An Importance Measure of a Group of Components - Relations among Groups II (Multi-State Case) -

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Abstract

An importance measure of a component in a system gives us an instruction for reliability maintenance. Z.W. Birnbaum[4] was the first to propose an importance concept called the Birnbaum importance measure. After that, several concepts of an importance measure have been proposed and discussed variously. Discussions in a context of binary-state system have been organized by W. Kuo and X. Zhu[10]. M.J. Armstrong[1] has proposed the joint importance measure for two components. F. Ohi[18, 20, 21] have examined a general concept of an importance measure for a group of components in a context of multi-state system, but not enough. In particular, the relationship between the results of the set theoretical calculations on groups and the importance measures of the groups have not been cleared so far. For a binary-state system, F.Ohi[22] has proved that the set of critical state vectors of a group is the union of the sets of critical state vectors of the components of which the group consists. This result plays a crucial role for examinations of importance measures of groups, but is not proved for multi-state systems. In this paper, extending the result to multi-state case, i.e., the state spaces of components and a system composed of the components are mathematically finite partially ordered sets with minimum elements, we give some discussions which are the extensions of F.Ohi[22]. Simple probabilistic bounds for the importance measure of a group is given by applying the inclusion and exclusion principle to the theorem, which is considered to be rough one.

1 Introduction

Importance measures of components constructing a system give us preference order for the preventive maintenance of components, ensuring the long-run reliability of the system. Z.W. Birnbaum[4] was the first to propose an importance concept which is usually called the Birnbaum importance measure in the context of sensitive analysis. Reliability of a system is given by a reliability function, which is a polynomial function of the reliabilities of the components, when the components are stochastically independent. The Birnbaum importance measure of a component is defined to be a partial differentiation of the reliability function with respect to the reliability of the component. When the components are not necessarily stochastically independent, the Birnbaum importance measure of a component is defined to be the probability of the set of critical state vectors. Refer to F. Ohi[17].

After the work of Z.W. Birnbaum[4], several concepts of an importance measure have been proposed and discussed in various contexts : R.E. Barlow, et al.[2], S. Bisanovic, et al.[5], M.C. Cheok, et al.[6], J. Espiritu, et al.[7], J.B. Fussell[8], F.Ohi[14, 15, 16, 17, 19], Y. Shimada, et al.[23, 24], W.E. Vesely, et al.[25]. Examinations of importance measures of a component in a context of a binary-state system have been organized by W. Kuo and X. Zhu[10].

M.J. Armstrong[1] has considered a joint importance measure for two components. F. Ohi[18, 20, 21] have examined a general concept of an importance measure of a group of components in a multi-state system, however, not sufficiently. In particular, the relationship between the results of the set theoretical calculations on groups and the importance measures of the groups have not been cleared so far.

The importance measure of a group A, $\boldsymbol{P}(Cr(A))$, is defined to be the probability of the set of critical state vectors of the group written as $Cr(A)$. For example, for $Cr(A)$, $Cr(B)$ and $Cr(A \cup B)$ of groups A, B and $A \cup B$, roughly speaking, $Cr(A \cup B) = Cr(A) \cup Cr(B)$ holds in the multi-state case, which is proved accurately in this paper.

For a binary-state system, F.Ohi[22] has proved that the set of critical state vectors of a group A is the union of the sets of critical state vectors of components composing the group, i.e., $Cr(A) = \cup_{i \in A} Cr(\{i\})$. This result plays a crucial role for examinations of importance measures of groups, but has not been proved in multi-state case. In this paper, extending the result to multi-state case, i.e., the state spaces of components and a system composed of the components are mathematically finite partially ordered sets with minimum elements, we give some discussions which are the extensions of F.Ohi[22]. Simple probabilistic bounds for the importance measure of a group is given by applying the inclusion and exclusion principle to the theorem, which is considered to be rough one. Examples are given for binary-state systems as k-out-of-n:G and the bridge systems for making the situation clear. R.E. Barlow and F. Proschan[2] has given the concrete theoretical basis for a binary-state system.

2 Preliminary Definitions and Notations

For the first, following F.Ohi[11, 13], we present a definition of a multi-state system, which is usually called a system. The definition makes how the reliability system is basically composed of mathematical elements.

Definition 2.1 (Definition of a System) A system is a triplet (Ω_C, S, φ) consisting of the following elements.

(1) C is a non-empty finite set, denoting the set of all the components of which the system consists. Each element of C denotes a label of a component, and the cardinal number $|C|$ of C is the number of the components, i.e., assuming $n = |C|$, we may write $C = \{1, 2, 3, \cdots, n\}$.

(2) Ω_i $(i \in C)$ and S are finite ordered sets, each of which denotes the state space of the component $i \in C$ and the system, respectively. Each state space Ω_i $(i \in C)$ is assumed to have the minimum element (the least element) written as m_i, which denotes the perfect failure state.

(3) $\Omega_C = \prod_{i \in C} \Omega_i$ is the product ordered set of Ω_i $(i \in C)$. Each element $x_i \in \Omega_i$ of the state vector $\boldsymbol{x} = (x_1, \cdots, x_n) \in \Omega_C$ denotes the state of the component i. $\boldsymbol{m} = (m_1, \cdots, m_n)$ is the minimum element of Ω_C and then $\forall \boldsymbol{x} \in \Omega_C,\ \boldsymbol{m} \leqq \boldsymbol{x}$.

(4) φ is an increasing surjection from Ω_C to S, called a structure function denoting the structure of the system, for example, k-out-of-n:G system, the bridge system and so on.

(5) When $\Omega_i = S = \{0, 1\}$ $(i \in C)$ with the order relation $0 < 1$, the system (Ω_C, S, φ) is called a binary-state system. ■

Generally, a subset X of an ordered set W, $X \subseteqq W$, is called as the following :

(i) an increasing set (an upper set), when for $x \in X$, $x \leqq y$ implies $y \in X$,

(ii) a decreasing set (a lower set), when for $x \in X$, $y \leqq x$ implies $y \in X$.

The following necessary and sufficient relation is easily proved.

$$X \subseteqq W \text{ is an increasing set } \iff \forall \boldsymbol{x} \in X,\ \{\ \boldsymbol{y} \mid \boldsymbol{x} \leqq \boldsymbol{y}\ \} \subseteqq X\ . \tag{1}$$

We may consider the case of the decreasing set in the dual manner.

Notations 2.1 For a system (Ω_C, S, φ), we utilize the following notations :

(1) For a subset $A \subseteqq C$, generally, $\boldsymbol{x}_A$ is an element of the product ordered set $\Omega_A = \prod_{i \in A} \Omega_i$. The symbols $\leqq$, $<$, $\geqq$, $>$ are used in a well-known manner for the orders. For $\boldsymbol{x}_A,\ \boldsymbol{y}_A \in \Omega_A$,

$$\begin{aligned} \boldsymbol{x}_A \leqq \boldsymbol{y}_A &\iff x_i \leqq y_i\ (\forall i \in A), \\ \boldsymbol{x}_A < \boldsymbol{y}_A &\iff \boldsymbol{x}_A \leqq \boldsymbol{y}_A, \text{ and } x_j \neq y_j\ (\exists j \in A). \end{aligned}$$

(2) For $F \subseteqq E \subseteqq C,\ X \subseteqq \Omega_E$, $X_F = \{\ \boldsymbol{x}_F \mid \boldsymbol{x} \in X\ \}$, i.e., the section of X at F. For example, when

$$E = \{1,2,3,4\},\ F = \{1,2\},\ X = \{(1_1,3_2,2_3,2_4),\ (3_1,3_2,1_3,2_4),\ (3_1,2_2,1_3,4_4)\},\ \boldsymbol{x} = (1_1,3_2,2_3,2_4),$$

we have

$$\boldsymbol{x}_F = (1_1,3_2), \quad X_F = \{(1_1,3_2),\ (3_1,3_2),\ (3_1,2_2)\}\ ,$$

where 3_2 denotes $3 \in \Omega_2$ and the state of the component 2 is 3. Then, $(1_1,3_2)$ and $(3_2,1_1)$ are the same state vectors.

(3) For mutually disjoint $A \subset C$, $B \subset C$ $(A \cap B = \phi)$, $\boldsymbol{x}_A \in \Omega_A$ and $\boldsymbol{x}_B \in \Omega_B$, $(\boldsymbol{x}_A, \boldsymbol{x}_B)$ is an element of $\Omega_{A\cup B}$ such that

$$(\boldsymbol{x}_A, \boldsymbol{x}_B)_A = \boldsymbol{x}_A,\ (\boldsymbol{x}_A, \boldsymbol{x}_B)_B = \boldsymbol{x}_B.$$

(4) For an arbitrarily fixed $\boldsymbol{x}_{A^c} \in \Omega_{A^c}$, $\varphi(\cdot_A, \boldsymbol{x}_{A^c})$ is an increasing mapping from Ω_A to S, since the structure function φ is assumed to be increasing. For $V \subseteqq S$, the inverse image of V with respect to $\varphi(\cdot_A, \boldsymbol{x}_{A^c})$ is written as $\varphi(\cdot_A, \boldsymbol{x}_{A^c})^{-1}(V)$, i.e.

$$\varphi(\cdot_A, \boldsymbol{x}_{A^c})^{-1}(V) = \{\ \boldsymbol{x}_A \in \Omega_A \mid \varphi(\boldsymbol{x}_A, \boldsymbol{x}_{A^c}) \in V\ \}.$$

When V is an increasing (decreasing) subset of S, the inverse image $\varphi(\cdot_A, \boldsymbol{x}_{A^c})^{-1}(V)$ is an increasing (decreasing) subset of Ω_A.

(5) Generally, for an ordered set W, $MI(W)$ is the set of all the minimal elements of W and $MA(W)$ is the set of all the maximal elements of W.

(6) For a probability $\boldsymbol{P}$ on Ω_C and a subset $A \subseteqq C$, $\boldsymbol{P}_A$ is the marginal probability of $\boldsymbol{P}$ on Ω_A, in other words, the restriction of $\boldsymbol{P}$ on to Ω_A. ■

3 A Critical State Vector

Definition 3.1 (F.Ohi[21], A Definition of a Critical State Vector) For a subset $A \subset C$ and a non-empty increasing subset $V \subset S$ such that $V^c \neq \phi$, $\boldsymbol{x}_{A^c} \in \Omega_{A^c}$ is called an A-V-critical state vector, when the following "non-empty condition" is satisfied :

$$\varphi(\cdot_A, \boldsymbol{x}_{A^c})^{-1}(V) \neq \phi \quad \text{and} \quad \varphi(\cdot_A, \boldsymbol{x}_{A^c})^{-1}(V^c) \neq \phi,$$

which may also be written equivalently as follows :

$$\exists \boldsymbol{u}_A \in \Omega_A,\ \exists \boldsymbol{v}_A \in \Omega_A, \quad \varphi(\boldsymbol{u}_A, \boldsymbol{x}_{A^c}) \in V,\ \varphi(\boldsymbol{v}_A, \boldsymbol{x}_{A^c}) \in V^c.$$

We write the set of all the A-V-critical state vectors as $Cr(A,V)$, called as the A-V-critical set.

When $A = \{i\}$ $(i \in C)$, the A-V-critical set (or i-V-critical set) is written as $Cr(i,V)$, which is the set of the critical state vectors of the component $i \in C$ and a subset of $\Omega_{C\setminus\{i\}}$.

Setting $U \subset \Omega_A$ as

$$U \equiv \varphi(\cdot_A, \boldsymbol{x}_{A^c})^{-1}(V), \quad U^c \equiv \varphi(\cdot_A, \boldsymbol{x}_{A^c})^{-1}(V^c), \tag{2}$$

$U \subseteqq \Omega_A$ is called an A-V contributing set of the grouop A at the critical state vector $\boldsymbol{x}_{A^c}$, which is more definitely written as $U(\boldsymbol{x}_{A^c})$. ■

An A-V-critical state vector, $\boldsymbol{x}_{A^c}$, denotes that when the components of A^c are at this critical state vector, the transition of the group A from U to U^c makes the system's transition from V to V^c.

The following theorem given by F.Ohi[21] shows us a necessary and sufficient condition for $\boldsymbol{x}_{A^c}$ to be an A-V-critical state vector for a group $A \subseteqq C$.

Theorem 3.1 (F.Ohi[21]) For a group $A \subseteqq C$ and a non-empty increasing subset $V \subseteqq S$ such that $V^c \neq \phi$, setting

$$\mathcal{P}(A,V) = \left\{(\boldsymbol{m}, \boldsymbol{M})\ \middle|\ \boldsymbol{m} \in MI\left(\varphi^{-1}(V)\right),\ \boldsymbol{M} \in MA\left(\varphi^{-1}(V^c)\right),\ \boldsymbol{m}_{A^c} \leqq \boldsymbol{M}_{A^c}\right\},$$

we have

$$Cr(A,V) = \bigcup_{(\boldsymbol{m},\ \boldsymbol{M})\in\mathcal{P}(A,V)} \left[\boldsymbol{m}_{A^c},\ \boldsymbol{M}_{A^c}\right] ,$$

in other words, $\boldsymbol{x}_{A^c}$ is an A-V-critical state vector if and only if

$$\exists \boldsymbol{m} \in MI\left(\varphi^{-1}(V)\right),\ \exists \boldsymbol{M} \in MA\left(\varphi^{-1}(V^c)\right) \quad \text{such that} \quad \boldsymbol{m}_{A^c} \leqq \boldsymbol{x}_{A^c} \leqq \boldsymbol{M}_{A^c} , \tag{3}$$

where

$$\left[\boldsymbol{m}_{A^c},\ \boldsymbol{M}_{A^c}\right] = \{\ \boldsymbol{x}_{A^c} \mid \boldsymbol{m}_{A^c} \leqq \boldsymbol{x}_{A^c} \leqq \boldsymbol{M}_{A^c}\}$$

is an closed interval. ■

The following Theorem 3.2 is the basis for our examinations, which was proved under the assumption of binary-state system by F. Ohi[22]. In this paper, the state spaces are assumed to be partially ordered sets having the minimum elements.

Theorem 3.2 Supposing that V and $V^c = C\backslash V$ are respectively an increasing and a decreasing non-empty subsets of S of a system (Ω_C, S, φ), we have the following relations :

(1) For $A \subset B \subset C,\ Cr(A,V)_{B^c} \subsetneqq Cr(B,V)$,

(2) $\forall \boldsymbol{x}_{B^c} \in Cr(B,V),\ \exists i \in B,\ \exists \boldsymbol{y}_{C\backslash\{i\}} \in Cr(i,V), \quad \boldsymbol{x}_{B^c} = (\boldsymbol{y}_{C\backslash\{i\}})_{B^c}$.

Noticing (1), the above (2) may be written as the following :

$$Cr(B,V) = \bigcup_{i\in B} \left\{\ (\boldsymbol{y}_{C\backslash\{i\}})_{B^c}\ \middle|\ \boldsymbol{y}_{C\backslash\{i\}} \in Cr(i,V)\ \right\} = \bigcup_{i\in B} Cr(i,V)_{B^c}.$$

The environment in which the group B contribute critically to V consists of the state vectors cut off at B^c from the critical state vectors of the components of B.

Proof : (1) $Cr(A,V)_{B^c} \subsetneqq Cr(B,V)$ is obvious by the Theorem 3.1.

(2) The proposition (2) is in a sense the inverse of (1). We note that the definition of a system assumes the existence of the minimum element of the state space of each component, i.e., $m_i = \min \Omega_i\ (i \in C)$.

For a B-V-critical state vector $\boldsymbol{x}_{B^c} \in Cr(B,V)$, reminding the B-V-contributing set,

$$U \equiv \varphi(\cdot_B, \boldsymbol{x}_{B^c})^{-1}(V) \subset \Omega_B, \quad U^c \equiv \varphi(\cdot_B, \boldsymbol{x}_{B^c})^{-1}(V^c) \subset \Omega_B,$$

for which $U \neq \phi$ and $U^c \neq \phi$, and then,

$$\forall \boldsymbol{a} \in MI(U),\ \boldsymbol{m}_B < \boldsymbol{a}, \quad \text{since } \boldsymbol{m}_B \text{ is the minimum element of } \Omega_B,$$
$$\exists i \in B, \quad m_i < a_i, \quad \text{and then,} \quad \boldsymbol{a} \in MI(U) \subsetneqq U \text{ and } (m_i, \boldsymbol{a}) \in U^c.$$

Hence, we have

$$\varphi(\boldsymbol{a}, \boldsymbol{x}_{B^c}) \in V \quad \text{and} \quad \varphi((m_i, \boldsymbol{a}), \boldsymbol{x}_{B^c}) \in V^c,$$

which denotes that $((\cdot_i, \boldsymbol{a}), \boldsymbol{x}_{B^c})$ is a critical state vector of the component i and $\boldsymbol{x}_{B^c}$ is a state vector cut out from the former critical state vector at B^c. ■

The Theorem 3.2 (2) is basis through our examination. (1) derives a simple monotonic property of the importance measure as the next Theorem 3.3 which is easily derived from Theorem 3.2 (1).

Theorem 3.3 Under the assumptions same to the ones of Theorem 3.2 (1), the following including relation holds :

$$A \subsetneqq B, \quad Cr(A,V) \subsetneqq \Omega_{B\cap A^c} \times Cr(B,V). \qquad ■$$

Taking the probability of the inclusion relation of the Theorem 3.3, we have

$$\boldsymbol{P}_{A^c}(Cr(A,V)) \leqq \boldsymbol{P}_{B^c}(Cr(B,V)),$$

which is a monotonic property of the group importance, and means that the larger the group is, the more important it is, in other words, the majority takes the initiative within the system.

Theorem 3.4 For non-empty subsets A and B of C, not necessarily mutually disjoint, Theorem 3.2 implies the following equality.

$$Cr(A \cup B, V) \quad = \quad Cr(A,V)_{(A \bigcup B)^c} \bigcup Cr(B,V)_{(A \bigcup B)^c} \ .$$

Proof : By the Theorem 3.2 (2), we may have the following decomposition :

$$\begin{aligned} Cr(A \cup B, V) &= \bigcup_{i \in A \cup B} Cr(i,V)_{(A \cup B)^c} \qquad \text{by (2) of the Theorem 3.2} \\ &= \left(\bigcup_{i \in A} Cr(i,V)_{(A \cup B)^c} \right) \bigcup \left(\bigcup_{i \in B} Cr(i,V)_{(A \cup B)^c} \right) \ . \end{aligned}$$

For example, $\bigcup_{i \in A} Cr(i,V)_{(A \cup B)^c} = Cr(A,V)_{(A \cup B)^c}$ holds, since

$$\begin{aligned} Cr(A,V) &= \bigcup_{i \in A} Cr(i,V)_{A^c} \quad \text{by Theorem 3.2 (2),} \\ Cr(A,V)_{A^c} &= Cr(A,V) \quad \text{since } Cr(A,V) \subseteqq \Omega_{A^c}\text{, a kind of property of the section.} \end{aligned}$$

Then, for $H \subseteqq A^c$, we have the following :

$$\begin{aligned} Cr(A,V)_H &= \left(\bigcup_{i \in A} Cr(i,V)_{A^c} \right)_H \\ &= \bigcup_{i \in A} (Cr(i,V)_{A^c})_H \quad \text{(taking the section at } A^c\text{, and then at } H\text{, i.e., the section at } A^c \cap H \text{)} \\ &= \bigcup_{i \in A} Cr(i,V)_{A^c \cap H} \quad (H \subseteqq A^c \text{ and then } H \cap A^c = H) \\ &= \bigcup_{i \in A} Cr(i,V)_H \ , \\ Cr(A \cup B, V) &= \bigcup_{i \in A} Cr(i,V)_{(A \cup B)^c} \bigcup \bigcup_{i \in B} Cr(i,V)_{(A \cup B)^c} \\ &= Cr(A,V)_{(A \cup B)^c} \bigcup Cr(B,V)_{(A \cup B)^c} \ . \qquad \blacksquare \end{aligned}$$

Theorem 3.4 denotes that a critical state vector of $A \cup B$, which is an element of $\Omega_{(A \cup B)^c}$, is cut off from a critical state vector of A or B at $(A \cup B)^c$.

The following corollary is clear from the Theorem3.4.

Corollary 3.1 Supposing $\{A_i\}_{i=1}^a$ to be a family of nonempty subsets of $B(\subseteqq C)$ such that $\bigcup_{i=1}^a A_i = B$, we have :

$$Cr(B,V) = Cr\left(\bigcup_{i=1}^a A_i, V \right) = \bigcup_{i=1}^a Cr(A_i,V)_{\bigcap_{i=1}^a A_i^c} = \bigcup_{i=1}^a Cr(A_i,V)_{B^c} \ . \qquad \blacksquare$$

Definition 3.2 For $A \subseteqq B$, noticing $Cr(A,V)_{B^c} \subseteqq Cr(B,V)$,

$$\frac{|Cr(A,V)_{B^c}|}{|Cr(B,V)|}$$

denotes a structural relative importance measure of the group A with respect to B, which is called as relative A-B group structural importance measure. $\blacksquare$

4 Examples

We restrict our examinations to the binary case for making the situation clear. In this section, for a k-out-of-n:G system and the bridge system, which are typical examples of binary-state systems, we present critical state vectors, basis for the group importance, by utilizing the methods given in the previous section.

4.1 A Binary-State k-out-of-n:G System

The structure function φ of a binary-state k-out-of-n:G system is defined as the following :

$$\text{for } \boldsymbol{y} \in \Omega_C,\ \varphi(\boldsymbol{y}) = \begin{cases} 1, & \sum_{j=1}^{n} y_j \geqq k \ , \\ 0, & \sum_{j=1}^{n} y_j < k \ . \end{cases} \tag{4}$$

The set of the critical state vectors of the component $i \in C$ is given by the following :

$$Cr(i, 1) \quad = \quad \left\{ (\cdot_i, \boldsymbol{x}) \ \middle|\ \sum_{i \neq j} x_i = k - 1 \right\} \ . \tag{5}$$

By the Theorem 3.2, $Cr(A, V)$, the set of the critical state vectors of a group A of the binary-state k-out-of-n:G system is given as follows, where $V = \{1\} \subseteqq S$.

Theorem 4.1 $Cr(A, V)$ of the binary-state k-out-of-n:G system is given as the following :

$$\begin{aligned} Cr(A, V) \quad = \quad \Bigg\{ \boldsymbol{x}_{A^c} \ \Bigg|\ & \sum_{i \in A^c} x_i \leqq k - 1, \quad \sum_{i \in A^c} (1 - x_i) \leqq n - k, \\ & |A| + \sum_{i \in A^c} x_i \geqq k, \quad |A| + \sum_{i \in A^c} (1 - x_i) \geqq n - k + 1 \Bigg\} . \end{aligned} \tag{6}$$

Proof : <u>The proof of $Cr(A, V) \subseteqq$ the right hand side of (6).</u> Following the Definition 3.1, the definition of $Cr(A, V)$ is give as the following :

$$Cr(A, V) = \{ \ \boldsymbol{x}_{A^c} \mid \exists \boldsymbol{a} \in \Omega_A,\ \exists \boldsymbol{b} \in \Omega_A,\ \varphi(\boldsymbol{a}, \boldsymbol{x}_{A^c}) = 1,\ \varphi(\boldsymbol{b}, \boldsymbol{x}_{A^c}) = 0 \ \} \ . \tag{7}$$

For $\boldsymbol{x}_{A^c} \in Cr(A, V)$, by the monotonic property of φ, we have

$$\varphi(\mathbf{1}_A, \boldsymbol{x}_{A^c}) = 1 \ , \quad \varphi(\mathbf{0}_A, \boldsymbol{x}_{A^c}) = 0 \ . \tag{8}$$

Furthermore, since the system is k-out-of-n:G system,

$$\begin{aligned} &\text{from the first equality,} && |A| + \sum_{i \in A^c} x_i \geqq k \ , \quad \sum_{i \in A^c} (1 - x_i) \leqq n - k \ , \\ &\text{from the second equality,} && |A| + \sum_{i \in A^c} (1 - x_i) \geqq n - k + 1 \ , \quad \sum_{i \in A^c} x_i \leqq k - 1 \ . \end{aligned}$$

Then, $Cr(A, V)$ is a subset of or equals to the right hand side of (6).
<u>The proof of $Cr(A, V) \supseteqq$ the right hand side of (6).</u> For an element $\boldsymbol{x}_{A^c}$ of the right hand side of (6),

$$\begin{aligned} \varphi(\mathbf{1}_A, \boldsymbol{x}_{A^c}) = 1, &\quad \text{since} \quad |A| + \sum_{i \in A^c} x_i \geqq k \ , \\ \varphi(\mathbf{0}_A, \boldsymbol{x}_{A^c}) = 0, &\quad \text{since} \quad |A| + \sum_{i \in A^c} (1 - x_i) \geqq n - k + 1 \ . \end{aligned}$$

Then the inclusion relation holds. ■

By the Theorem 4.1,

when $k = 1$, i.e., the system is a parallel system, $Cr(A, V) = \{\mathbf{0}_{A^c}\}$,
when $k = n$, i.e., the system is a series system, $Cr(A, V) = \{\mathbf{1}_{A^c}\}$,
when the system is 2-out-of-4:G system,

$$\begin{aligned} Cr(1,1) &= \{(1_2, 0_3, 0_4), (0_2, 1_3, 0_4), (0_2, 0_3, 1_4)\} \ , \\ Cr(\{1,2\},1) &= \{(0_3, 0_4), (1_3, 0_4), (0_3, 1_4)\} \ , \\ Cr(\{1,2,3\},1) &= \{(0_4), (1_4)\} \ . \end{aligned}$$

4.2 The Binary-State Bridge System

The minimal path and cut vectors of the binary-state bridge system are given, respectively, as

$$\begin{aligned} MI\left(\varphi^{-1}(1)\right) &= \big\{(1_1, 0_2, 0_3, 1_4, 0_5),\ (1_1, 0_2, 1_3, 0_4, 1_5),\ (0_1, 1_2, 1_3, 1_4, 0_5),\ (0_1, 1_2, 0_3, 0_4, 1_5)\big\}, \\ MA\left(\varphi^{-1}(0)\right) &= \big\{(0_1, 0_2, 1_3, 1_4, 1_5),\ (0_1, 1_2, 0_3, 1_4, 0_5),\ (1_1, 0_2, 0_3, 0_4, 1_5),\ (1_1, 1_2, 1_3, 0_4, 0_5)\big\}. \end{aligned}$$

Referring to the Fig.1, the structure function of the bridge system is given as

$$\varphi(x_1, x_2, x_3, x_4, x_5) = \max\big\{\min\{x_1, x_4\},\ \min\{x_1, x_3, x_5\},\ \min\{x_2, x_3, x_4\},\ \min\{x_2, x_5\}\big\}.$$

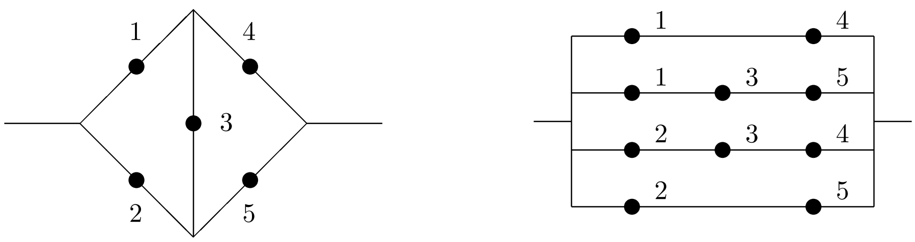

Figure 1: The reliability diagram and the max-min formula of the binary-state bridge system.

Following the characterization of the critical state vectors of components (3), $Cr(i, 1)$ $i \in C$ are given as the following :

$$\begin{aligned} Cr(1,1) &= \left\{ \begin{array}{l} (0_2, 0_3, 1_4, 0_5), (0_2, 1_3, 1_4, 0_5), (0_2, 0_3, 1_4, 1_5), (0_2, 1_3, 1_4, 1_5), \\ (1_2, 0_3, 1_4, 0_5), (0_2, 1_3, 0_4, 1_5) \end{array} \right\}, \\ Cr(2,1) &= \left\{ \begin{array}{l} (0_1, 1_3, 1_4, 0_5), (0_1, 1_3, 1_4, 1_5), (0_1, 0_3, 0_4, 1_5), (0_1, 1_3, 0_4, 1_5), \\ (0_1, 0_3, 1_4, 1_5), (1_1, 0_3, 0_4, 1_5) \end{array} \right\}, \\ Cr(3,1) &= \left\{ \ (1_1, 0_2, 0_4, 1_5), (0_1, 1_2, 1_4, 0_5) \ \right\}, \\ Cr(4,1) &= \left\{ \begin{array}{l} (1_1, 0_2, 0_3, 0_5), (1_1, 0_2, 0_3, 1_5), (1_1, 1_2, 0_3, 0_5), (1_1, 0_2, 1_3, 0_5), \\ (1_1, 1_2, 1_3, 0_5), (0_1, 1_2, 1_3, 0_5) \end{array} \right\}, \\ Cr(5,1) &= \left\{ \begin{array}{l} (0_1, 1_2, 0_3, 0_4), (0_1, 1_2, 0_3, 1_4), (1_1, 1_2, 0_3, 0_4), (0_1, 1_2, 1_3, 0_4), \\ (1_1, 1_2, 1_3, 0_4), (1_1, 0_2, 1_3, 0_4) \end{array} \right\}. \end{aligned}$$

Then, by the Theorem 3.2, we have the critical state vectors of groups of components from the critical state vectors of components, for example, as the following :

$$\begin{aligned} Cr(\{1,2\},1) &= Cr(1,1)_{\{3,4,5\}} \cup Cr(2,1)_{\{3,4,5\}} \\ &= \left\{ \ (0_3, 1_4, 0_5), (1_3, 1_4, 0_5), (1_3, 1_4, 1_5), (0_3, 0_4, 1_5), (1_3, 0_4, 1_5), (0_3, 1_4, 1_5) \ \right\}, \\ Cr(\{1,3\},1) &= Cr(1,1)_{\{2,4,5\}} \cup Cr(3,1)_{\{2,4,5\}} \\ &= \left\{ \ (0_2, 1_4, 0_5), (0_2, 1_4, 1_5), (1_2, 1_4, 0_5), (0_2, 0_4, 1_5) \ \right\}, \end{aligned}$$

$$
\begin{aligned}
Cr(\{1,4\},1) &= Cr(1,1)_{\{2,3,5\}} \cup Cr(4,1)_{\{2,3,5\}} \\
&= \left\{ \begin{array}{l} (0_2,0_3,0_5),(0_2,1_3,0_5),(0_2,0_3,1_5),(0_2,1_3,1_5),(1_2,0_3,0_5),(1_2,0_3,0_5), \\ (1_2,1_3,0_5) \end{array} \right\}, \\
Cr(\{1,5\},1) &= Cr(1,1)_{\{2,3,4\}} \cup Cr(5,1)_{\{2,3,4\}} \\
&= \left\{ \ (0_2,0_3,1_4),(0_2,1_3,1_4),(1_2,0_3,1_4),(0_2,1_3,0_4),(1_2,0_3,0_4),(1_2,1_3,0_4) \ \right\},
\end{aligned}
$$

$$
\begin{aligned}
Cr(\{1,2,3\},1) &= Cr(1,1)_{\{4,5\}} \cup Cr(2,1)_{\{4,5\}} \cup Cr(3,1)_{\{4,5\}} \\
&= \left\{ \ (1_4,0_5),(1_4,1_5),(0_4,1_5) \ \right\}, \\
Cr(\{1,2,4\},1) &= Cr(1,1)_{\{3,5\}} \cup Cr(2,1)_{\{3,5\}} \cup Cr(4,1)_{\{3,5\}} \\
&= \left\{ \ (0_3,0_5),(1_3,0_5),(0_3,1_5),(1_3,1_5) \ \right\} \ = \ \Omega_3 \times \Omega_5 \ , \\
Cr(\{1,2,5\},1) &= Cr(1,1)_{\{3,4\}} \cup Cr(2,1)_{\{3,4\}} \cup Cr(4,1)_{\{3,4\}} \\
&= \left\{ \ (0_3,0_4),(1_3,0_4),(0_3,1_4),(1_3,1_4) \ \right\} \ = \ \Omega_3 \times \Omega_4 \ , \\
Cr(\{1,2,3,4\},1) &= Cr(1,1)_{\{5\}} \cup Cr(2,1)_{\{5\}} \cup Cr(3,1)_{\{5\}} \cup Cr(4,1)_{\{5\}} \\
&= \left\{ \ (0_5),(1_5) \ \right\} \ = \ \Omega_5 \ .
\end{aligned}
$$

Taking the probability of each of the above formulae, we obtain the group importance measures, which are presented in the next section. ■

5 A Group Importance Measure

5.1 A Definition of a Group Importance Measure

We suppose $\boldsymbol{P}$ to be a probability on Ω_C, of which restriction to Ω_A $(A \subsetneqq C)$ is written as $\boldsymbol{P}_A$. The stochastic performance of the component i $(i \in C)$ is given by $\boldsymbol{P}_{\{i\}}$.

Definition 5.1 (A Definition of a group importance measure) The A-V-group importance measure of the group $A \subsetneqq C$ is defined to be $\boldsymbol{P}_{A^c}\big(Cr(A,V)\big)$. ■

Remark 5.1 (The Birnbaum importance measure) For a binary-state system (Ω_C, S, φ), $Cr(i,V)$ $(i \in C)$ is the set of all the critical state vectors of the component i, and then $\boldsymbol{P}_{\{i\}^c}(Cr(A,V))$ is the well-known Birnbaum importance measure of the component i. Refer to Z.W. Birnbaum[4] and F. Ohi[17]. ■

Theorem 5.1 When $A \subsetneqq B \subset C$, $\boldsymbol{P}_{A^c}(Cr(A,V)) \leqq \boldsymbol{P}_{B^c}(Cr(B,V))$ holds for every probability $\boldsymbol{P}$, which denotes that the importance of a large group is greater.

Proof : From Theorem 3.2 (1), $Cr(A,V) \subsetneqq \Omega_{B\cap A^c} \times Cr(B,V)$ holds and then, taking the probability, the stochastic inequality holds. ■

For non-empty groups A and B, the Theorem 5.1 implies the following very apparent inequalities :

$$
\left. \begin{array}{l} \boldsymbol{P}_{A^c} Cr(A,V) \\ \boldsymbol{P}_{B^c} Cr(B,V) \end{array} \right\} \leqq \boldsymbol{P}_{A^c \cap B^c} Cr(A \cup B, V) \ . \tag{9}
$$

When $\boldsymbol{P}$ is associated, F.Ohi[12], applying the method of F.Ohi[20] to $\mathcal{P}(A,V)$ of the Theorem 3.1, we may have a bound for A-V-group importance measure, $\boldsymbol{P}(Cr(A,V))$, which is omitted here. We may have another bound by Theorem 3.2. Applying the inclusion and exclusion principle to the Theorem 3.2 (2),

$$
\boldsymbol{P}_{A^c}(Cr(A,V)) \leqq \left\{ 1 \ , \ \sum_{i \in A} \boldsymbol{P}_{A^c}(Cr(i,V)_{A^c}) \right\}
$$

is given and then furthermore applying the method of F.Ohi [20] to the importance measure of the component i, we have the two kinds of bounds for the group importance measure, however, it remains for future which bound is strict.

5.2 Examples

5.2.1 Binary-state bridge system (Continuation of the subsection 4.2)

Critical state vectors for the binary-state bridge system are presented in the subsection 4.2. Taking probabilities of these state vectors, we have the following group importance measures.

$$
\begin{aligned}
\text{the importance of the group } \{1,\,2\} \quad &: \quad \boldsymbol{P}_{\{3.4,5\}}(Cr\{1,2\},1) \\
&= \quad \boldsymbol{P}_{\{3,4,5\}}(0_3,1_4,0_5) + \boldsymbol{P}_{\{3,4,5\}}(1_3,1_4,0_5) + \boldsymbol{P}_{\{5\}}(1) \\
&= \quad \boldsymbol{P}_{\{4,5\}}(1_4,0_5) + \boldsymbol{P}_{\{5\}}(1_5)\ , \\
\text{the importance of the group } \{1,2,3\} \quad &: \quad \boldsymbol{P}_{\{4,5\}}(Cr\{1,2,3\},1) \\
&= \quad \boldsymbol{P}_{\{4,5\}}(1_4,0_5) + \boldsymbol{P}_{\{4,5\}}(1_4,1_5) + \boldsymbol{P}_{\{4,5\}}(0_4,1_5) \\
&= \quad \boldsymbol{P}_{\{4,5\}}(1_4,0_5) + \boldsymbol{P}_{\{5\}}(1_5)\ , \\
\text{the importance of the group } \{1,2,3,4\} \quad &: \quad \boldsymbol{P}_{\{5\}}(Cr(\{1,2,3,4\},1) \\
&= \quad \boldsymbol{P}_{\{5\}}(\Omega_5) = 1\ , \\
\max\left\{ 1,\ \sum_{i\in\{1,2,3,4\}} \boldsymbol{P}_5\left(Cr(i,V)_{\{5\}}\right) \right\} \quad &= \quad 1\ .
\end{aligned}
$$

The stochastic bounds for the group importance measure seem to be rough and we require more precise examination of the measure.

6 Concluding Remarks

In this paper it has proved in the context of a multi-state system of which state spaces are mathematically finite partially ordered sets having the minimum elements that the set of the critical state vectors of a group is the union of the sets of critical state vectors of components constructing the group, which denotes that the importance measures of components are the basis for the examination of importance measures, in other words, Theorem 3.2 is the basis.

We have also shown that greater the group is, more the importance of the group is, i.e., a monotonic property of an importance measure of a group.

A diverse examination of importance measures of groups is remained for the future work, especially it is an essential issue to extend the state spaces to the ones not necessarily having the minimum elements.

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