_n(x)$ in the stopping region. (see (3.4))
- $V^{[m]}_n(x^{[m]*}_n) = g^{[m]}_n(x^{[m]*}_n)$. (see Lemma 3.11)
- $\partial^-_x V^{[m]}_n(x^{[m]*}_n) \le \partial^+_x V^{[m]}_n(x^{[m]*}_n)$. (Lemma 3.12)
- The process $(\alpha^n V^{[m]}_{N-n}(X_{N-n}))_{n\in\mathcal{N}_0}$ is a supermartingale. (from (3.4))
- The stopped process $(\alpha^{n\wedge\tau^{[m]*}} V^{[m]}_{N-(n\wedge\tau^{[m]*})}(X_{N-(n\wedge\tau^{[m]*})}))_{n\in\mathcal{N}_0}$ is a martingale. (from (3.4))

The next lemma characterizes the one-sided contact at the free boundary for the discrete time American multiple stopping put option.

Lemma 3.12 (One-sided contact at the boundary)**.** *Fix $m \in \mathcal{N}$ and $n \in \mathcal{N}_0$. Define the payoff function $g^{[m]}_n$ and the continuation value function $C^{[m]}_n$ by $g^{[m]}_n(x) := (K-x)^+ + \alpha\mathbb{E}_x[V^{[m-1]}_{n-1}(X_{n-1})]$, $C^{[m]}_n(x) := \alpha\mathbb{E}_x[V^{[m]}_{n-1}(X_{n-1})]$. At the optimal boundary $x^{[m]*}_n$, the following inequality holds: $\partial^-_x V^{[m]}_n(x^{[m]*}_n) \le \partial^+_x V^{[m]}_n(x^{[m]*}_n)$.*

In this abbreviated version, Section 4 on American put options with random maturity and Section 5 on Russian options are omitted.

Acknowledement

This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

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